On quasi-isometric liftings for operators on Hilbert spaces

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Contents

1. An introduction on the theory of bounded linear operators
2. Matrix Representation of a bounded linear operator
3. Extensions and liftings for operators
4. Some classes of operators related to dilation theory
5. Quasi-isometries, quasicontractions and operators similar to contractions
6. References

An introduction on the theory of bounded linear operators

Let $\mathcal{H}$ be a complex Hilbert space. Some examples of complex Hilbert spaces are: $\ell^2_+(\mathbb{C})$, $\ell^2(\mathbb{C})$, $L^2(\mathbb{T})$ (where $\mathbb{T}$ denotes the unit circle about the origin of the complex plane), etc.

Definition 1.1

*Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. The **orthogonal complement subspace** $\mathcal{M}^\perp$ of $\mathcal{M}$ is defined by*

$$\mathcal{M}^\perp = \{z \in \mathcal{H} : (z, k) = 0 \text{ for all } k \in \mathcal{M}\}.$$

Theorem 1.2 (Orthogonal decomposition.)

Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Any vector h in $\mathcal{H}$ has the unique representation as follows :

$$h = k \bigoplus z$$

where $k \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. Briefly it can be expressed as $\mathcal{H} = \mathcal{M} \bigoplus \mathcal{M}^\perp$.

Definition 1.3

*A mapping T from a Hilbert space $\mathcal{H}$ to $\mathcal{H}$ is said to be a **linear operator** if T satisfies:*

(i) *additive :* $T(h+k) = Th + Tk$ *for any $h, k \in \mathcal{H}$.*

(ii) *homogeneous :* $T(\alpha h) = \alpha Th$ *for any $h \in \mathcal{H}$ and any complex number α.*

The identity operator I is defined by $\mathrm{I}h = h$ for all $h \in \mathcal{H}$, and the zero operator 0 is defined by $0h = 0$ for all $h \in \mathcal{H}$.

Definition 1.4

*A linear operator T on a Hilbert space $\mathcal{H}$ is said to be **bounded** if there exists $c > 0$ such that $\|Th\| \leq c\|h\|$ for all $h \in \mathcal{H}$. The norm of T is defined by*

$$\|T\| = \inf\{c > 0 : \|Th\| \leq c\|h\| \text{ for all } h \in \mathcal{H}\}.$$

Proposition 1.5

For any bounded linear operator T,

$$\|T\| = \sup\{\|Th\| : \|h\| = 1\} = \sup\{\|Th\| : \|h\| \leq 1\}.$$

Definition 1.6

*$\mathcal{B}(\mathcal{H})$ means the **set of all bounded linear operators on a Hilbert space** $\mathcal{H}$. Needless to say, $\mathcal{B}(\mathcal{H})$ can be regarded as an extension of the set of all $n \times n$ matrices.*

Theorem 1.7

For any linear operator T on a Hilbert space $\mathcal{H}$, the following statements are mutually equivalent:

(i) *T is bounded;*

(ii) *T is continuous on the whole space $\mathcal{H}$;*

(iii) *T is continuous on some point h_0 on $\mathcal{H}$.*

Let T be an operator in $\mathcal{B}(\mathcal{H})$.
For each fixed $k \in \mathcal{H}$, consider a functional f defined by $f(h) = (\ Th, k\ )$ on $\mathcal{H}$.
According to Riesz's representation theorem, there exists uniquely $u \in \mathcal{H}$ such that $f(h) = (Th, k) = (h, u)$ for all $h \in \mathcal{H}$.
Hence, we may define T^*, the **adjoint operator** of T, by

$$(Th, k) = (h, u) = (h, T^*k)\,, \text{ for } h, k \in \mathcal{H}.$$

Proposition 1.8

Let $T \in \mathcal{B}(\mathcal{H})$ be an operator on a Hilbert space $\mathcal{H}$. Then T^* is also in $\mathcal{B}(\mathcal{H})$, and the following properties hold:

- $\|T^*\| = \|T\|$
- $(T^*)^* = T$
- $(T_1 + T_2)^* = T_1^* + T_2^*$
- $(ST)^* = T^*S^*$.

Corollary 1.9

Let $T \in \mathcal{B}(\mathcal{H})$ be an operator. Then

- $\|T^*T\| = \|TT^*\| = \|T\|^2$.
- $T^*T = 0$ *if and only if* $T = 0$.

Definition 1.10

*$\mathcal{R}(T)$, the **range** of T, is defined by $\mathcal{R}(T) = \{Th : h \in \mathcal{H}\}$, and $\mathcal{N}(T)$, the **kernel** of T, is defined by $\mathcal{N}(T) = \{h \in \mathcal{H} : Th = 0\}$.*

Matrix Representation of a bounded linear operator

Let $T \in \mathcal{B}(\mathcal{H})$. Any orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ induces for the operator T the following representation:

$$T = \begin{pmatrix} T_0 & T_1 \\ T_2 & T_3 \end{pmatrix} \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix}.$$

This matrix form indicates the following relations: $\begin{cases} T_0 : \mathcal{M} \to \mathcal{M}, \\ T_1 : \mathcal{M}^\perp \to \mathcal{M}, \\ T_2 : \mathcal{M} \to \mathcal{M}^\perp, \\ T_3 : \mathcal{M}^\perp \to \mathcal{M}^\perp. \end{cases}$

Moreover, its adjoint T^* is represented as:

$$T^* = \begin{pmatrix} T_0^* & T_2^* \\ T_1^* & T_3^* \end{pmatrix} \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix}.$$

This matrix form indicates the following relations: $\begin{cases} T_0^* : \mathcal{M} \to \mathcal{M}, \\ T_1^* : \mathcal{M} \to \mathcal{M}^\perp, \\ T_2^* : \mathcal{M}^\perp \to \mathcal{M}, \\ T_3^* : \mathcal{M}^\perp \to \mathcal{M}^\perp. \end{cases}$

Remark 2.1

Any operator $T \in \mathcal{B}(\mathcal{H})$ admits the following matrix representations with respect to the decompositions $\mathcal{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$:

- $T = \begin{pmatrix} 0 & T_1 \\ 0 & T_3 \end{pmatrix} \begin{bmatrix} \mathcal{N}(T) \\ \overline{\mathcal{R}(T^*)} \end{bmatrix},$
- $T = \begin{pmatrix} T_0 & T_1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T^*) \end{bmatrix}.$

Definition 2.2

Let T be an operator on a Hilbert space $\mathcal{H}$.

(i) *A closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to be **invariant under** T if $T\mathcal{M} \subset \mathcal{M}$, that is, $Th \in \mathcal{M}$ whenever $h \in \mathcal{M}$.*

(ii) *A closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to **reduce** T if $T\mathcal{M} \subset \mathcal{M}$ and $T\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$, that is, if $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are both invariant under T.*

Theorem 2.3

Let T be an operator on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then the following conditions are mutually equivalent:

(i) $T\mathcal{M} \subset \mathcal{M}$.

(ii) $T^*\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$.

Definition 2.4

Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$. Some special types of operators are defined as follows:

1. ***Normal**: $TT^* = T^*T$;*
2. ***Self-adjoint**: $T = T^*$;*
3. ***Positive operator** (denoted by $T \geq 0$): $(Th, h) \geq 0$ for all $h \in \mathcal{H}$;*
4. ***Isometry**: $T^*T = \mathrm{I}$;*
5. ***Coisometry**: an operator whose adjoint is an isometry;*
6. ***Unitary operator**: $T^*T = TT^* = \mathrm{I}$ i.e. $||Th|| = ||T^*h|| = ||h||$, for all $h \in \mathcal{H}$;*
7. ***Contraction**: $\mathrm{I} \geq T^*T$, i.e. $||T|| \leq 1$.*

Examples

Consider the Hilbert space $L^2(\mathbb{T})$.
Let $\varphi : \mathbb{T} \to \mathbb{C}$ be a bounded function.
We consider the mapping $M_\varphi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$M_\varphi f = \varphi f \quad \text{for every} \quad f \in L^2(\mathbb{T}).$$

This mapping is called the **multiplication operator** on $L^2(\mathbb{T})$.

1. M_φ is linear and bounded (i.e. $M_\varphi \in \mathcal{B}(L^2(\mathbb{T}))$).
2. $||M_\varphi|| = ||\varphi||_\infty = sup_{z\in\mathbb{T}}|\varphi(z)|$.
3. $M_\varphi^* g = \bar{\varphi} g$ for every $g \in L^2(\mathbb{T})$.
4. M_φ is a normal operator, because $M_\varphi M_\varphi^* g = \varphi\overline{\varphi} g = \overline{\varphi}\varphi g = M_\varphi^* M_\varphi g$, for every $g \in L^2(\mathbb{T})$.
5. M_φ is unitary if and only if $\varphi(z) \in \mathbb{T}$ for all $z \in \mathbb{T}$.
6. M_φ is self-adjoint if and only if $\varphi(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$.
7. M_φ is positive if and only if $\varphi(z) \geq 0$ for all $z \in \mathbb{T}$.

Definition 2.5

*Let $\mathcal{H} = \ell^2_+(\mathbb{C})$ be a Hilbert space with the canonical orthonormal basis $(e_n)_{n\geq 0}$. An operator $W \in \mathcal{B}(\mathcal{H})$ is called a **weighted shift** (**forward weighted shift**) if there exists a bounded sequence of nonnegative scalars $(w_n)_{n\geq 0}$ (called **weights**) such that*

$$We_n = w_n e_{n+1}, \quad \forall n \geq 0.$$

*The adjoint operator $W^* : \mathcal{H} \to \mathcal{H}$ called the **backward weighted shift** is defined by*

$$W^* e_0 = 0, \quad W^* e_{n+1} = w_n e_n, \quad n \geq 0.$$

Remark 2.6

- *If all $w_n = 1$, W is the classical **unilateral shift**, which is an isometry.*
- *The operator is completely determined by the sequence of weights (w_n).*
- $||W|| = sup_{n\in\mathbb{N}} w_n$.
- *W is a contraction if and only if $w_n \in [0,1]$, $\forall n \geq 0$.*

- **Matrix Representation of W:**

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ w_0 & 0 & 0 & 0 & \cdots \\ 0 & w_1 & 0 & 0 & \cdots \\ 0 & 0 & w_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- **Matrix Representation of W^*:**

$$W^* = \begin{pmatrix} 0 & w_0 & 0 & 0 & \cdots \\ 0 & 0 & w_1 & 0 & \cdots \\ 0 & 0 & 0 & w_2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Extensions and liftings for operators

Let $\mathcal{H}$, and $\mathcal{K}$ be complex Hilbert spaces such that $\mathcal{H} \subset \mathcal{K}$ as a closed subspace.

Definition 3.1

*For $T \in \mathcal{B}(\mathcal{H})$, $S \in \mathcal{B}(\mathcal{K})$ we say that S is an **extension** of T if S has the matrix relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ in the form $S = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix}$. In this case $S\mathcal{H} \subset \mathcal{H}$.*

Definition 3.2

*For T and S, as in the Definition 3.1 we say that S is a **lifting** of T if S has the matrix relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ of the form $S = \begin{pmatrix} T & 0 \\ * & * \end{pmatrix}$. In this case $S\mathcal{H}^{\perp} \subset \mathcal{H}^{\perp}$.*

Remark 3.3

S is a lifting (extension) of $T \Leftrightarrow S^*$ is an extension (lifting) of T^*.

Definition 3.4

*If S is an extension of T then T is called a **restriction** of S.*
*In the case, when, S is a lifting of T then T is called a **lifting compression** of S.*

A lifting (extension) S of T is called unitary, isometric or coisometric, respectively, if S is a unitary operator, an isometry or a coisometry.

Some classes of operators related to dilation theory

Definition 4.1

Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$.
Left and right invertible operators:

- *T is **left-invertible** if there exists S such that $ST = \mathrm{I}$. In this case S will be a left inverse of T.*
- *T is **right-invertible** if there exists S such that $TS = \mathrm{I}$. In this case S will be a right inverse of T.*
- *T is **invertible** if it is left and right- invertible.*

Proposition 4.2

The following assertions are equivalent in pairs:

- *T is left-invertible;*
- *$T^*T \geq \alpha I$, i.e. T is bounded from below;*
- *T is injective with closed range.*

Proposition 4.3

The following assertions are equivalent in pairs:

- *T is right-invertible;*
- *T is surjective.*

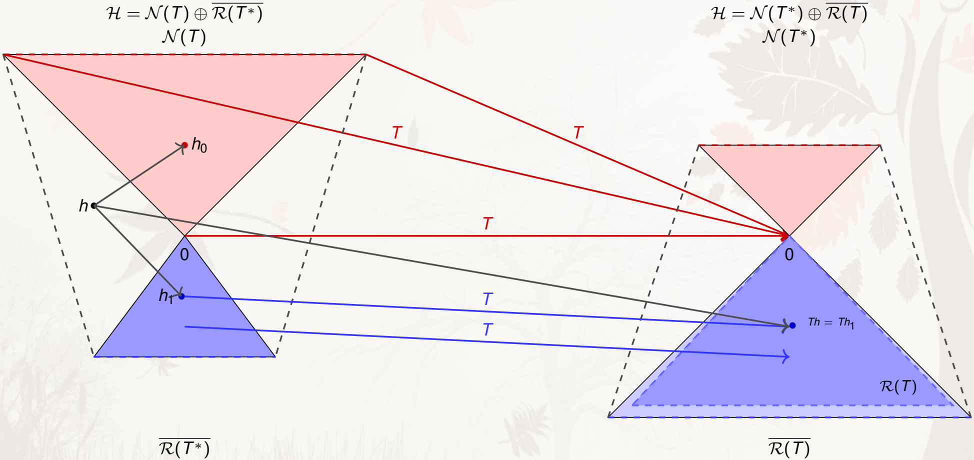
Remark 4.4

T is left-invertible if and only if T^* is right-invertible.

Example 1

Here are some examples of operators with invertible properties:

- An isometry is always left-invertible.
  More generally, a weighted shift W is left-invertible if and only if $\inf_{n\geq 0} w_n > 0$.
- A coisometry is consequently right-invertible.
- Finally, a unitary operator is invertible.



Quasi-isometries, quasicontractions and operators similar to contractions

An important objective in this thesis will be to characterize and study from the point of view of dilation theory operators on a Hilbert space that are similar to contractions.

Definition 5.1

*We say that $T \in \mathcal{B}(\mathcal{H})$ is **similar to a contraction** on $\mathcal{H}$ if there exists an invertible operator $L \in \mathcal{B}(\mathcal{H})$ such that*

$$\left\| LTL^{-1} \right\| \leq 1.$$

Proposition 5.2

It can be proved that T is similar to a contraction if and only if there exists an invertible $A \geq 0$, $A \in \mathcal{B}(\mathcal{H})$ such that

$$T^* AT \leq A \geq T^* T.$$

Proposition 5.3

A weighted shift W with weights $(w_n)_{n \geq 0}$ is similar to a contraction if and only if

$$\sup_{n,k \geq 0} \left| w_n w_{n+1} \cdots w_{n+k-1} \right| < \infty.$$

Example 2

Consider the weighted shift W on a Hilbert space with weights $(w_n)_{n\geq 0}$ defined by $w_n = \begin{cases} 2, & \text{if } n \text{ is even,} \\ \frac{1}{2}, & \text{if } n \text{ is odd.} \end{cases}$

- The operator satisfies $\|W\| = 2 > 1$, so W is not a contraction, but it is similar to a contraction.

- Matrix Representation of W:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Matrix Representation of W^*:

$$W^* = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 3

Let W be a weighted shift satisfying $\lim_{n\to\infty} w_n = \ell \in [0,1)$. Then W is similar to a contraction.

As a concrete example we consider : $w_n = \frac{3}{4} + \frac{1}{n+1}, \quad n \geq 0.$

$$w_0 = \frac{7}{4}, \quad w_1 = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}, \quad w_2 = \frac{3}{4} + \frac{1}{3} = \frac{13}{12}, \ldots$$

- $\lim\limits_{n\to\infty} w_n = \dfrac{3}{4} \in (0,1)$, so the weights converge to a value strictly less than 1.
- W is **not a contraction** because $\|W\| = w_0 = 7/4 > 1$, but it is **similar to a contraction.**

Definition 5.4

*A continuous linear Hilbert space operator S is said to be a **quasi-isometry** if the operator S and its adjoint S^* satisfy the relation*

$$S^{*2}S^2 = S^*S.$$

A quasi-isometry can be represented on the decomposition $\mathcal{H} = \overline{\mathcal{R}(S)} \oplus \mathcal{N}(S^*)$ by the following matrix

$$S = \begin{pmatrix} V & X \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \overline{\mathcal{R}(S)} \\ \mathcal{N}(S^*) \end{bmatrix}$$

with $V^*V = \mathrm{I}$ and X is an arbitrary operator.
*If the inequality $S^{*2}S^2 \leq S^*S$ is satisfied then S is called a **quasicontraction**.*
A quasicontraction can be represented on the decomposition $\mathcal{H} = \overline{\mathcal{R}(S)} \oplus \mathcal{N}(S^*)$ by the following matrix

$$S = \begin{pmatrix} C & X \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \overline{\mathcal{R}(S)} \\ \mathcal{N}(S^*) \end{bmatrix}$$

with $C^*C \leq \mathrm{I}$ and X is an arbitrary operator.

- Quasi-isometry:

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Quasicontraction:

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 5.5 (Square root of a positive operator)

For any positive operator A, there exists the unique positive operator S such that

$$S^2 = A.$$

We denote the unique positive square root S of A by $A^{1/2}$.

For any contraction T on $\mathcal{H}$, we have $I - T^*T \geq 0$, and we denote

$$D_T = (I - T^*T)^{1/2} \text{ and}$$
$$\boldsymbol{D}_T = \overline{\mathcal{R}(D_T)}$$

which are the defect operator and the defect space of T, respectively.
Next, we consider the Hilbert space $\ell^2_+(\boldsymbol{D}_T) = \bigoplus_0^\infty \boldsymbol{D}_T$, and define the natural shift S_T on $\ell^2_+(\boldsymbol{D}_T)$ as follows:

$$S_T(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) = (0, x_0, x_1, \ldots, x_{n-1}, x_n, \ldots)$$

where $x_i \in \boldsymbol{D}_T$ for $i \in \mathbb{N}$.
The matrix representation of S_T on $\ell^2_+(\boldsymbol{D}_T)$ is given by:

$$S_T = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ \mathrm{I} & 0 & 0 & \ldots \\ 0 & \mathrm{I} & 0 & \ldots \\ 0 & 0 & \mathrm{I} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \vdots \end{bmatrix}.$$

The operator S_T is an isometry.

The following isometry V_T is called the isometric lifting (or the isometric Sz.Nagy-Foias's dilation) of the contraction T.

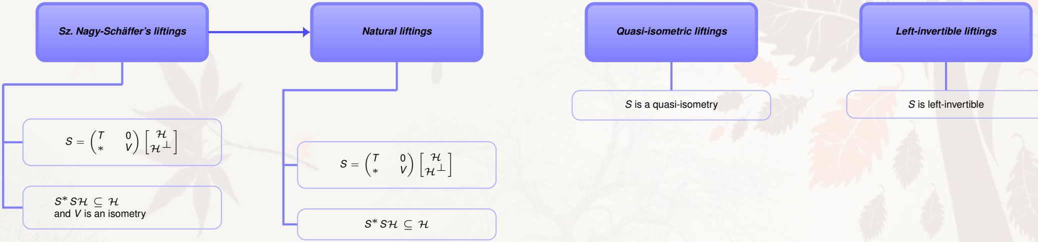
The matrix representation of V_T (the Sz.-Nagy-Schaffer's form) on the space $\mathcal{H}_T = \mathcal{H} \oplus \ell_+^2 \left(\boldsymbol{D}_T\right)$ is given by

$$V_T = \begin{pmatrix} T & 0 & 0 & 0 & \dots \\ D_T & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ 0 & 0 & 0 & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} \mathcal{H} \\ \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \boldsymbol{D}_T \\ \vdots \end{bmatrix} = \begin{pmatrix} T & 0 \\ \widetilde{D}_T & S_T \end{pmatrix} \begin{bmatrix} \mathcal{H} \\ \ell_+^2 \left(\boldsymbol{D}_T\right) \end{bmatrix},$$

where I denotes the identity operator on $\boldsymbol{D}_T$.

Types of Special liftings

For an operator $T \in \mathcal{B}(\mathcal{H})$ we have just seen that it is important to highlight liftings S with desirable properties, such as being left-invertible.
We now define types of liftings S that are well-behaved with respect to T, liftings that arise from Nagy-Foias's dilation theory.
Let S on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$.



The isometric liftings for contractions are of the Sz. Nagy-Schäffer's form.

Theorem 5.6

For an operator $T \in \mathcal{B}(\mathcal{H})$ the following assertions are equivalent :

1. *T is similar to a contraction;*
2. *T admits a natural quasi-isometric lifting S_0 on a space $\mathcal{K}_0 \supset \mathcal{H}$;*
3. *T admits a left-invertible quasi-isometric lifting S_1 on a space $\mathcal{K}_1 \supset \mathcal{H}$.*

Corollary 5.7

Every operator similar to a contraction admits a quasi-isometric lifting, which is similar to an isometry.

Theorem 5.8

An operator $T \in \mathcal{B}(\mathcal{H})$ has a quasi-isometric lifting S on a space $\mathcal{K} \supset \mathcal{H}$ that is both natural and left-invertible, if and only if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$A \geq T^*T, A \geq T^*AT, \mathcal{R}\left[(A - T^*T)^{1/2}\right] = \mathcal{R}\left[(A - T^*AT)^{1/2}\right]. \tag{1}$$

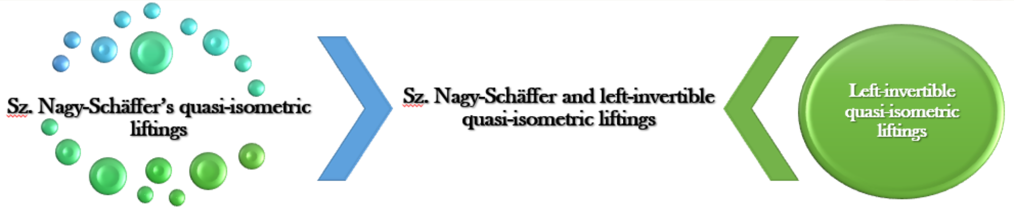
Examples

1. If $T \in \mathcal{B}(\mathcal{H})$ has $\mathbf{r}(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} < 1$, then T admits a quasi-isometric lifting that is both natural and left-invertible.

   For W a weighted shift satisfying $\lim_{n\to\infty} w_n = \ell$ then
   $$\mathbf{r}(W) = \ell.$$
   If ℓ is strictly less than 1 then W is an example for Theorem 5.8.

2. On the space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ we consider the operators J, A_0 and T, having the block matrices
   $$J = \begin{pmatrix} 0 & J_0 \\ J_0^* & 0 \end{pmatrix}, A_0 = \begin{pmatrix} \frac{1}{4}\mathrm{I} & 0 \\ 0 & \mathrm{J} \end{pmatrix}, T = \begin{pmatrix} \mathbf{0} & 2J_0 \\ \frac{1}{2}J_0^* & \mathbf{0} \end{pmatrix}$$
   where $J_0 : \{0\} \oplus \mathcal{H} \to \mathcal{H} \oplus \{0\}$ is the natural embedding $J_0(0,h) = (h,0)$.
   Clearly, $J^2 = \mathrm{I}$ with $||J|| = 1$ and $A_0{}^{1/2}T = JA_0{}^{1/2}$.
   Therefore T is not a contraction which is similar with the unitary operator J, by $A_0 \in \mathcal{B}(\widetilde{\mathcal{H}})$.
   For any invertible operator $A \in B(\widetilde{\mathcal{H}})$ with
   $$T^*AT \leq A \geq T^*T,$$
   together T and A cannot satisfy the range condition from (1).

   Hence for every left-invertible quasi-isometric lifting S for T, one has $S^*S\mathcal{H} \not\subset \mathcal{H}$.



Theorem 5.9

For an operator $T \in \mathcal{B}(\mathcal{H})$ the following assertions are equivalent :

1. *T is a quasicontraction;*
2. *T has a quasi-isometric lifting S that is both in Sz. Nagy-Schäffer's form and left-invertible.*

Proof.

Let T be a quasicontraction on $\mathcal{H}$. Then T and T^*T have on $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$ the matrix representation

$$T = \begin{pmatrix} C & G \\ 0 & 0 \end{pmatrix}, \quad T^*T = \begin{pmatrix} C^*C & C^*G \\ G^*C & G^*G \end{pmatrix} \tag{2}$$

We aim to obtain a left-invertible quasi-isometric lifting for T, using an isometric lifting for the contraction C and an invertible operator in $\mathcal{B}(\mathcal{N}(T^*))$.

Let $\mathrm{D} \in \mathcal{B}(\mathcal{N}(T^*))$ be an invertible operator such that $D^*D \geq G^*G + \frac{1}{2}\mathrm{I}$, which subsequently yields

$$G^*CC^*G \leq G^*\mathrm{I}G = G^*G \leq \frac{1}{2}\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right) = \frac{1}{2}\left[\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)\right]^{\frac{1}{2}}\left[\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)\right]^{\frac{1}{2}}$$

Hence, using Douglas' Criterion for Range Inclusion:

Let A, B be bounded operators on a Hilbert space $\mathcal{H}$.
The following statements are equivalent:
(1) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;
(2) $\exists\, E \in \mathcal{B}(\mathcal{H})$ such that $A = BE$;
(3) $\exists\, \lambda \geq 0$ such that $AA^* \leq \lambda BB^*$;
Moreover, when these conditions hold, one can choose $||E|| \leq \sqrt{\lambda}$ and $E^* : \overline{\mathcal{R}(B^*)} \to \overline{\mathcal{R}(A^*)}$.
,

there exists a contraction

$$C_0 : \overline{\mathcal{R}\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)} = \overline{\mathcal{R}\left(\left[\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)\right]^{\frac{1}{2}}\right)} \to \overline{\mathcal{R}(C^*G)} \subset \overline{\mathcal{R}(T)}$$

with $\|C_0\| \leq \frac{1}{\sqrt{2}}$ and satisfying the relation

$$C_0\left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)^{\frac{1}{2}} = C^*G, \qquad (3)$$

where we considered $A = G^*C$, $\lambda = \frac{1}{2}$, $B = \left(G^*G + D^*D - \frac{1}{2}\mathrm{I}\right)^{\frac{1}{2}}$ and $C = C_0$.

Let now $D_C = (\mathrm{I} - C^* C)^{\frac{1}{2}}$ and $\boldsymbol{D}_C = \overline{\mathcal{R}(D_C)}$ be the defect operator, and the defect space of C respectively. Denote $\mathcal{K} = \mathcal{L} \oplus \mathcal{H}$ where $\mathcal{L} = \ell_+^2(\boldsymbol{D}_C \oplus \mathcal{N}(T^*))$, and let $S \in \mathcal{B}(\mathcal{K}_0)$ be the operator with the following matrix representation on $\mathcal{K}_0 = \mathcal{L} \oplus \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*) = \mathcal{L} \oplus \mathcal{H}$,

$$S = \begin{pmatrix} S_C & \widetilde{D_c} & D \\ 0 & C & G \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} S_C & \widetilde{D} \\ 0 & T \end{pmatrix}, \quad \widetilde{D} = \begin{pmatrix} \widetilde{D_c} & D \end{pmatrix} : \mathcal{H} \to \mathcal{L}, S_C \text{ is the forward shift on } \mathcal{L}. \quad (4)$$

Since $V = S|_{\mathcal{L} \oplus \overline{\mathcal{R}(T)}} = \begin{pmatrix} S_C & \widetilde{D_C} \\ 0 & C \end{pmatrix}$ is an isometry, it follow that S is quasi-isometric for T.

We would like now to state that S is left invertible and of Sz.-Nagy-Schäffer's form.
By the representations of S in (4), T and $T^* T$ in (2), we obtain

$$S^* S = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & C^* G \\ 0 & G^* C & D^* D + G^* G \end{pmatrix} = \mathrm{I} \oplus B$$

respectively on $\mathcal{K} = \mathcal{L} \oplus \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*) = \mathcal{L} \oplus \mathcal{H}$, where $B = S^* S|_{\mathcal{H}}$.

So $S^* S \mathcal{H} \subset \mathcal{H}$ and because $S_C = S|_{\mathcal{L}}$ is an isometry, the lifting S has the Sz.-Nagy-Schäffer's form.

On the other hand, we have on $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$ that

$$B - \frac{1}{2}\mathrm{I} = \begin{pmatrix} \frac{1}{2}\mathrm{I} & C^*G \\ G^*C & D^*D + G^*G - \frac{1}{2}\mathrm{I} \end{pmatrix}.$$

Since the relation (3) can be expressed in the form $C^*G = \frac{1}{\sqrt{2}}\mathrm{I}\left(\sqrt{2}C_0\right)\left(D^*D + G^*G - \frac{1}{2}\mathrm{I}\right)^{\frac{1}{2}}$,
where $\sqrt{2}C_0$ is a contraction, from the characterization theorem for positive 2 by 2 operator matrices (see [11], Ch. XVI, theorem 1-1.1)

$$\begin{cases} \text{The operator } T = \begin{pmatrix} \Delta & \Theta \\ \Theta^* & \Omega \end{pmatrix} \text{ on } \mathcal{G}_1 \oplus \mathcal{G}_2 \text{ is positive if and only if } \Delta \text{ and } \Omega \text{ are both positive,} \\ \text{and there exists a contraction } \Gamma : \overline{\mathcal{R}(\Omega)} \to \overline{\mathcal{R}(\Delta)} \text{ satisfying } \Theta = \Delta^{1/2}\Gamma\Omega^{1/2}, \quad (\|\Gamma\| \leq 1) \end{cases},$$

where we considered $\Theta = C^*G$, $\Delta = \frac{1}{2}\mathrm{I}$, $\Omega = D^*D + G^*G - \frac{1}{2}\mathrm{I}$ and thus $\Gamma = \sqrt{2}C_0$,

we infer that

$$B - \frac{1}{2}\mathrm{I} \geq 0.$$

Hence $S^*S \geq \frac{1}{2}\mathrm{I}$, which ensures that S is left-invertible in $\mathcal{B}(\mathcal{K}_0)$.
Thus S has the properties required in (2), hence (1) implies (2).
The reverse implication is simple.

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