On the existence of stationary solutions for certain systems of integro-differential equations with the double scale anomalous diffusion

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Existence of stationary solutions of nonlocal reaction- diffusion equations: existence of biological species.

1. Introduction

The integro-differential systems: nonlocal consumption of resources, intra-specific competition. Nonlocal interaction of neurons. $\mathbb{R}^3$, $N \geq 2$.

$$\frac{\partial u_m}{\partial t} =$$

$$-D_m[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m + \int_{\mathbb{R}^3} K_m(x-y)g_m(u(y,t))dy + f_m(x), \quad (1)$$

$$1 \leq m \leq N, \quad s_{1,m} < s_{2,m} < 1, \quad \frac{1}{4} < s_{1,m} < \frac{3}{4}$$

from the cell population dynamics. Cell genotype is x, cell density distributions for various groups of cells as functions of their genotype and time are $u_m(x,t)$, such that

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The evolution of the cell densities is due to the cell proliferation, mutations and cell influx/efflux. The change of genotype due to small random mutations-double scale anomalous diffusion terms. Large mutations are via the integral production terms. $g_m(u)$ are the rates of cell birth depending on u (density dependent proliferation). $K_m(x-y)$ are the proportions of newly born cells changing their genotype from y to x, depend on the distance between the genotypes.

$f_m(x)$ are the influxes/effluxes of cells for different genotypes. Proved the existence of a stationary solution in $H^2(\mathbb{R}^3, \mathbb{R}^N)$. The space variable corresponds to the cell genotype.

Double scale anomalous diffusion problem with

$$l_m := (-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}} : H^{2s_{2,m}}(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad 1 \le m \le N \tag{2}$$

defined via the spectral calculus, namely for $\phi(x) \in H^{2s_{2,m}}(\mathbb{R}^3)$.

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\phi}(p) e^{ipx} dp,$$

$$l_m \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} (|p|^{2s_{1,m}} + |p|^{2s_{2,m}}) \widehat{\phi}(p) e^{ipx} dp, \quad 1 \le m \le N.$$

The single equation with a single fractional Laplacian in the diffusion term: V.V., V. Volpert, J. Pseudo Differ. Oper. Appl. (2015).

Anomalous diffusion: plasma physics and turbulence.

B.Carreras, V.Lynch, G.Zaslavsky, Phys. Plasmas (2001).

Surface diffusion.

J.Sancho, A. Lacasta, K.Lindenberg, I.Sokolov, A.Romero, Phys. Rev. Lett. (2004).

Semiconductors.

H.Scher, E.Montroll, Phys. Rev. B (1975).

Physical meaning: the random process occurs with longer jumps in comparison with normal diffusion.

Normal diffusion: finite moments of jump length distribution.

Anomalous diffusion: not the case.

R. Metzler, J. Klafter, Phys. Rep. (2000).

The existence of stationary solutions.

For $1 \le m \le N, \; D_m = 1, \; s_{1,m} < s_{2,m} < 1, \; \frac{1}{4} < s_{1,m} < \frac{3}{4}$

$$-[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m + \int_{\mathbb{R}^3} K_m(x-y)g_m(u(y))dy + f_m(x) = 0. \quad (3)$$

Set $K_m(x) = \varepsilon_m H_m(x), \; \varepsilon_m \ge 0$ small parameters, $\varepsilon = \max_{1 \le m \le N} \varepsilon_m$.

Sobolev inequality for the fractional negative Laplacian.

E.H. Lieb, Ann. of Math., (1983).

$$\|f_m(x)\|_{L^{\frac{6}{4s_{1,m}-1}}(\mathbb{R}^3)} \leq c_{sob}\|(-\Delta)^{1-s_{1,m}}f_m(x)\|_{L^2(\mathbb{R}^3)},$$

$$\frac{1}{4} < s_{1,m} < \frac{3}{4}.$$

Hence

$$f_m(x) \in L^1(\mathbb{R}^3) \cap L^{\frac{6}{4s_{1,m}-1}}(\mathbb{R}^3).$$

Standard interpolation argument: $f_m(x) \in L^2(\mathbb{R}^3), \quad 1 \leq m \leq N.$

Fractional Sobolev norms

$$\|\phi\|^2_{H^{2s_{2,m}}(\mathbb{R}^3)} := \|\phi\|^2_{L^2(\mathbb{R}^3)} + \|(-\Delta)^{s_{2,m}}\phi\|^2_{L^2(\mathbb{R}^3)}. \tag{4}$$

Standard Sobolev embedding

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \le c_e \|\phi\|_{H^2(\mathbb{R}^3)}, \quad c_e > 0.$$

Here c_e is the constant of the embedding.

When all the parameters ε_m vanish, we obtain the generalized Poisson type equations

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}] u_m(x) = f_m(x), \quad 1 \le m \le N. \tag{5}$$

The standard Fourier transform

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}^3. \tag{6}$$

Upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{(2\pi)^{\frac{3}{2}}}\|\phi(x)\|_{L^1(\mathbb{R}^3)}. \tag{7}$$

2. Solvability conditions for the generalized Poisson equation.

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]\phi(x) = f(x), \quad 0 < s_1 < s_2 < 1. \tag{8}$$

Let solution $\phi(x) \in L^2(\mathbb{R}^3)$, assume $f(x) \in L^2(\mathbb{R}^3)$. Apply Fourier transform (6) to equation (8). Hence

$$(|p|^{2s_1} + |p|^{2s_2})\widehat{\phi}(p) = \widehat{f}(p) \in L^2(\mathbb{R}^3),$$

so that

$$\int_{\mathbb{R}^3}[|p|^{2s_1} + |p|^{2s_2}]^2|\widehat{\phi}(p)|^2 dp < \infty.$$

The simple identity

$$\|(-\Delta)^{s_2}\phi\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |p|^{4s_2} |\widehat{\phi}(p)|^2 dp < \infty.$$

Hence $(-\Delta)^{s_2}\phi \in L^2(\mathbb{R}^3)$. Recall norm definition (4). Then $\phi(x) \in H^{2s_2}(\mathbb{R}^3)$ as well.

Uniqueness.

Let $\phi_1(x),\ \phi_2(x) \in H^{2s_2}(\mathbb{R}^3)$ solve (8).
$w(x) := \phi_1(x) - \phi_2(x) \in H^{2s_2}(\mathbb{R}^3)$.

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]\phi_1 = f(x), \quad [(-\Delta)^{s_1} + (-\Delta)^{s_2}]\phi_2 = f(x).$$

Hence

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]w = 0.$$

$l = (-\Delta)^{s_1} + (-\Delta)^{s_2} : H^{2s_2}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ (see (2)) no nontrivial zero modes. Then w vanishes in $\mathbb{R}^3$.

Apply Fourier transform (6) to equation (8).

$$\widehat{\phi}(p)=\frac{\widehat{f}(p)}{|p|^{2s_1}+|p|^{2s_2}}=\frac{\widehat{f}(p)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}+\frac{\widehat{f}(p)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|>1\}}. \tag{9}$$

Second term

$$\left|\frac{\widehat{f}(p)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|>1\}}\right|\le\frac{|\widehat{f}(p)|}{2}\in L^2(\mathbb{R}^3)$$

as assumed. First term via (7)

$$\left|\frac{\widehat{f}(p)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}\right|\le\frac{\|f\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}|p|^{2s_1}}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^3)$$

for $s_1\in\left(0,\frac{3}{4}\right)$.

No orthogonality conditions are required for right side $f(x)$.

Solvability of (8) in $H^{2s_2}(\mathbb{R}^3)$ for $s_1<s_2<1, \quad s_1\in\left(0,\frac{3}{4}\right)$.

Then let $s_1 \in \left[\frac{3}{4}, 1\right)$. Write

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}(q,\sigma)}{\partial q} dq.$$

σ-the angle variables on the sphere.

First term in the right side of (9)

$$\frac{\widehat{f}(0)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p|\leq 1\}} + \frac{\int_0^{|p|} \frac{\partial \widehat{f}(q,\sigma)}{\partial q} dq}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p|\leq 1\}}. \tag{10}$$

From the definition of the standard Fourier transform (6)

$$\left| \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \leq \frac{\|xf(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}}.$$

Then for the second term in (10)

$$\left|\frac{\int_0^{|p|}\frac{\partial \widehat{f}(q,\sigma)}{\partial q}dq}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}\right|\le \frac{\|xf(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}}|p|^{1-2s_1}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^3).$$

Remains to analyze

$$\frac{\widehat{f}(0)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}. \tag{11}$$

(11) is contained in $L^2(\mathbb{R}^3)$ if and only if

$$\widehat{f}(0)=0. \tag{12}$$

Clearly,

$$\widehat{f}(0)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}f(x)dx=\frac{1}{(2\pi)^{\frac{3}{2}}}(f(x),1)_{L^2(\mathbb{R}^3)}.$$

Then (12) is equivalent to orthogonality condition

$$(f(x),1)_{L^2(\mathbb{R}^3)}=0, \quad s_1\in\left[\frac{3}{4},1\right). \tag{13}$$

The space variable corresponds to cell genotype, not the usual physical space. The space dimension is not limited to $d = 3$. Let all $\varepsilon_m = 0$. Then

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m = f_m(x), \quad 1 \le m \le N.$$

Unique $u_{0,m}(x) \in H^{2s_{2,m}}(\mathbb{R}^3)$ with $s_{1,m} < s_{2,m} < 1$, $\dfrac{1}{4} < s_{1,m} < \dfrac{3}{4}$. Then

$$[-\Delta + (-\Delta)^{1+s_{2,m}-s_{1,m}}]u_{0,m}(x) = (-\Delta)^{1-s_{1,m}} f_m(x) \in L^2(\mathbb{R}^3).$$

Apply the Fourier transform (6) to deduce
$\Delta u_{0,m}(x) \in L^2(\mathbb{R}^3), \ 1 \le m \le N$. Obtain

$$u_{0,m}(x) \in H^2(\mathbb{R}^3).$$

Recall the definition of the norm

$$\|u\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{H^2(\mathbb{R}^3)} = \sum_{m=1}^{N} \{\|u_m\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u_m\|^2_{L^2(\mathbb{R}^3)}\}.$$

We have

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^2(\mathbb{R}^3, \mathbb{R}^N).$$

3. Fixed point argument

Seek the resulting solution of the stationary nonlinear problem (3) as

$$u(x) = u_0(x) + u_p(x), \tag{14}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Perturbative system of equations with

$s_{1,m} < s_{2,m} < 1, \ \frac{1}{4} < s_{1,m} < \frac{3}{4}, \ 1 \le m \le N$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{p,m}(x) =$$

$$\varepsilon_m \int_{\mathbb{R}^3} H_m(x-y) g_m(u_0(y) + u_p(y)) dy. \tag{15}$$

The Fixed Point argument in a closed ball in the Sobolev space:

$$B_\rho = \{u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N) \mid \|u\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{16}$$

Seek the solution of (15) as the fixed point of the auxiliary nonlinear problem with $s_{1,m} < s_{2,m} < 1$, $\dfrac{1}{4} < s_{1,m} < \dfrac{3}{4}$, $1 \le m \le N$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m(x) = \varepsilon_m \int_{\mathbb{R}^3} H_m(x-y) g_m(u_0(y) + v(y))dy \tag{17}$$

in ball (16). Non-Fredholm operators in the left side of (17) (see (2))

$$l_m : H^{2s_{2,m}}(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad 1 \le m \le N.$$

The essential spectrum $\sigma_{ess}(l_m) = [0, +\infty)$, no bounded inverse.

V.V., V.Volpert, Doc. Math. (2011), Anal. Math. Phys. (2012) relied on the orthogonality relations.

via the fixed point technique.

Y. Chen, V.V., Pure Appl. Funct. Anal., (2021).

The Schrödinger operator involved had the Fredholm property.

When each kernel $K_m(x) = \delta(x)$ is the Dirac's delta measure, the system of standard nonlinear heat equations.

The existence of the stationary solution in the case of the single fractional Laplacian in the diffusion term of the single equation.

V.V., V. Volpert, J. Pseudo Differ. Oper. Appl. (2015).

The operator τ_g via the auxiliary nonlinear system (17), such that $u = \tau_g v, \quad u$ is a solution. Our main result is as follows.

Theorem 1. Under our technical assumptions problem (17) defines the map $\tau_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon \leq \varepsilon^*$ for a certain $\varepsilon^* > 0$. The unique fixed point $u_p(x)$ of the map τ_g is the only solution of system (15) in B_ρ.

The resulting stationary solution of (3) is nontrivial: the source terms $f_m(x)$ are nontrivial for some $1 \leq m \leq N$ and $g_m(0) = 0$ as assumed.

Proof. Choose arbitrarily $v(x) \in B_\rho$, denote

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \leq m \leq N.$$

Apply the standard Fourier transform (6) to (17). Thus, for $1 \leq m \leq N$

$$\widehat{u_m}(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{H_m}(p)\widehat{G_m}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}, \quad s_{1,m} < s_{2,m} < 1, \quad \frac{1}{4} < s_{1,m} < \frac{3}{4}.$$

The norm

$$\|u_m\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 \varepsilon_m^2 \int_{\mathbb{R}^3} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_m}(p)|^2}{[|p|^{2s_{1,m}} + |p|^{2s_{2,m}}]^2} dp.$$

Express $\displaystyle\int_{\mathbb{R}^3} dp = \int_{|p| \leq R} dp + \int_{|p| > R} dp$ with $R \in (0, +\infty)$ and minimize over R.

We derive

$$\|u\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \varepsilon C \leq \rho$$

for all $\varepsilon > 0$ small enough, such that $u(x) \in B_\rho$ as well.

Uniqueness.

Suppose for some $v(x) \in B_\rho$ there are two solutions $u_{1,2}(x) \in B_\rho$ of (17).

Clearly, their difference $w(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N)$ solves the homogeneous system

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]w_m(x) = 0,$$

$s_{1,m} < s_{2,m} < 1,\ \frac{1}{4} < s_{1,m} < \frac{3}{4},\ 1 \leq m \leq N.$

The operators $l_m : H^{2s_{2,m}}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ (see (2)). No nontrivial zero modes, $w(x)$ vanishes in $\mathbb{R}^3$.

Then (17) defines a map $\tau_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

To show that this map is a strict contraction.

Choose arbitrarily $v_{1,2}(x) \in B_\rho$. Then $u_{1,2} := \tau_g v_{1,2} \in B_\rho$ as well, ε small enough. For $s_{1,m} < s_{2,m} < 1$, $\frac{1}{4} < s_{1,m} < \frac{3}{4}$, $1 \le m \le N$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{1,m}(x) = \varepsilon_m \int_{\mathbb{R}^3} H_m(x-y)g_m(u_0(y)+v_1(y))dy,$$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{2,m}(x) = \varepsilon_m \int_{\mathbb{R}^3} H_m(x-y)g_m(u_0(y)+v_2(y))dy.$$

Introduce
$G_{1,m}(x) := g_m(u_0(x)+v_1(x)),\ G_{2,m}(x) := g_m(u_0(x)+v_2(x)),\ 1 \le m \le N$.
Apply the standard Fourier transform (6). Arrive at

$$\widehat{u_{1,m}}(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{H_m}(p)\widehat{G_{1,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}},$$

$$\widehat{u_{2,m}}(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{H_m}(p)\widehat{G_{2,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}.$$

Write the norm

$$\|u_{1,m}-u_{2,m}\|_{L^2(\mathbb{R}^3)}^2 = \varepsilon_m^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,m}}(p)-\widehat{G_{2,m}}(p)|^2}{[|p|^{2s_{1,m}}+|p|^{2s_{2,m}}]^2} dp.$$

Express

$$\int_{\mathbb{R}^3} dp = \int_{|p|\le R} dp + \int_{|p|>R} dp,$$

minimize over $R \in (0,+\infty)$. Estimate the norm

$$\|u_1 - u_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \le \varepsilon C \|v_1 - v_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

The map $\tau_g : B_\rho \to B_\rho$ defined by (17) is a strict contraction for all $\varepsilon > 0$ small enough. Unique fixed point $u_p(x)$, the only solution of the perturbative system (15) in B_ρ.

The resulting solution of the stationary problem (3):

$$u(x) = u_0(x) + u_p(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N),$$

where $u_{0,m}(x),\ 1 \le m \le N$ solve the generalized Poisson equations (5).

Also proved: cumulative $u(x)$ is continuous in $H^2(\mathbb{R}^3, \mathbb{R}^N)$ with respect to the nonlinear, twice continuously differentiable vector function $g(z)$.

4. Discussion of the future work.

1. The global well-posedness of the integro-differential equations for $x \in \mathbb{R},\ t \geq 0$. Anomalous diffusion and transport. M.Efendiev, V.V., Math. Methods Appl. Sci. (2025).

Bi-Laplacian and transport. Z. Angew. Math. Phys. (2025).

To generalize these results to the periodic case, to the systems of coupled equations, to the problems with two kernels, to the double scale anomalous diffusion.

2. To perform the iterations of the kernels of integro-differential equations and the systems of such equations to show the existence of their stationary solution in the sense of sequences.

V.V., V.Volpert, J.Math.Sci.(NY) (2018), Mediterr. J. Math. (2018).
M.Efendiev, V.V., Osaka J. Math. (2020).