Quantum theory and mathematics

A 100-year success story.

Christoph Schweigert, Karlstad, May 28, 2025

Collaborative research center „Higher structures, moduli spaces and integrability“

Federal cluster of excellence „Quantum Universe“

100 years ago

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Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen.

Von **W. Heisenberg** in Göttingen.

(Eingegangen am 29. Juli 1925.)

In der Arbeit soll versucht werden, Grundlagen zu gewinnen für eine quantentheoretische Mechanik, die ausschließlich auf Beziehungen zwischen prinzipiell beobachtbaren Größen basiert ist.

Bekanntlich läßt sich gegen die formalen Regeln, die allgemein in der Quantentheorie zur Berechnung beobachtbarer Größen (z. B. der

Werner Heisenberg

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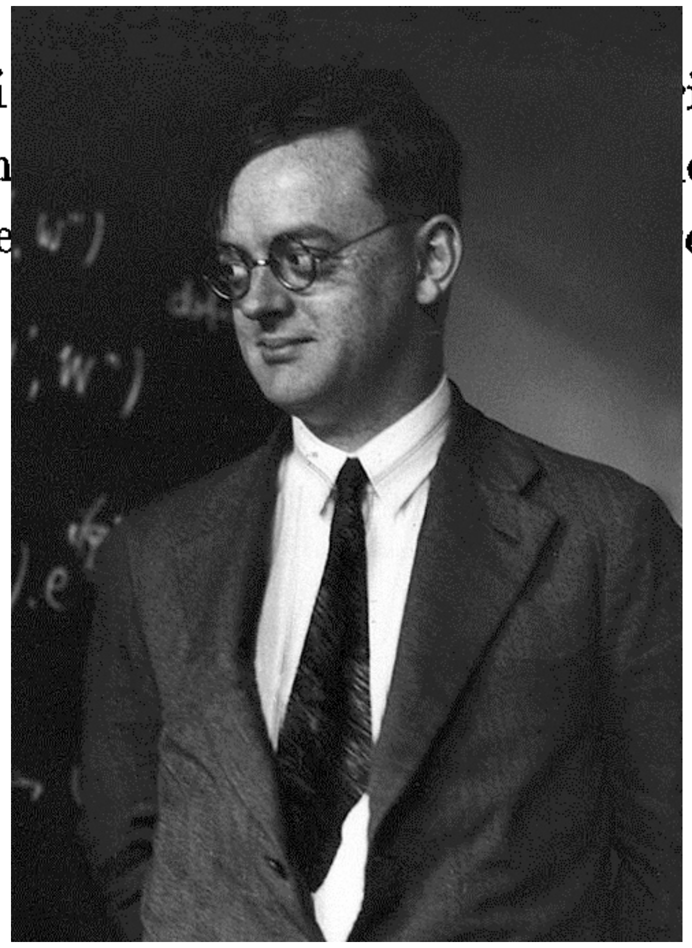
Zur Quantenmechanik.

Von **M. Born** und **P. Jordan** in Göttingen.

(Eingegangen am 27. September 1925.)

Die kürzlich von Heisenberg gegebenen Ansätze werden (zunächst für Systeme von einem Freiheitsgrad) zu einer systematischen Theorie der Quantenmechanik entwickelt. Das mathematische Hilfsmittel ist die Matrizenrechnung. Nachdem diese kurz dargestellt ist, werden die mechanischen Bewegungsgleichungen aus einem Variationsprinzip abgeleitet und der Beweis geführt, daß auf Grund der Heisenbergschen Quantenbedingung der Energiesatz und die Bohrsche Frequenzbedingung aus den mechanischen Gleichungen folgen. Am Beispiel des anharmonischen Oszillators wird die Frage der Eindeutigkeit der Lösung und die Bedeutung der Phasen in den Partialschwingungen erörtert. Den Schluß bildet ein Versuch, die Gesetze des elektromagnetischen Feldes der neuen Theorie einzufügen.

kürzlich von Hei... ift
einer neuen Kin... en
Quantentheorie e... on



3. ***Quantisierung als Eigenwertproblem;***
von E. Schrödinger.

(Erste Mitteilung.)

§ 1. In dieser Mitteilung möchte ich zunächst an dem einfachsten Fall des (nichtrelativistischen und ungestörten) Wasserstoffatoms zeigen, daß die übliche Quantisierungsvorschrift sich durch eine andere Forderung ersetzen läßt, in der kein Wort von „ganzen Zahlen" mehr vorkommt. Vielmehr ergibt sich

Erwin Schrödinger

unesco

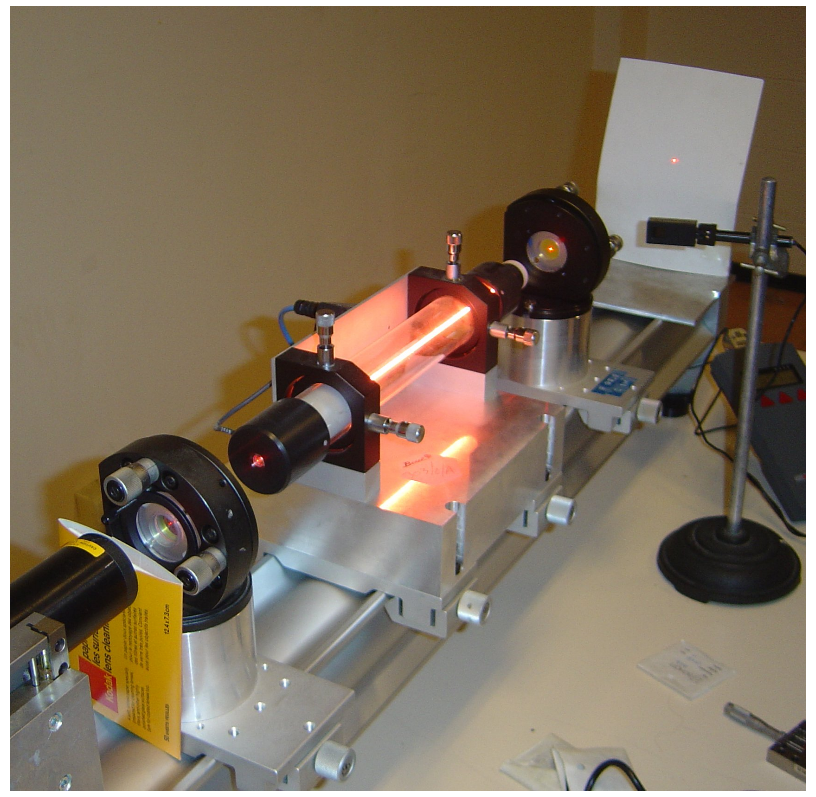
2025: International year of quantum science and technology (based on fundamental research!)

Quantum science and mathematics

- Mathematics provides technology for the study of quantum theories
- Quantum theories provide technology for mathematics

This is really remarkable, given that quantum theory is indispensable for the description of our world:

- quantum chemistry
- quantum computing
- quantum optics ....



Quantum mechanics on our roofs

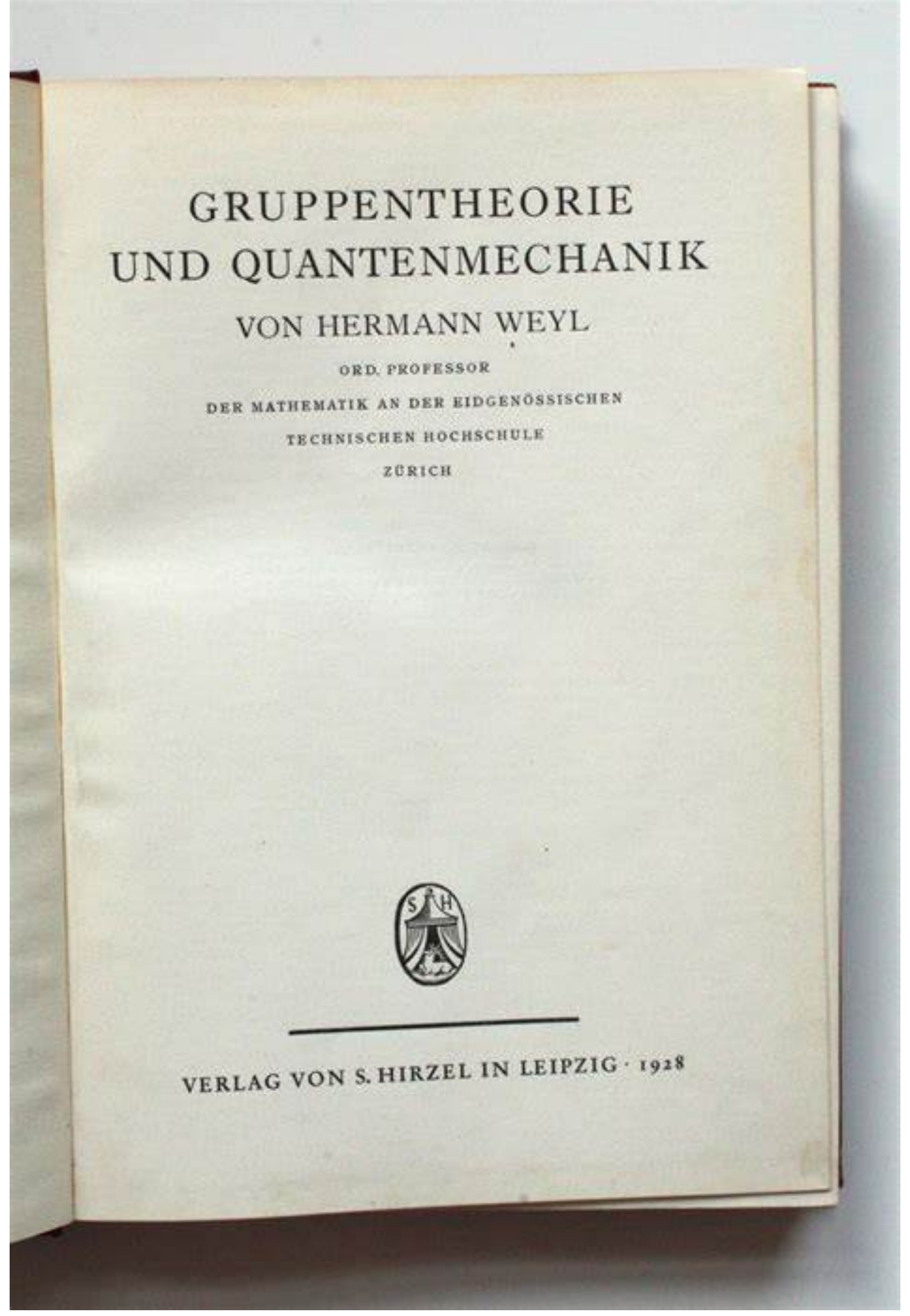
Ruska, commercial version 1938

and many modern devices:

- diodes, laser, electron microscopy, photovoltaics.....

A success story

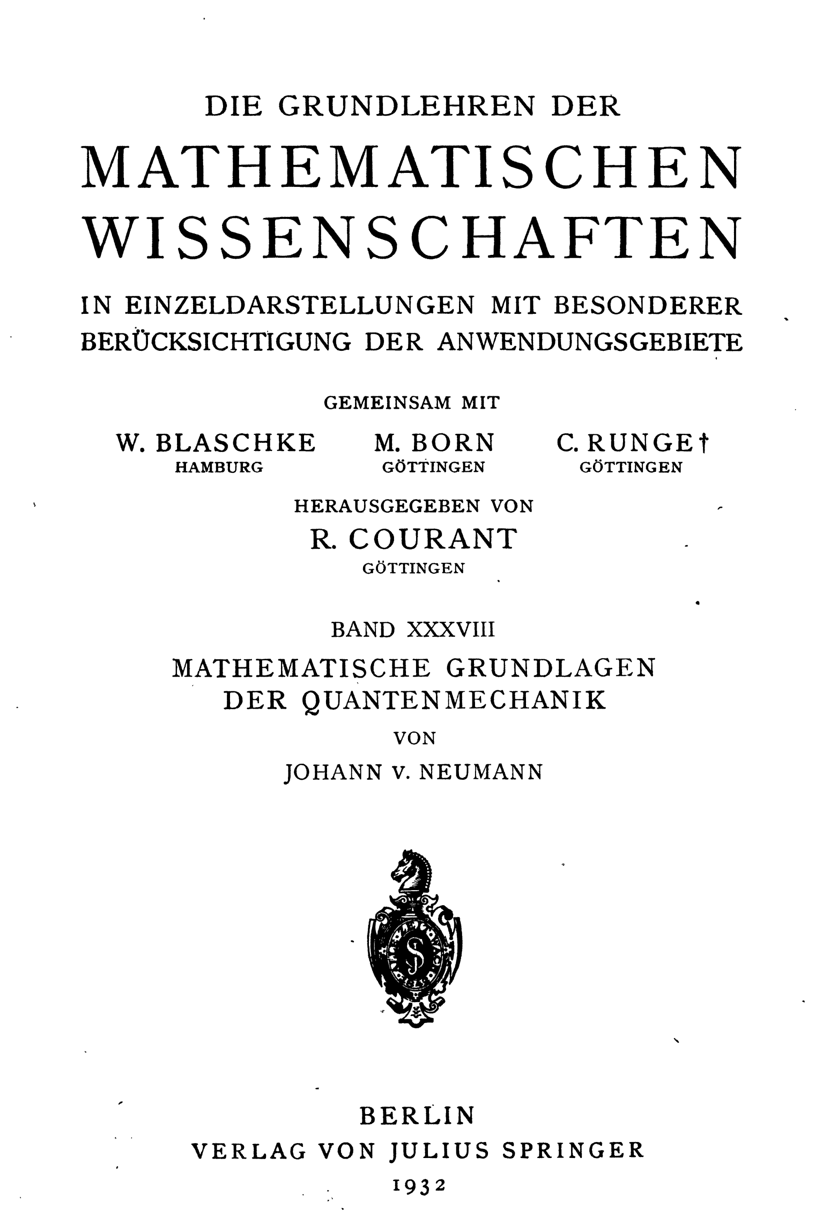
- 1925 Heisenberg, Born, Jordan, Schrödinger
- 1928 Textbook Hermann Weyl (mathematician): „Gruppentheorie und Quantenmechanik". Non-commutativity.
- 1927 Paul Dirac: relativistic quantum mechanics
- 1930 Paul Dirac „Principles of quantum mechanics" (bra-ket formalism)
- 1932 Textbook John von Neumann (mathematician): „Mathematische Grundlagen der Quantenmechanik"



GRUPPENTHEORIE
UND QUANTENMECHANIK
VON HERMANN WEYL
ORD. PROFESSOR
DER MATHEMATIK AN DER EIDGENÖSSISCHEN
TECHNISCHEN HOCHSCHULE
ZÜRICH

VERLAG VON S. HIRZEL IN LEIPZIG · 1928

Hermann Weyl



DIE GRUNDLEHREN DER
MATHEMATISCHEN
WISSENSCHAFTEN
IN EINZELDARSTELLUNGEN MIT BESONDERER
BERÜCKSICHTIGUNG DER ANWENDUNGSGEBIETE
GEMEINSAM MIT
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HERAUSGEGEBEN VON
R. COURANT
GÖTTINGEN
BAND XXXVIII
MATHEMATISCHE GRUNDLAGEN
DER QUANTENMECHANIK
VON
JOHANN V. NEUMANN

BERLIN
VERLAG VON JULIUS SPRINGER
1932

John von Neumann

A mathematical success story

Why was such a quick development possible?

- Newton's classical mechanics (1687) started an impressive amount of mathematical work on classical mechanics (Huygens, Leibniz, Bernoulli, Euler, d'Alembert, Lagrange, Laplace, Cauchy, Hamilton) lasting for more than two centuries.

- Mathematical structure of classical mechanics was extremely well understood; all formulations of quantum mechanics (old: Bohr-Sommerfeld, modern: Heisenberg-Schrödinger) build on this.

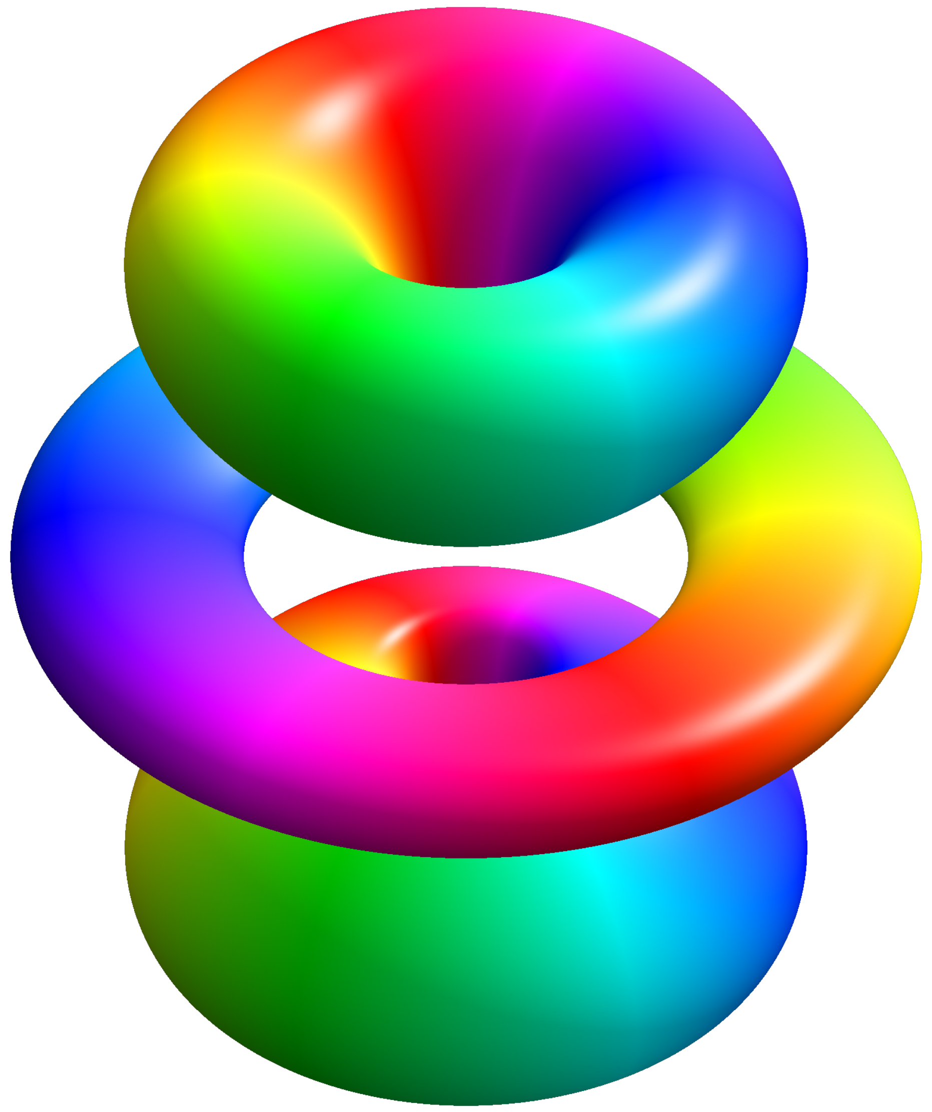
Understanding the mathematical structure of physical theories pays off!

Mathematical payoff:

- Operator algebras (C^*-algebras, von Neumann algebras)
- Quantum groups
- Non-commutative geometry

A remark on symmetry

New physical theories have a good start
if systems with many symmetries
are to a good approximation realized in nature:



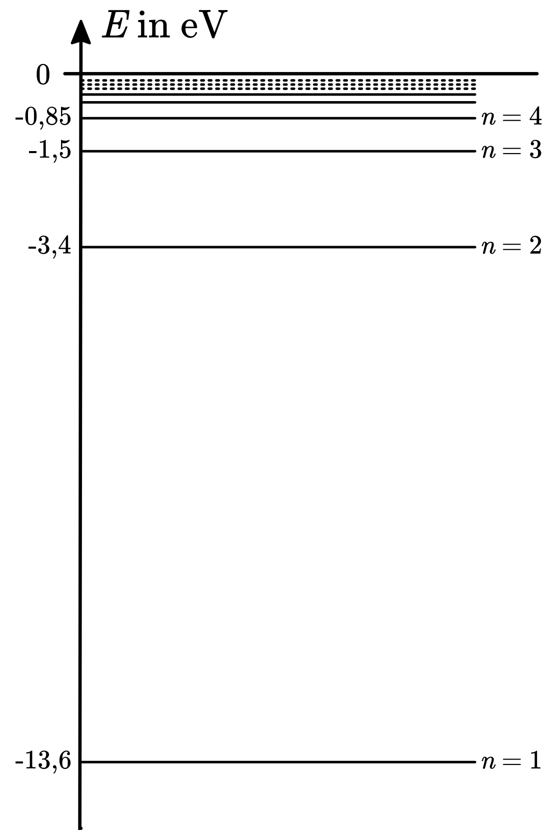
- Classical mechanics: orbits of the planets, Kepler's laws
- Quantum mechanics: hydrogen atom

In fact, the potential

$$V(r) = -\frac{k}{r}$$

has beyond rotation symmetry the Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$



as an additional symmetry
leading to the degeneracy of the energy levels

$$E_n = -13,6 \text{ eV} \cdot \frac{1}{n^2}$$

Quantum field theory

> 6. *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt;*
> *von A. Einstein.*
>
> Zwischen den theoretischen Vorstellungen, welche sich die Physiker über die Gase und andere ponderable Körper gebildet haben, und der Maxwellschen Theorie der elektromagnetischen Prozesse im sogenannten leeren Raume besteht ein tiefgreifender formaler Unterschied. Während wir uns

- With Einstein's explanation of the photoelectric effect from 1905, a (relativistic) quantum theory of fields was clearly needed.
- In quantum mechanics, time is a (one-dimensional) parameter, but the phase space (including position functions) is „quantized“: replaced by a non-commutative algebra.
- In quantum field theory, the parameter space is higher-dimensional. There exist various relevant geometries, leading to a variety of notions of quantum field theories. In particular, the relevant geometry can be Lorentzian or Riemannian.
- Classical field theory is much less developed than classical mechanics. The mathematical point of departure for quantum field theory is very different from the situation in quantum mechanics.

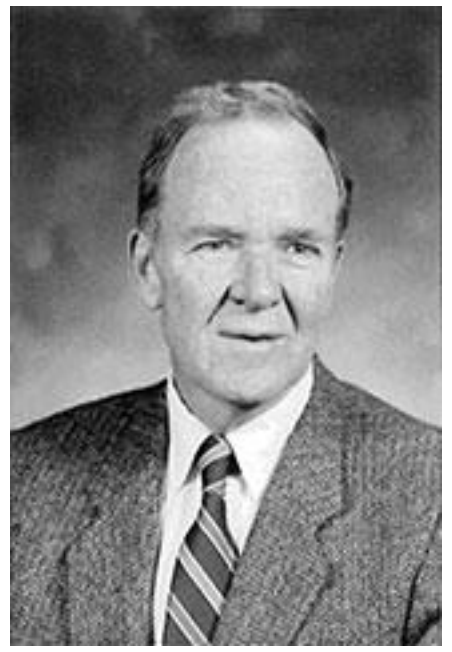
Success and frameworks

- Quantum electrodynamics was an early success of perturbative quantum field theory, leading to the 1965 Nobel prize in physics for Feynman, Schwinger and Tomonaga.

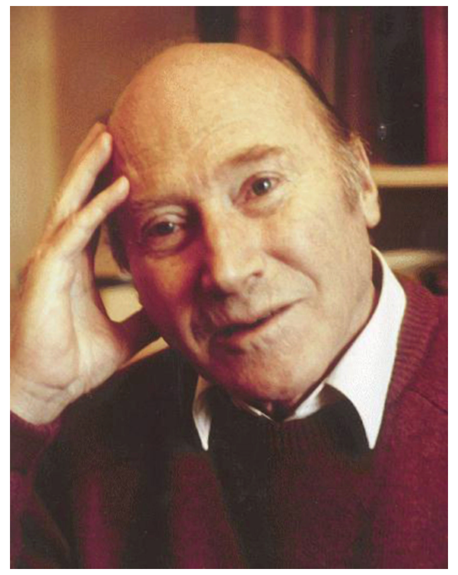
- In particular within the framework of the standard model, perturbative quantum field theory has many successes
  (and leads to interesting mathematical structures: renormalization, resummation,…)

Various axiomatic approaches:

- Wightman axioms
- Haag-Kastler local quantum field theory
- Euclidean: Osterwalder-Schrader axioms



Wightman (1922-2013)



Haag (1922-2016)

Kastler (1926-2015)

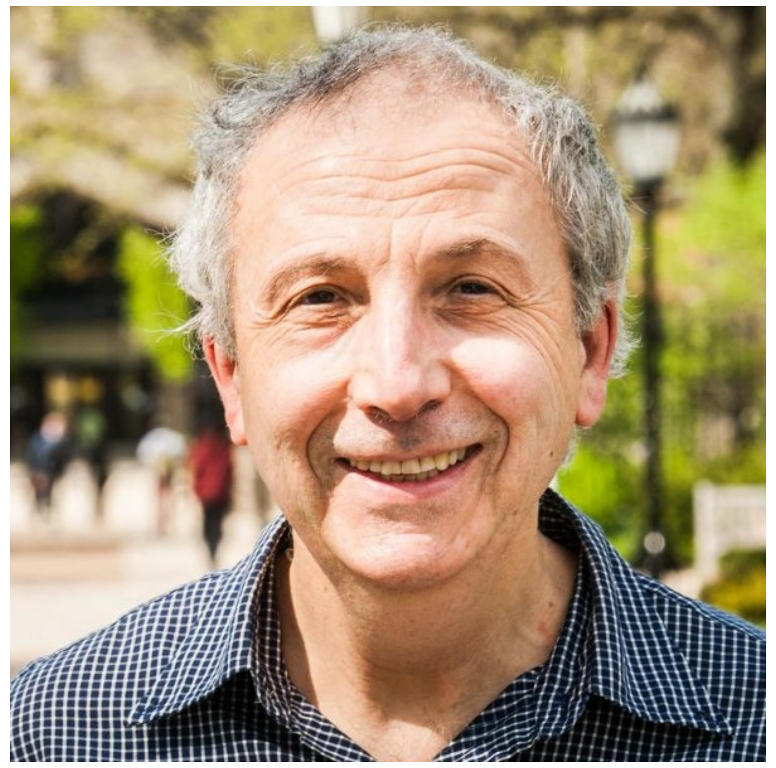
Osterwalder (*1942)

Schrader (1939-2015)

Mathematically amenable classes of quantum field theories

- Integrable quantum field theories have many conserved symmetries
- Topological quantum field theories: various mathematical constructions, various dimensions. Important tool to understand higher categorical structures in mathematics and fault tolerant topological quantum computing.

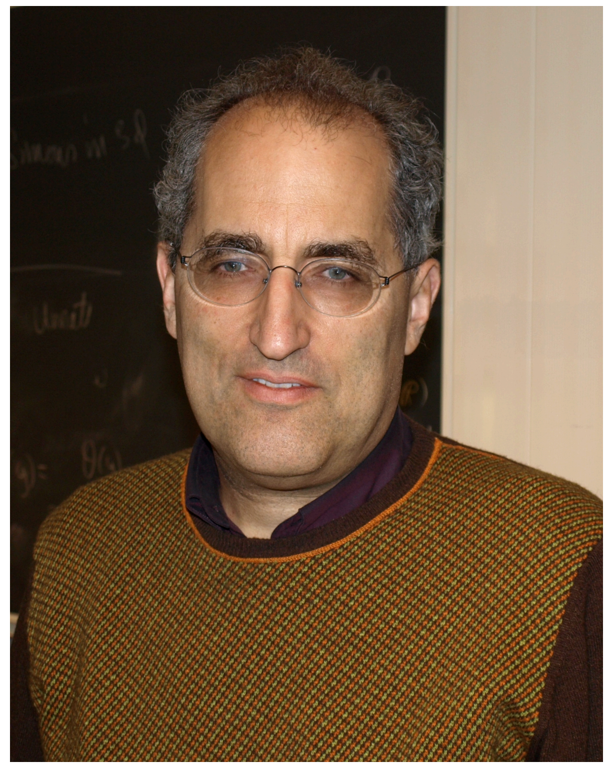
The „quantum Fields medal“ 1990, ICM Kyoto



Vladimir Drinfeld

Vaughan Jones

Shegefumi Mori



Edward Witten

More classes of quantum field theories

- Gauge theories have also important mathematical successes:
  Donaldson theory, Seiberg-Witten theory
- String theory has implied many new angles on quantum field theory: holography, theories without Lagrangian,…
- String theory also brought many more mathematical insights and challenges: Mirror symmetry, enumerative invariants in algebraic geometry,…
- Two-dimensional (rational) conformal field theories
  are particularly well understood.

Two-dimensional conformal field theory

Two-dimensional local conformal (quantum) field theories

- World-sheet theory for perturbative string theory
- Two-dimensional critical systems (Ising, Potts,….)


Ingo Runkel


Jürgen Fuchs


Jens Fjelstad

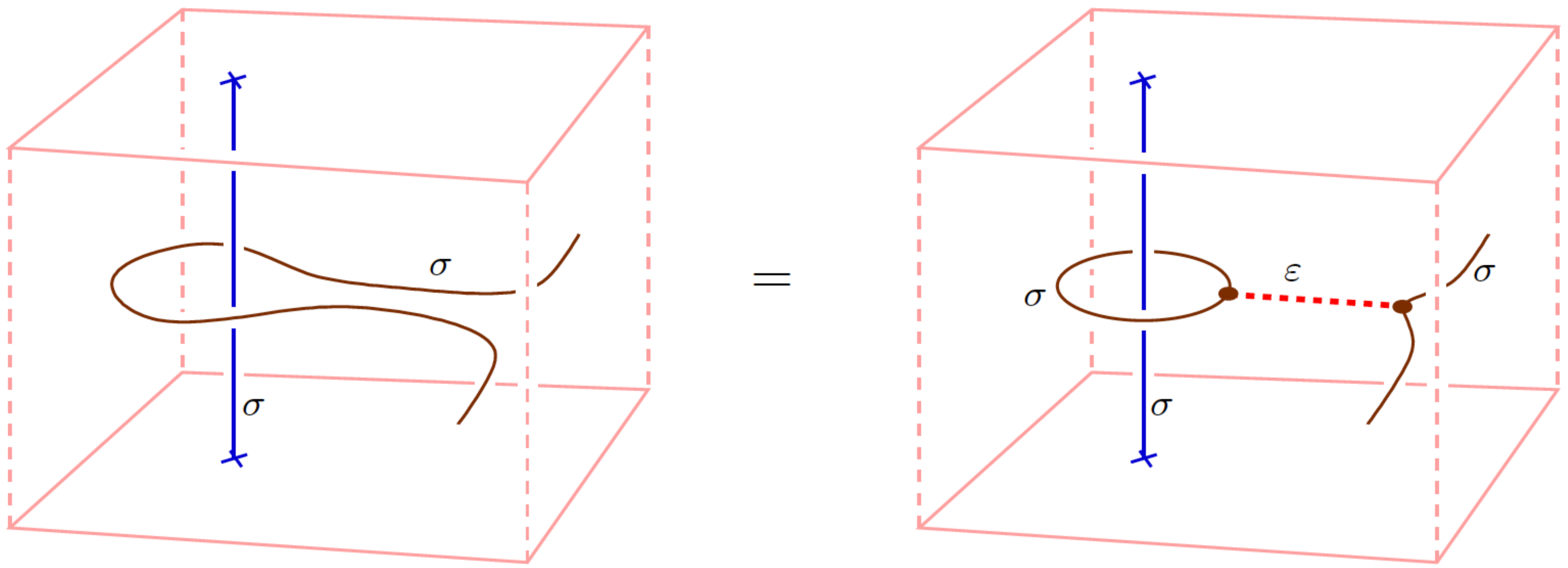
Mathematical strategy:

- Chiral symmetries (vertex operator algebra or net of observable algebras) give a representation category $\mathcal{C}$
- Chiral symmetries give Ward identities, solutions: conformal blocks. Holographic setting: controlled by three-dimensional topological field theory (TFT)
- [Fuchs, Runkel, S] Use this TFT to ensure existence of consistent sets of correlators. Solutions in bijection to special symmetric Frobenius algebras internal to $\mathcal{C}$.
- Englobes all results found so far about these theories

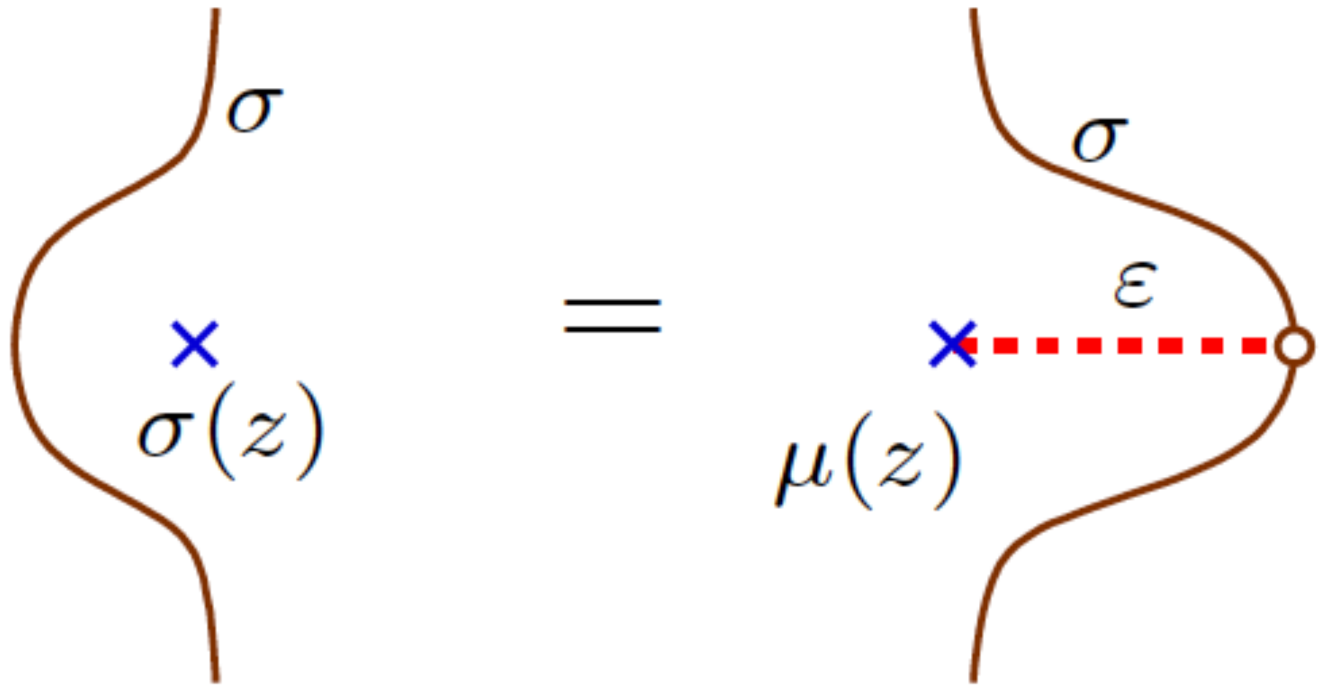
Non-invertible symmetries

We can really learn something from „toy models“!

First use the mathematical setting to show an identity:

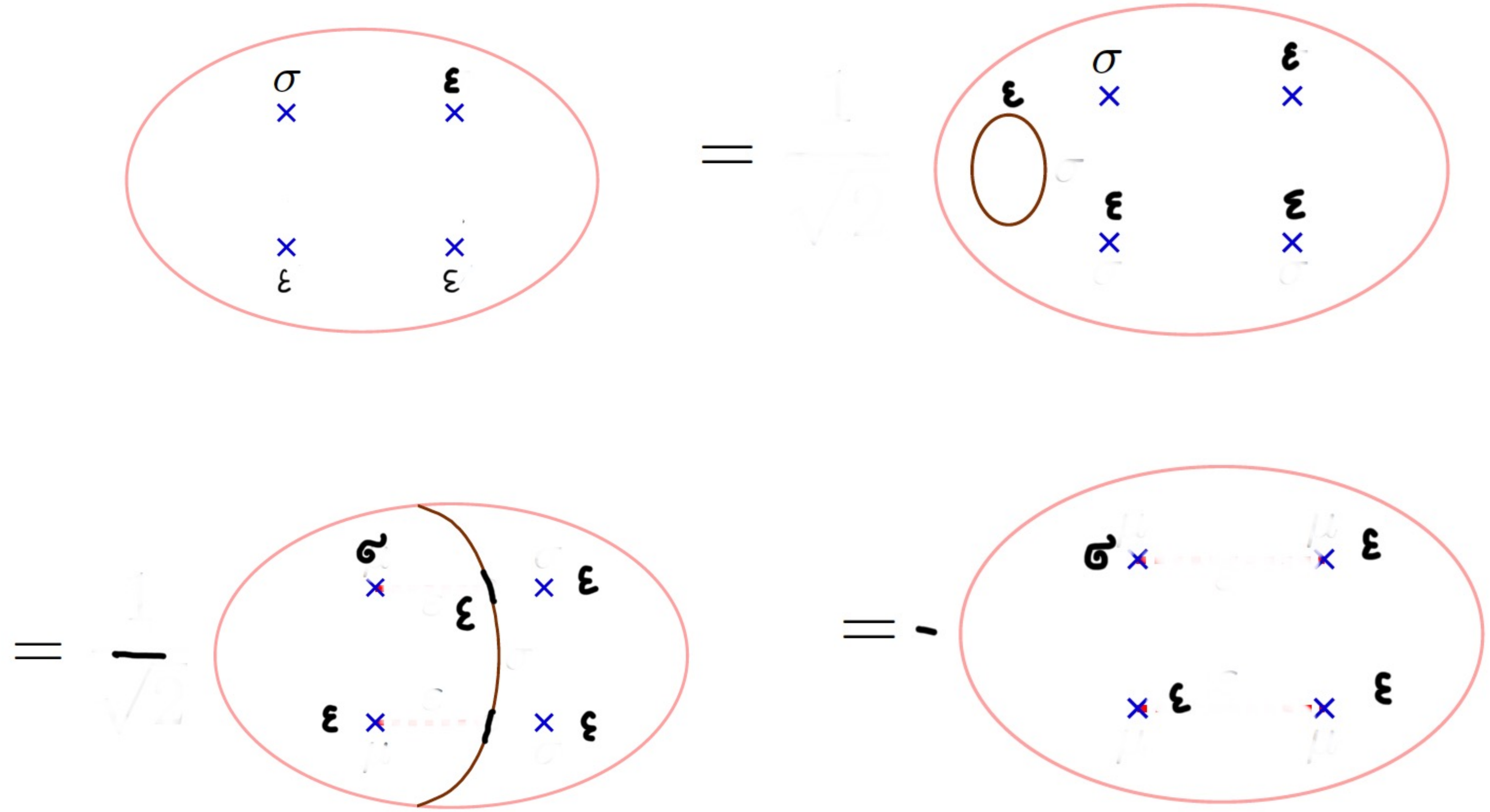


Which in the two-dimensional field theory reads gives an identity for fields and line defect:



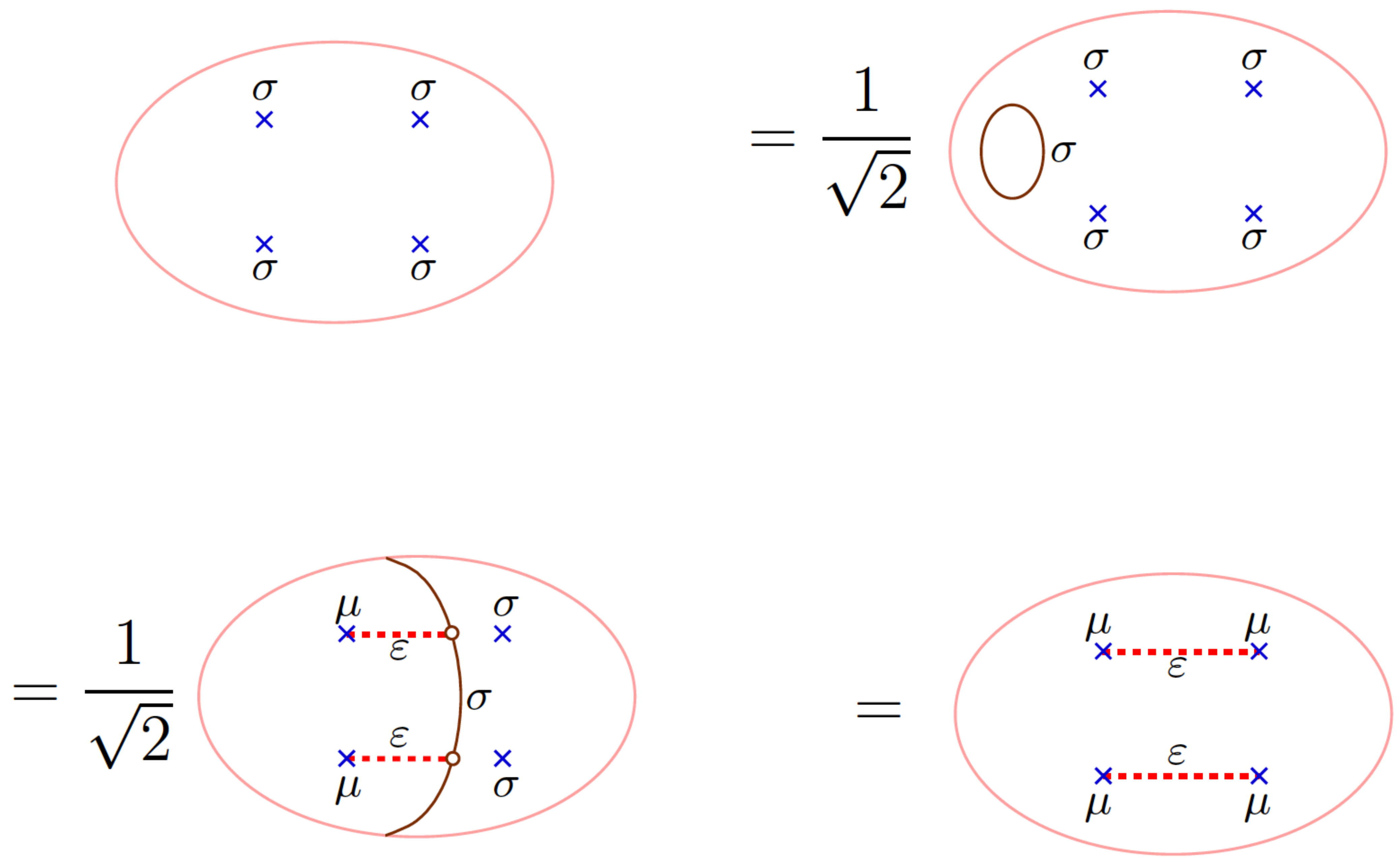
Symmetries from topological defects

An equality of correlation functions from invertible defect



Symmetry implies a selection rule

Kramers-Wannier duality from duality defects



Order disorder duality

Outlook

Quantum mechanics:

- Active field of mathematical physics
- Basis of much of modern technology, including quantum computation

Quantum field theory:

- Tension between general frameworks and computability
- Many challenges and a huge potential for mathematics