Many faces of Lebesgue spaces

What may be done with the simplest function space ever

Martin Křepela

Faculty of Electrical Engineering

Czech Technical University in Prague

There is only one space and Lebesgue is its prophet.

Martin Lind

There is only one space and Lebesgue is its prophet.

Martin Lind

Based on the Master thesis of Pavel Berkman
(Charles University in Prague)

Lebesgue spaces

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Denote by $\mathcal{M}_\mu(\Omega)$ the set of all μ-measurable functions on Ω.

Let $p \in (0, \infty]$. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{L^p_\mu} := \begin{cases} \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} & \text{if } p < \infty; \\ \operatorname*{ess\,sup}_{\Omega,\mu} |f| & \text{if } p = \infty. \end{cases}$$

Then denote

$$L^p_\mu := \left\{ f \in \mathcal{M}_\mu(\Omega); \|f\|_{L^p_\mu} < \infty \right\}.$$

Lebesgue spaces

- Is L^p_μ a normed space?

Lebesgue spaces

- Is L^p_μ a normed space?
- Really?

Lebesgue spaces

- Is L^p_μ a normed space?
- Really?
- What to do about it?

Lebesgue spaces

- Is L^p_μ a normed space?
- Really?
- What to do about it?

For $f \in \mathcal{M}_\mu(\Omega)$ define

$$[f]_\mu := \{g \in \mathcal{M}_\mu(\Omega);\ g = f\ \mu\text{-a.e.}\}$$

and set

$$\mathbf{L}^p_\mu := \left\{[f]_\mu;\ f \in L^p_\mu\right\}.$$

Hölder inequality

Let $p \in (1, \infty)$ and $p' = \frac{p}{p-1}$.

For every $f, g \in \mathcal{M}_\mu(\Omega)$ we have

$$\|fg\|_{L^1_\mu} \leq \|f\|_{L^p_\mu} \|g\|_{L^{p'}_\mu}.$$

Hölder inequality

Let $p \in (1, \infty)$ and $p' = \frac{p}{p-1}$.

For every $f, g \in \mathcal{M}_\mu(\Omega)$ we have

$$\|fg\|_{L^1_\mu} \leq \|f\|_{L^p_\mu} \|g\|_{L^{p'}_\mu}.$$

Moreover, for any $f \in \mathcal{M}_\mu(\Omega)$,

$$\|f\|_{L^p_\mu} = \sup_{\|g\|_{L^{p'}_\mu} \leq 1} \|fg\|_{L^1_\mu}.$$

Detour 1: Five commandments of Luxemburg

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

(P1) for every $f, g \in \mathcal{M}_\mu(\Omega)$ and every $a \in \mathbb{R}$ there holds $\|f+g\|_X \leq \|f\|_X + \|g\|$, $\|af\|_X = |a|\|f\|_X$, and $(\|f\|_X = 0 \Leftrightarrow f = 0$ μ-a.e.);

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

(P1) for every $f, g \in \mathcal{M}_\mu(\Omega)$ and every $a \in \mathbb{R}$ there holds $\|f+g\|_X \leq \|f\|_X + \|g\|$, $\|af\|_X = |a|\|f\|_X$, and $(\|f\|_X = 0 \Leftrightarrow f = 0\ \mu\text{-a.e.})$;

(P2) for every $f, g \in \mathcal{M}_\mu(\Omega)$ such that $|f| \leq |g|$ μ-a.e. there holds $\|f\|_X \leq \|g\|_X$;

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

(P1) for every $f, g \in \mathcal{M}_\mu(\Omega)$ and every $a \in \mathbb{R}$ there holds $\|f+g\|_X \leq \|f\|_X + \|g\|$, $\|af\|_X = |a|\|f\|_X$, and $(\|f\|_X = 0 \Leftrightarrow f = 0\ \mu\text{-a.e.})$;

(P2) for every $f, g \in \mathcal{M}_\mu(\Omega)$ such that $|f| \leq |g|$ μ-a.e. there holds $\|f\|_X \leq \|g\|_X$;

(P3) for every $f \in \mathcal{M}_\mu(\Omega)$ and every sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{M}_\mu(\Omega)$ such that $0 \leq f_n \uparrow f$ μ-a.e. there holds $\|f_n\|_X \uparrow \|f\|_X$;

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

(P1) for every $f, g \in \mathcal{M}_\mu(\Omega)$ and every $a \in \mathbb{R}$ there holds $\|f+g\|_X \le \|f\|_X + \|g\|$, $\|af\|_X = |a|\|f\|_X$, and $(\|f\|_X = 0 \Leftrightarrow f = 0\ \mu\text{-a.e.})$;

(P2) for every $f, g \in \mathcal{M}_\mu(\Omega)$ such that $|f| \le |g|$ μ-a.e. there holds $\|f\|_X \le \|g\|_X$;

(P3) for every $f \in \mathcal{M}_\mu(\Omega)$ and every sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{M}_\mu(\Omega)$ such that $0 \le f_n \uparrow f$ μ-a.e. there holds $\|f_n\|_X \uparrow \|f\|_X$;

(P4) for every $E \in \mathcal{A}$ such that $\mu(E) < \infty$ there holds $\|\chi_E\|_X < \infty$;

Detour 1: Five commandments of Luxemburg

A mapping $\|\cdot\|_X : \mathcal{M}_\mu(\Omega) \to [0,\infty]$ is a **Banach function norm** if:

(P1) for every $f, g \in \mathcal{M}_\mu(\Omega)$ and every $a \in \mathbb{R}$ there holds $\|f+g\|_X \leq \|f\|_X + \|g\|$, $\|af\|_X = |a|\|f\|_X$, and $(\|f\|_X = 0 \Leftrightarrow f = 0\ \mu\text{-a.e.})$;

(P2) for every $f, g \in \mathcal{M}_\mu(\Omega)$ such that $|f| \leq |g|$ μ-a.e. there holds $\|f\|_X \leq \|g\|_X$;

(P3) for every $f \in \mathcal{M}_\mu(\Omega)$ and every sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{M}_\mu(\Omega)$ such that $0 \leq f_n \uparrow f$ μ-a.e. there holds $\|f_n\|_X \uparrow \|f\|_X$;

(P4) for every $E \in \mathcal{A}$ such that $\mu(E) < \infty$ there holds $\|\chi_E\|_X < \infty$;

(P5) for every $E \in \mathcal{A}$ such that $\mu(E) < \infty$ there exists a constant $C_E \in (0,\infty)$ such that $\int_E |f|\,\mathrm{d}\mu \leq C_E\|f\|_X$ for every $f \in \mathcal{M}_\mu(\Omega)$.

Banach function spaces

If $\|\cdot\|_X$ is a Banach function norm, the collection

$$X := \{f \in \mathcal{M}_\mu(\Omega),\ \|f\|_X < \infty\}$$

is called a **Banach function space (BFS)**.

Banach function spaces

If $\|\cdot\|_X$ is a Banach function norm, the collection

$$X := \{f \in \mathcal{M}_\mu(\Omega),\ \|f\|_X < \infty\}$$

is called a **Banach function space (BFS)**.

- BFS axioms control "a.e.-properties" of functions.

Banach function spaces

If $\|\cdot\|_X$ is a Banach function norm, the collection

$$X := \{f \in \mathcal{M}_\mu(\Omega),\ \|f\|_X < \infty\}$$

is called a **Banach function space (BFS)**.

- BFS axioms control "a.e.-properties" of functions.
- After invoking equivalence classes, we get a Banach space.

Banach function spaces

If $\|\cdot\|_X$ is a Banach function norm, the collection

$$X := \{f \in \mathcal{M}_\mu(\Omega),\ \|f\|_X < \infty\}$$

is called a **Banach function space (BFS)**.

- BFS axioms control "a.e.-properties" of functions.
- After invoking equivalence classes, we get a Banach space.
- A Banach function space is *not* just a Banach space consisting of functions.

Banach function spaces – relaxations

Possible generalizations:

- **Banach function quasi-space:**
  relax (P1), remove (P5)

Banach function spaces – relaxations

Possible generalizations:

- **Banach function quasi-space:**
  relax (P1), remove (P5)
- **Banach function lattice:**
  keep only homogeneity and (P2)

Associated Banach function space

Let X be a BFS (over $(\Omega, \mathcal{A}, \mu)$). For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X'} := \sup_{\|g\|_X \leq 1} \int_\Omega |fg| \, \mathrm{d}\mu.$$

Associated Banach function space

Let X be a BFS (over $(\Omega, \mathcal{A}, \mu)$). For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X'} := \sup_{\|g\|_X \leq 1} \int_\Omega |fg| \, \mathrm{d}\mu.$$

Then $\|\cdot\|_{X'}$ is a Banach function norm generating the BFS X', called the **associate space** of X.

Associated Banach function space

Let X be a BFS (over $(\Omega, \mathcal{A}, \mu)$). For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X'} := \sup_{\|g\|_X \leq 1} \int_\Omega |fg| \, \mathrm{d}\mu.$$

Then $\|\cdot\|_{X'}$ is a Banach function norm generating the BFS X', called the **associate space** of X.

For every $f, g \in \mathcal{M}_\mu(\Omega)$ we have

$$|fg|_{L^1_\mu} \leq \|f\|_X \|g\|_{X'}.$$

Embeddings

Let X, Y be BFS with underlying σ-finite measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{A}, \nu)$, respectively.

Embeddings

Let X, Y be BFS with underlying σ-finite measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{A}, \nu)$, respectively. Then

$$X \hookrightarrow Y$$

if there exists $C \in (0, \infty)$ such that

$$\|f\|_Y \leq C\|f\|_X$$

for all $f \in \mathcal{M}_\mu(\Omega) = \mathcal{M}_\nu(\Omega)$.

Embeddings

Let X, Y be BFS with underlying σ-finite measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{A}, \nu)$, respectively. Then

$$X \hookrightarrow Y$$

if there exists $C \in (0, \infty)$ such that

$$\|f\|_Y \leq C\|f\|_X$$

for all $f \in \mathcal{M}_\mu(\Omega) = \mathcal{M}_\nu(\Omega)$. Also define

$$\|\mathrm{id}\|_{X \to Y} := \sup\left\{\|f\|_Y;\ \|f\|_X \leq 1\right\},$$

the **optimal constant** C above.

Embeddings: a necessary condition

Let X, Y be BFS with underlying σ-finite measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{A}, \nu)$, respectively.

Embeddings: a necessary condition

Let X, Y be BFS with underlying σ-finite measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{A}, \nu)$, respectively.

If $X \hookrightarrow Y$, then $\nu \ll \mu$.

Embeddings of Lebesgue spaces

Assume $\nu \ll \mu$. When does $L^p_\mu \hookrightarrow L^q_\nu$ hold?

Embeddings of Lebesgue spaces

Assume $\nu \ll \mu$. When does $L^p_\mu \hookrightarrow L^q_\nu$ hold?

Kabaila (1982):

(i) If $0 < p \leq q < \infty$, then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \sup_{0<\mu(E)<\infty} \frac{(\nu(E))^{\frac{1}{q}}}{(\mu(E))^{\frac{1}{p}}}.$$

For $p < q$, this is finite only if ν is completely atomic or trivial.

Embeddings of Lebesgue spaces

Assume $\nu \ll \mu$. When does $L^p_\mu \hookrightarrow L^q_\nu$ hold?

Kabaila (1982):

(i) If $0 < p \leq q < \infty$, then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \sup_{0<\mu(E)<\infty} \frac{(\nu(E))^{\frac{1}{q}}}{(\mu(E))^{\frac{1}{p}}}.$$

For $p < q$, this is finite only if ν is completely atomic or trivial.

(ii) If $0 < q < p < \infty$, then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \left(\int_\Omega \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)^{\frac{p}{p-q}} \mathrm{d}\mu \right)^{\frac{p-q}{pq}}.$$

Sums and intersections of BFS

Let X, Y be BFS with the same underlying measure space. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X\cap Y} := \max\left\{\|f\|_X, \|f\|_Y\right\}$$

Sums and intersections of BFS

Let X, Y be BFS with the same underlying measure space. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X\cap Y} := \max\left\{\|f\|_X, \|f\|_Y\right\}$$

and

$$\|f\|_{X+Y} := \inf\left\{\|f_1\|_X + \|f_2\|_Y;\ f = f_1 + f_2\right\}.$$

Sums and intersections of BFS

Let X, Y be BFS with the same underlying measure space. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X\cap Y} := \max\left\{\|f\|_X, \|f\|_Y\right\}$$

and

$$\|f\|_{X+Y} := \inf\left\{\|f_1\|_X + \|f_2\|_Y;\ f = f_1 + f_2\right\}.$$

Both these functionals are Banach function norms.

Sums and intersections of BFS

Let X, Y be BFS with the same underlying measure space. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{X\cap Y} := \max\left\{\|f\|_X, \|f\|_Y\right\}$$

and

$$\|f\|_{X+Y} := \inf\left\{\|f_1\|_X + \|f_2\|_Y;\ f = f_1 + f_2\right\}.$$

Both these functionals are Banach function norms.

They are not equivalent to $\|\cdot\|_X$, $\|\cdot\|_Y$ unless $X \hookrightarrow Y$ or $Y \hookrightarrow X$.

The main problem

P. Berkman's task was to characterize these embeddings:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu \cap L^r_\nu$$

$$L^p_\mu + L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

The main problem

P. Berkman's task was to characterize these embeddings:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu \cap L^r_\nu$$

$$L^p_\mu + L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- The measures are different in general.

The main problem

P. Berkman's task was to characterize these embeddings:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu \cap L^r_\nu$$

$$L^p_\mu + L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- The measures are different in general.
- Optimal constants also have to be found.

The main problem

P. Berkman's task was to characterize these embeddings:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu \cap L^r_\nu$$

$$L^p_\mu + L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- The measures are different in general.
- Optimal constants also have to be found.
- For which measures is this interesting?

The main problem

P. Berkman's task was to characterize these embeddings:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu \cap L^r_\nu$$

$$L^p_\mu + L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- The measures are different in general.
- Optimal constants also have to be found.
- For which measures is this interesting?
- Which embeddings are "easy"?

The main problem

We are thus left with these problems:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

The main problem

We are thus left with these problems:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- What do the norms control?

The main problem

We are thus left with these problems:

$$L^p_\mu \cap L^r_\mu \hookrightarrow L^q_\nu$$

$$L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$$

- What do the norms control?
- What about Orlicz spaces?

Detour 2: Orlicz spaces

Detour 2: Orlicz spaces

Let $\varphi : (0, \infty) \to (0, \infty)$ be nondecreasing, right-continuous, let $\lim_{s \to 0+} \varphi(s) = 0$ and $\lim_{s \to \infty} \varphi(s) = \infty$.

The function

$$\Phi(t) = \int_0^t \varphi(s) \, \mathrm{d}s, \qquad t > 0,$$

is then called a **Young function**.

Detour 2: Orlicz spaces

The **Orlicz modular** of $f \in \mathcal{M}_\mu(\Omega)$ is given by

$$\varrho_\Phi(f) := \int_\Omega \Phi(|f|)\,\mathrm{d}\mu.$$

Detour 2: Orlicz spaces

The **Orlicz modular** of $f \in \mathcal{M}_\mu(\Omega)$ is given by

$$\varrho_\Phi(f) := \int_\Omega \Phi(|f|)\,\mathrm{d}\mu.$$

The set

$$\mathcal{L}_\mu^\Phi := \{f \in \mathcal{M}_\mu(\Omega);\ \varrho_\Phi(f) < \infty\}$$

is called the **Orlicz class**.

Detour 2: Orlicz spaces

The **Luxemburg norm** of $f \in \mathcal{M}_\mu(\Omega)$ is given by

$$\|f\|_\Phi := \inf\left\{\lambda > 0;\ \varrho_\Phi(f/\lambda) \leq 1\right\}.$$

Detour 2: Orlicz spaces

The **Luxemburg norm** of $f \in \mathcal{M}_\mu(\Omega)$ is given by

$$\|f\|_\Phi := \inf\{\lambda > 0;\ \varrho_\Phi(f/\lambda) \leq 1\}.$$

The set

$$L^\Phi_\mu := \{f \in \mathcal{M}_\mu(\Omega);\ \|f\|_\Phi < \infty\}$$

is called the **Orlicz space**.

It is a BFS, as $\|\cdot\|_\Phi$ is a Banach function norm.

Lebesgue vs. Orlicz

Orlicz spaces generalize the Lebesgue spaces.

Lebesgue vs. Orlicz

Orlicz spaces generalize the Lebesgue spaces.

Let $p \in (1, \infty)$. Take the Young function

$$\Phi(t) = t^p, \qquad t > 0.$$

Then $L^\Phi_\mu = L^p_\mu$ with equivalent norms.

Lebesgue vs. Orlicz

Lebesgue intersection as an Orlicz space:

Let $1 < p < r < \infty$. Take the Young function

$$\Phi(t) = \begin{cases} t^p, & t \in (0,1); \\ \frac{r-p}{r} + \frac{p}{r} t^r, & t \in [1,\infty). \end{cases}$$

Then $L^\Phi_\mu = L^p_\mu \cap L^r_\mu$ with equivalent norms.

Lebesgue vs. Orlicz

Lebesgue sum as an Orlicz space:

Let $1 < p < r < \infty$. Take the Young function

$$\Phi(t) = \begin{cases} t^r, & t \in (0,1); \\ \frac{p-r}{p} + \frac{r}{p} t^p, & t \in [1,\infty). \end{cases}$$

Then $L_\mu^\Phi = L_\mu^p + L_\mu^r$ with equivalent norms.

Lebesgue vs. Orlicz

The main problem is a special case of $L^{\Phi}_{\mu} \hookrightarrow L^{\Psi}_{\nu}$.

Lebesgue vs. Orlicz

The main problem is a special case of $L^{\Phi}_{\mu} \hookrightarrow L^{\Psi}_{\nu}$.

Issues with this approach:

- Orlicz does not cover L^p for $p \leq 1$ (not a problem though)

Lebesgue vs. Orlicz

The main problem is a special case of $L^{\Phi}_{\mu} \hookrightarrow L^{\Psi}_{\nu}$.

Issues with this approach:

- Orlicz does not cover L^p for $p \leq 1$ (not a problem though)
- results for Orlicz spaces? (Krbec, Pick 1991)

Lebesgue vs. Orlicz

The main problem is a special case of $L^{\Phi}_{\mu} \hookrightarrow L^{\Psi}_{\nu}$.

Issues with this approach:

- Orlicz does not cover L^p for $p \leq 1$ (not a problem though)
- results for Orlicz spaces? (Krbec, Pick 1991)
- optimal constants?

Lebesgue vs. Orlicz

The main problem is a special case of $L^{\Phi}_{\mu} \hookrightarrow L^{\Psi}_{\nu}$.

Issues with this approach:

- Orlicz does not cover L^p for $p \leq 1$ (not a problem though)
- results for Orlicz spaces? (Krbec, Pick 1991)
- optimal constants?

There is another way to express the norms of $L^p_\mu \cup L^r_\mu$ and $L^p_\mu \cap L^r_\mu$.

Detour 3: Rearrangements

Detour 3: Rearrangements

Let $f \in \mathcal{M}_\mu(\Omega)$. Its **distribution function** is defined by

$$\mu_f(s) := \mu\left(\{x \in \Omega; |f(x)| > s\}\right), \qquad s > 0.$$

Detour 3: Rearrangements

Let $f \in \mathcal{M}_\mu(\Omega)$. Its **distribution function** is defined by

$$\mu_f(s) := \mu\left(\{x \in \Omega; |f(x)| > s\}\right), \qquad s > 0.$$

The **nonincreasing rearrangement** of f is then given by

$$f^*(t) = \inf\{s > 0;\ \mu_f(s) \leq t\}, \quad t > 0.$$

Detour 3: Rearrangements

Let $f \in \mathcal{M}_\mu(\Omega)$. Its **distribution function** is defined by

$$\mu_f(s) := \mu\left(\{x \in \Omega; |f(x)| > s\}\right), \qquad s > 0.$$

The **nonincreasing rearrangement** of f is then given by

$$f^*(t) = \inf\{s > 0;\ \mu_f(s) \leq t\}, \quad t > 0.$$

If needed, write f^*_μ instead of f^*.

Rearrangement-invariant spaces

A Banach function norm $\|\cdot\|_X$ is **rearrangement-invariant (r.i.)** if it satisfies:

$$(f^* = g^*) \quad \Rightarrow \quad (\|f\|_X = \|g\|_X).$$

Rearrangement-invariant spaces

A Banach function norm $\|\cdot\|_X$ is **rearrangement-invariant (r.i.)** if it satisfies:

$$(f^* = g^*) \quad \Rightarrow \quad (\|f\|_X = \|g\|_X).$$

The BFS generated by it is called an **r.i. space**.

Hardy–Littlewood inequality

For any $f, g \in \mathcal{M}_\mu(\Omega)$, we have

$$\int_\Omega fg \, \mathrm{d}\mu \leq \int_0^\infty f^*(t)\, g^*(t) \, \mathrm{d}t.$$

Hardy–Littlewood inequality

For any $f, g \in \mathcal{M}_\mu(\Omega)$, we have

$$\int_\Omega fg \, \mathrm{d}\mu \leq \int_0^\infty f^*(t)\, g^*(t) \, \mathrm{d}t.$$

The measure space $(\Omega, \mathcal{A}, \mu)$ is called **resonant** if, for every pair $f, g \in \mathcal{M}_\mu(\Omega)$,

$$\int_0^\infty f^*(t)\, g^*(t) \, \mathrm{d}t = \sup \left\{ \int_\Omega |fh| \, \mathrm{d}\mu; \;\; h \in \mathcal{M}_\mu(\Omega), \; h^* = f^* \right\}.$$

Hardy–Littlewood inequality

For any $f, g \in \mathcal{M}_\mu(\Omega)$, we have

$$\int_\Omega fg \, \mathrm{d}\mu \leq \int_0^\infty f^*(t)\, g^*(t) \, \mathrm{d}t.$$

The measure space $(\Omega, \mathcal{A}, \mu)$ is called **resonant** if, for every pair $f, g \in \mathcal{M}_\mu(\Omega)$,

$$\int_0^\infty f^*(t)\, g^*(t) \, \mathrm{d}t = \sup \left\{ \int_\Omega |fh| \, \mathrm{d}\mu; \ h \in \mathcal{M}_\mu(\Omega), \ h^* = f^* \right\}.$$

A measure space is resonant if and only if it is either nonatomic or purely atomic with all atoms of the same measure.

Holmstedt's formula

- How to reformulate the Lebesgue sum norm using the rearrangement?

Holmstedt's formula

- How to reformulate the Lebesgue sum norm using the rearrangement?

Holmstedt (1970):

Let $0 < p < r < \infty$ and $f \in \mathcal{M}_\mu(\Omega)$. Then

$$\|f\|_{L^p_\mu + L^r_\mu} = \inf\left\{\|f_1\|_{L^p_\mu} + \|f_2\|_{L^r_\mu};\ f_1, f_2 \in \mathcal{M}_\mu(\Omega),\ f = f_1 + f_2\right\}$$

$$\approx \left(\int_0^1 (f^*_\mu(s))^p \,\mathrm{d}s\right)^{\frac{1}{p}} + \left(\int_1^\infty (f^*_\mu(s))^r \,\mathrm{d}s\right)^{\frac{1}{r}}.$$

Consequences of Ryff's theorem

"Old" part:

Let $(\Omega, \mathcal{A}, \mu)$ be nonatomic, $f \in \mathcal{M}_\mu(\Omega)$ and $0 < t \leq \mu(\Omega)$. Then

$$\int_0^t f_\mu^*(s)\,\mathrm{d}s = \sup_{\substack{E\in\mathcal{A}\\ \mu(E)=t}} \int_E |f|\,\mathrm{d}\mu.$$

Consequences of Ryff's theorem

"Old" part:

Let $(\Omega, \mathcal{A}, \mu)$ be nonatomic, $f \in \mathcal{M}_\mu(\Omega)$ and $0 < t \leq \mu(\Omega)$. Then

$$\int_0^t f_\mu^*(s)\,\mathrm{d}s = \sup_{\substack{E\in\mathcal{A}\\ \mu(E)=t}} \int_E |f|\,\mathrm{d}\mu.$$

Moreover, if

$$\lim_{s\to\infty} f_\mu^*(s) = 0,$$

then there exists a set $E \in \mathcal{A}$ with $\mu(E) = t$ such that

$$\int_0^t f_\mu^*(s)\,\mathrm{d}s = \int_E |f|\,\mathrm{d}\mu.$$

Consequences of Ryff's theorem

"New" part:

Let $(\Omega, \mathcal{A}, \mu)$ be nonatomic, $f \in \mathcal{M}_\mu(\Omega)$ and $0 < t \leq \mu(\Omega)$. Then

$$\int_t^{\mu(\Omega)} f_\mu^*(s)\,\mathrm{d}s = \inf_{\substack{E \in \mathcal{A} \\ \mu(E)=t}} \int_{\Omega \setminus E} |f|\,\mathrm{d}\mu.$$

Consequences of Ryff's theorem

"New" part:

Let $(\Omega, \mathcal{A}, \mu)$ be nonatomic, $f \in \mathcal{M}_\mu(\Omega)$ and $0 < t \leq \mu(\Omega)$. Then

$$\int_t^{\mu(\Omega)} f_\mu^*(s)\,\mathrm{d}s = \inf_{\substack{E\in\mathcal{A}\\ \mu(E)=t}} \int_{\Omega\setminus E} |f|\,\mathrm{d}\mu.$$

Moreover, if

$$\lim_{s\to\infty} f_\mu^*(s) = 0,$$

then there exists a set $E \in \mathcal{A}$ with $\mu(E) = t$ such that

$$\int_t^{\mu(\Omega)} f_\mu^*(s)\,\mathrm{d}s = \int_{\Omega\setminus E} |f|\,\mathrm{d}\mu.$$

An idea for the $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$ embedding

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic, $\nu \ll \mu$ and $0 < q < r < \infty$. Then

$$\|f\|_{L^q_\nu + L^r_\nu} \leq C\|f\|_{L^p_\mu} \qquad \text{for all } f \in \mathcal{M}_\mu(\Omega)$$

if and only if

An idea for the $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$ embedding

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic, $\nu \ll \mu$ and $0 < q < r < \infty$. Then

$$\|f\|_{L^q_\nu + L^r_\nu} \leq C \|f\|_{L^p_\mu} \qquad \text{for all } f \in \mathcal{M}_\mu(\Omega)$$

if and only if

$$\sup_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_E |f|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} + \inf_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_{\Omega \setminus E} |f|^r \, \mathrm{d}\nu \right)^{\frac{1}{r}} \lesssim C \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{M}_\mu(\Omega)$.

An idea for the $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$ embedding

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic, $\nu \ll \mu$ and $0 < q < r < \infty$. Then

$$\|f\|_{L^q_\nu + L^r_\nu} \leq C \|f\|_{L^p_\mu} \qquad \text{for all } f \in \mathcal{M}_\mu(\Omega)$$

if and only if

$$\sup_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_E |f|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} + \inf_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_{\Omega \setminus E} |f|^r \, \mathrm{d}\nu \right)^{\frac{1}{r}} \lesssim C \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{M}_\mu(\Omega)$.

- The constant in "$\lesssim$" does not depend on f, μ, ν.

An idea for the $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$ embedding

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic, $\nu \ll \mu$ and $0 < q < r < \infty$. Then

$$\|f\|_{L^q_\nu + L^r_\nu} \leq C \|f\|_{L^p_\mu} \qquad \text{for all } f \in \mathcal{M}_\mu(\Omega)$$

if and only if

$$\sup_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_E |f|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} + \inf_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_{\Omega \setminus E} |f|^r \, \mathrm{d}\nu \right)^{\frac{1}{r}} \lesssim C \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{M}_\mu(\Omega)$.

- The constant in "$\lesssim$" does not depend on f, μ, ν.
- The first term is fine.

An idea for the $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$ embedding

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic, $\nu \ll \mu$ and $0 < q < r < \infty$. Then

$$\|f\|_{L^q_\nu + L^r_\nu} \leq C \|f\|_{L^p_\mu} \qquad \text{for all } f \in \mathcal{M}_\mu(\Omega)$$

if and only if

$$\sup_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_E |f|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} + \inf_{\substack{E \in \mathcal{A} \\ \nu(E) = t}} \left(\int_{\Omega \setminus E} |f|^r \, \mathrm{d}\nu \right)^{\frac{1}{r}} \lesssim C \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{M}_\mu(\Omega)$.

- The constant in "$\lesssim$" does not depend on f, μ, ν.
- The first term is fine.
- The second one is worse.

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, convex case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q \leq r < \infty.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, convex case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q \leq r < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu} \approx \max \left\{ \sup_{0<\nu(E)\leq 1} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}, \sup_{1<\nu(E)} \frac{\nu(E)^{\frac{1}{r}}}{\mu(E)^{\frac{1}{p}}} \right\}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, convex case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q \leq r < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu} \approx \max \left\{ \sup_{0<\nu(E)\leq 1} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}, \sup_{1<\nu(E)} \frac{\nu(E)^{\frac{1}{r}}}{\mu(E)^{\frac{1}{p}}} \right\}.$$

Compare this to

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \sup_{0<\mu(E)<\infty} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, convex case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q \leq r < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu} \approx \max \left\{ \sup_{0<\nu(E)\leq 1} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}, \sup_{1<\nu(E)} \frac{\nu(E)^{\frac{1}{r}}}{\mu(E)^{\frac{1}{p}}} \right\}.$$

Compare this to

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \sup_{0<\mu(E)<\infty} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}.$$

If $p < q$, both the terms are infinite (ν is nonatomic).

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, convex case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q \leq r < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu} \approx \max \left\{ \sup_{0<\nu(E)\leq 1} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}, \sup_{1<\nu(E)} \frac{\nu(E)^{\frac{1}{r}}}{\mu(E)^{\frac{1}{p}}} \right\}.$$

Compare this to

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu} = \sup_{0<\mu(E)<\infty} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}.$$

If $p < q$, both the terms are infinite (ν is nonatomic).

This also gives a general necessary condition.

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, mixed case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < q < p \leq r < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu} \approx \max \left\{ \sup_{\nu(E)\leq 1} \left(\int_E \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)^{\frac{p}{p-q}} \mathrm{d}\mu \right)^{\frac{p-q}{pq}}, \sup_{\nu(E)>1} \frac{(\nu(E))^{\frac{1}{r}}}{(\mu(E))^{\frac{1}{p}}} \right\}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, Hölder case

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < q \leq r < p < \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^r_\nu}$$

$$\approx \max\left\{ \sup_{\nu(E)\leq 1} \left(\int_E \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)^{\frac{p}{p-q}} \mathrm{d}\mu \right)^{\frac{p-q}{pq}}, \inf_{\nu(E)\leq 1} \left(\int_{\Omega\setminus E} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)^{\frac{p}{p-r}} \mathrm{d}\mu \right)^{\frac{p-r}{pr}} \right\}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, weak cases

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < p \leq q < r = \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^\infty_\nu} \approx \sup_{\nu(E) \leq 1} \frac{\nu(E)^{\frac{1}{q}}}{\mu(E)^{\frac{1}{p}}}.$$

If $p < q$, this expression is infinite.

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, weak cases

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < q < p < r = \infty.$$

Then

$$\|\mathrm{id}\|_{L^p_\mu \to L^q_\nu + L^\infty_\nu} \approx \sup_{\nu(E) \le 1} \left(\int_E \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)^{\frac{p}{p-q}} \mathrm{d}\mu \right)^{\frac{p-q}{pq}}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, weak cases

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < q \leq r < p = \infty.$$

Then

$$\|\mathrm{id}\|_{L^\infty_\mu \to L^q_\nu + L^r_\nu} \approx \min\left\{(\nu(\Omega))^{\frac{1}{q}}, (\nu(\Omega))^{\frac{1}{r}}\right\}.$$

Results: $L^p_\mu \hookrightarrow L^q_\nu + L^r_\nu$, weak cases

Let $(\Omega, \mathcal{A}, \mu)$, $(\Omega, \mathcal{A}, \nu)$ be nonatomic with $\nu \ll \mu$ and

$$0 < q \leq r = p = \infty.$$

Then

$$\|\mathrm{id}\|_{L^\infty_\mu \to L^q_\nu + L^\infty_\nu} = 1.$$

Detour 4: Wiener–Luxemburg amalgams

Let X, Y be r.i. BFS over $(\Omega, \mathcal{A}, \mu)$. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{WL(X,Y)} := \|f^*_\mu \chi_{[0,1)}\|_X + \|f^*_\mu \chi_{[1,\infty)}\|_Y.$$

Detour 4: Wiener–Luxemburg amalgams

Let X, Y be r.i. BFS over $(\Omega, \mathcal{A}, \mu)$. For $f \in \mathcal{M}_\mu(\Omega)$ define

$$\|f\|_{WL(X,Y)} := \|f^*_\mu \chi_{[0,1)}\|_X + \|f^*_\mu \chi_{[1,\infty)}\|_Y.$$

Then $\|\cdot\|_{WL(X,Y)}$ is an r.i. Banach function norm generating the **Wiener–Luxemburg amalgam space** $WL(X, Y)$.

(Peša, 2022)

Detour 4: Wiener–Luxemburg amalgams

- Every r.i. BFS X is a WL-amalgam space, namely

$$X = WL(X, X).$$

Detour 4: Wiener–Luxemburg amalgams

- Every r.i. BFS X is a WL-amalgam space, namely

$$X = WL(X, X).$$

- Some sums and intersections of BFS are WL-amalgams. In particular, this is true for Lebesgue spaces.

Detour 4: Wiener–Luxemburg amalgams

- Every r.i. BFS X is a WL-amalgam space, namely

$$X = WL(X, X).$$

- Some sums and intersections of BFS are WL-amalgams. In particular, this is true for Lebesgue spaces.

- Using the WL-amalgam theory does not really simplify the main problem.

Last page

Last page

Thank you!