Fractional Boundary Value Problems: Analysis, Approximations and Applications

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*joint work with **M.Fečkan** (Comenius University in Bratislava, Slovakia),

J.R.Wang (Guizhou University, China)

Outline of my talk:

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1. **Fractional calculus:**
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3. **Work in progress and open questions**

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Fractional calculus

Origin and motivation

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Fractional calculus *is a branch of mathematics that generalizes the order of derivatives (and integrals) of a function to the non-integer numbers.*

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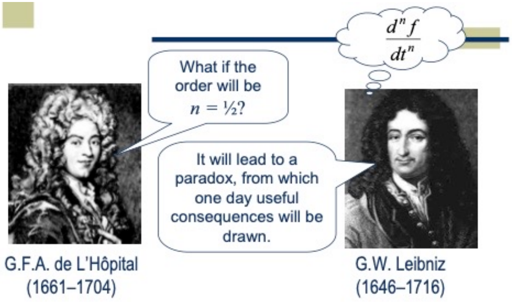
Its origin is dated by **1695** and can be traced back to **L'Hopital** and **Leibniz**.



Source: https://igor.podlubny.website.tuke.sk/USU/02_overview.pdf

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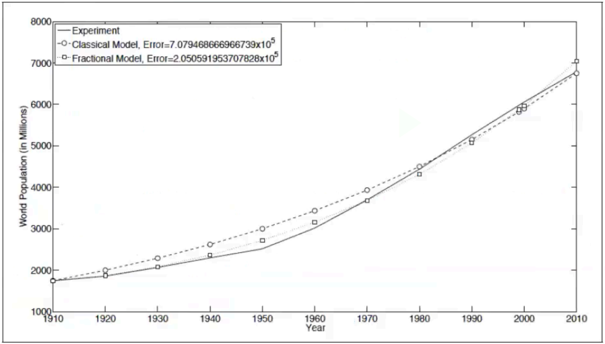
Range of **applications**:

- study of viscoelasticity and electrical circuits;
- control theory;
- modeling dynamical systems with memory;
- systems with anomalous diffusion.

- **Population model:**

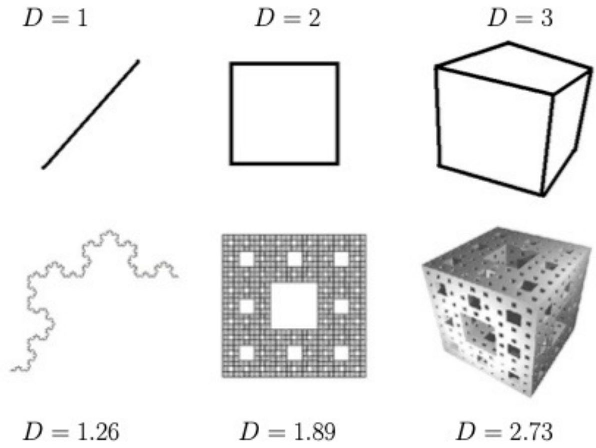
$$\begin{cases} \frac{d}{dt}u(t) = ku(t) & t > 0, \\ u(0) = u_0 \end{cases} \quad \text{VS} \quad \begin{cases} (D_{left})^{\alpha} u(t) = ku(t) & t > 0, \\ u(t) = u_0(t) & t \leq 0 \end{cases}$$



Ref. Almeida–Bastos–Monteiro: "Modeling some real phenomena by fractional differential equations", *Math. Methods Appl. Sci.* 2016

Source:
https://coursemedia.gmu.edu/media/CMAI_Colloquium/1_5lpprpgi

- **Porous media modeling:**



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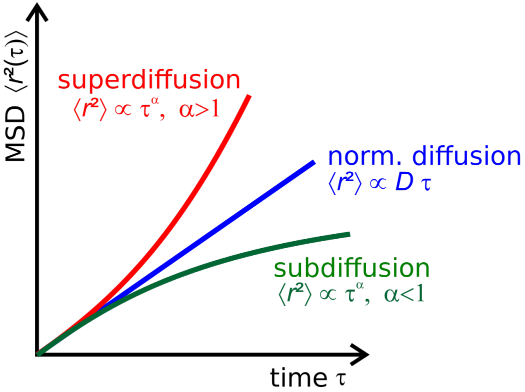
- **Anomalous diffusion:**



Source: https://en.wikipedia.org/wiki/Anomalous_diffusion

Fractional integrals and derivatives

Fractional integrals and derivatives

If f is a locally integrable function on (a, ∞), then the n-fold iterated integral is given by

$$
{}_{\mathbf{a}}\mathbf{D}_{\mathbf{t}}^{-\mathbf{n}} f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s) ds \tag{1}
$$

for almost all t with $-\infty \leq a < t < \infty$ and $n \in \mathbb{N}$.

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Using that $(n-1)! = \Gamma(n)$, *the integral of f of the fractional order $\alpha > 0$ (**Riemann-Liouville fractional integral**) reads:*

$$ {}_{\mathbf{a}}\mathbf{D}_{\mathbf{t}}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \quad (\textit{\textbf{left} hand}), \tag{2} $$

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and similarly for $-\infty < t < d \leq \infty$

$$ {}_{\mathbf{t}}\mathbf{D}_{\mathbf{b}}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds \quad (\textbf{\textit{right}}\ \textit{hand}). \tag{3} $$

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If $0 < \alpha < 1$ *then the* ***left*** *and* ***right*** ***Riemann-Liouville FDs*** *are defined as*

$$ {}_{\mathbf{a}}\mathbf{D}_{\mathbf{t}}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\alpha} f(s) ds \right), \quad t > a, \tag{4} $$

and

$$ {}_{\mathbf{t}}\mathbf{D}_{\mathbf{b}}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^b (s-t)^{-\alpha} f(s) ds \right), \quad t < b. \tag{5} $$

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Using this **principle of generalization**, the following **fractional derivatives** were derived:

- **Caputo**;
- Hilfer – as a generalization of the Riemann-Liouville and Caputo derivatives;
- further generatizations: Prabhakar, Hilfer-Prabhakar, etc.

Caputo fractional derivatives

*The **left** and **right** **Caputo FDs** of order* $\alpha \in (0,1)$ *are defined by*

$$ {}^{\mathbf{C}}_{\mathbf{a}}\mathbf{D}^{\alpha}_{\mathbf{t}} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} f'(s) ds \right), \quad t > a, $$

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"Memory effect" of the fractional derivative

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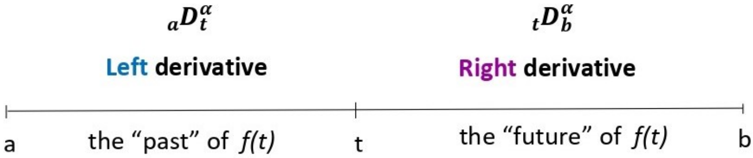

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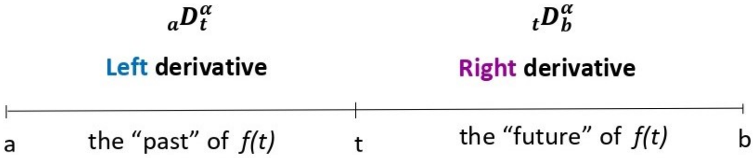

Thus, the present state of the process $f(t)$, started at $\tau = a$, depends on all its previous states $f(\tau)$ $(a \leq \tau < t)$ $\Longrightarrow$ it represents the "**memory effect**" of FDs. The same holds for the "future".

My research direction:

Study of **nonlinear fractional differential systems** of the form:

$$ {}_a\mathcal{D}_t^{p_i} x_i(t) = f_i\left(t, x_1(t), \ldots, x_n(t)\right), t \in (a, b), \quad (6) $$

for some $p_i \in (0, 1)$, $1 \leq i \leq m$, where ${}_a\mathcal{D}_t^{p}$ is a fractional differential operator with lower limit at 0,

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$+$ **boundary conditions**:

- **periodic**;
- linear;
- nonlinear (including integral boundary conditions).

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- approximation methods $\Longrightarrow$ **numerical-analytic technique**

Fractional differential equations

Fractional differential equations:

analysis and approximations

Main concept

Consider a **B**oundary **V**alue **P**roblem (**BVP**)[1]

$$ {}_{0}^{C}D_t^{p} x(t) = f\left(t, x\left(t\right)\right),\ t \in (0,T),\ p \in (0,1) \tag{7}$$

$$ x(0) = x(T). \tag{8}$$

[1] M. Fečkan and K.M., Approximation approach to periodic BVP for fractional differential systems, *European Physical Journal: Special Topics* (2017) 226, 3681-3692, doi: 10.1140/epjst/e2018-00017-9

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We perturb equation (7) by a constant term Δ and couple it with with the initial condition as follows:

$$ {}_0^C D_t^p x(t) = f\left(t, x\left(t\right)\right) + \Delta, \tag{9} $$

$$ x(0) = \xi. \tag{10} $$

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Using the integral representation of solution of the **I**nitial **V**alue **P**roblem (**IVP**) (9), (10) we get:

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To compute the perturbation Δ we require solution $x(t)$ to also satisfy periodic boundary conditions

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$$\Delta = \frac{p}{T^p} \int_0^T (T-s)^{p-1} f(s, x(s)) ds, \qquad (12)$$

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and thus, solution of the periodic BVP reads:

$$x(t, \xi) = \xi + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x(s, \xi)) ds$$
$$- \frac{t^p}{T^p \Gamma(p)} \int_0^T (T-s)^{p-1} f(s, x(s, \xi)) ds.$$

Higher-order fractional periodic BVPs

Let us now look at a higher order fractional differential system[2]

$$ {}_0^C D_t^p x(t) = f(t, x(t)), \ p \in (m, m+1), \ m \in \mathbb{N} \tag{13} $$

[2] M. Fečkan, K. M., J.R. Wang, Periodic boundary value problems for higher order fractional differential systems, *Mathematical Methods in Applied Sciences* (2019), 42, 3616-3632, doi: 10.1002/mma.5601

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where $t \in [0, T]$, $T > 0$, $x \in C^m([0, T], D)$, $D \subset \mathbb{R}^n$ is open, $f \in C(G, \mathbb{R}^n)$, $G := [0, T] \times D$ and ${}_{0}^{C}D_{t}^{p}$ is the generalized Caputo fractional derivative with lower limit at 0.

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Together with the BVP (13), (14) we consider a perturbed IVP:

$$ {}_0^C D_t^p x(t) = f\left(t, x\left(t\right)\right) + \Delta \tag{15} $$

$$ x(0) = \xi_0, \quad x'(0) = \xi_1, \quad \dots, \quad x^{(m)}(0) = \xi_m \tag{16} $$

where Δ and $\xi_i \in \mathbb{R}^n$ are unknown parameters.

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where Δ and $\xi_i \in \mathbb{R}^n$ are unknown parameters.

Then the solution of (15), (16) is determined as follows:

$$ x(t) = \sum_{k=0}^{m} \frac{t^k}{k!}\xi_k + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x(s)) ds + \frac{\Delta t^p}{\Gamma(p+1)}. \qquad (17)$$

We find values of the unknowns ξ_k, $k = \overline{1, m}$ and of the parameter Δ by substituting (17) into periodic conditions (14).

From (14) we define:

$$\xi_k = \sum_{j=k}^{m} \frac{T^{j-k-1} B_{j-k}}{(j-k)!} \Bigg[-\frac{1}{\Gamma(p-j+1)} \int_0^T (T-s)^{p-j} f(s, x(s)) ds + \frac{(p-m) T^{m-j+1}}{\Gamma(p-j+2)} \int_0^T (T-s)^{p-m-1} f(s, x(s)) ds \Bigg], k = \overline{1, m};$$

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$$\left. + \frac{(p-m) T^{m-j+1}}{\Gamma(p-j+2)} \int_0^T (T-s)^{p-m-1} f(s, x(s)) ds \right], k = \overline{1, m};$$

$$\Delta = - \frac{p-m}{T^{p-m}} \int_0^T (T-s)^{p-m-1} f(s, x(s)) ds, \tag{18}$$

with B_{j-k} being the Bernoulli numbers, resulting in the integral representation of solution:

$$x(t, \xi_0) = \xi_0 + \sum_{j=1}^{m} \frac{t^j}{j!} \xi_j + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x(s, \xi_0)) ds$$
$$- \frac{(p-m) t^p}{T^{p-m} \Gamma(p+1)} \int_0^T (T-s)^{p-m-1} f(s, x(s, \xi_0)) ds. \tag{19}$$

Mixed-order fractional periodic BVP

Problem setting

We consider a mixed-order periodic BVP[3] :

$$\begin{cases} {}_0^C D_t^p x = f(t, x(t), y(t)), \\ {}_0^C D_t^q y = g(t, x(t), y(t)) \end{cases} \tag{20}$$

$$x(0) = x(T), \quad y(0) = y(T), \tag{21}$$

for some $p, q \in (0, 1]$, where $f : G_f \to \mathbb{R}^{n_1}$, $g : G_g \to \mathbb{R}^{n_2}$ are continuous functions, $G_f := [0, T] \times D_f$, $G_g := [0, T] \times D_g$ and $D_f \subset \mathbb{R}^{n_1}$, $D_g \subset \mathbb{R}^{n_2}$ are closed and bounded domains.

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Applications of (20):

- dynamical macroeconomic model of two national economies;
- fractional Van der Pol oscillator;
- Duffing systems, etc.

[3] M. Fečkan, K.M., Approximation approach to periodic BVP for mixed fractional differential systems, *Journal of Comp. and Applied Mathematics* (2018) 339, 208-217, doi: 10.1016/j.cam.2017.10.028

Main assumptions on (20)-(21):

(A.1)

$$|f(t,x,y)| \leq M_f, \quad |g(t,x,y)| \leq M_g; \tag{22}$$

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$$|f(t,x,y)| \leq M_f, \quad |g(t,x,y)| \leq M_g; \tag{22}$$

(A.2)

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq K_{11}|x_1 - x_2| + K_{12}|y_1 - y_2|, \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq K_{21}|x_1 - x_2| + K_{22}|y_1 - y_2|; \end{aligned} \tag{23}$$

(A.3)

$$
\begin{aligned}
D_{\beta_f} &:= \left\{\xi_0 \in D_f : \{u \in \mathbb{R}^n : |u - \xi_0| \le \beta_f\} \subset D_f\right\} \neq \varnothing, \\
D_{\beta_g} &:= \{\xi_1 \in D_g : \{v \in \mathbb{R}^m : |v - \xi_1| \le \beta_g\} \subset D_g\} \neq \varnothing,
\end{aligned} \tag{24}
$$

where

$$
\beta_f := \frac{M_f T^p}{2^{2p-1}\Gamma(p+1)}, \quad \beta_g := \frac{M_g T^q}{2^{2q-1}\Gamma(q+1)};
$$

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(A.4) The spectral radius $r(Q)$ of the matrix $Q := K\Gamma_{pq}$ satisfies an inequality

$$r(Q) < 1,$$

where $\Gamma_{pq} := \max\left\{\frac{T^p}{2^{2p-1}\Gamma(p+1)}, \frac{T^q}{2^{2q-1}\Gamma(q+1)}\right\}$.

Approximation sequences

$$x_m(t,\xi_0,\xi_1) := \xi_0 + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, x_{m-1}(s,\xi_0,\xi_1), y_{m-1}(s,\xi_0,\xi_1))ds - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, x_{m-1}(s,\xi_0,\xi_1), y_{m-1}(s,\xi_0,\xi_1))ds\Bigg], \tag{25}$$

$$y_m(t,\xi_0,\xi_1) := \xi_1 + \frac{1}{\Gamma(q)}\Bigg[\int_0^t (t-s)^{q-1} g(s, x_{m-1}(s,\xi_0,\xi_1), y_{m-1}(s,\xi_0,\xi_1))ds - \left(\frac{t}{T}\right)^q \int_0^T (T-s)^{q-1} g(s, x_{m-1}(s,\xi_0,\xi_1), y_{m-1}(s,\xi_0,\xi_1))ds\Bigg], \tag{26}$$

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\end{aligned} \tag{26}
$$

where $t \in [0,T]$, $\xi_0 \in D_{\beta_f}$, $\xi_1 \in D_{\beta_g}$ and

$$x_0(t,\xi_0,\xi_1) = \xi_0, \quad y_0(t,\xi_0,\xi_1) = \xi_1.$$

Convergence of the sequence:

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Then for all fixed $\xi_0 \in D_{\beta_f}$, $\xi_1 \in D_{\beta_g}$:

1. *Functions of the sequence* (25), (26) *are continuous and satisfy periodic boundary conditions*

$$x_m(0, \xi_0, \xi_1) = x_m(T, \xi_0, \xi_1),$$
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TUDelft

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2. *The sequences of functions (25), (26) for* $t \in [0, T]$ *converge uniformly as* $m \to \infty$ *to the appropriate limit functions*

$$x_\infty(t, \xi_0, \xi_1) = \lim_{m\to\infty} x_m(t, \xi_0, \xi_1),$$
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3. *The limit functions x_∞, y_∞ satisfy periodic boundary conditions*

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3. *The limit functions* x_∞, y_∞ *satisfy periodic boundary conditions*

$$x_\infty(0,\xi_0,\xi_1) = x_\infty(T,\xi_0,\xi_1),$$
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4. *The limit functions* (27) *are the unique continuous solutions of the Cauchy problem:*

$$\begin{aligned} {}_0^C D_t^p x &= f(t,x(t),y(t)) + \Delta^p(\xi_0,\xi_1), \quad x(0) = \xi_1, \\ {}_0^C D_t^q y &= g(t,x(t),y(t)) + \Delta^q(\xi_0,\xi_1), \quad y(0) = \xi_1, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \Delta^p(\xi_0,\xi_1) &:= -\frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s,x_\infty(s,\xi_0,\xi_1),y_\infty(s,\xi_0,\xi_1))ds, \\ \Delta^q(\xi_0,\xi_1) &:= -\frac{q}{T^q}\int_0^T (T-s)^{q-1} g(s,x_\infty(s,\xi_0,\xi_1),y_\infty(s,\xi_0,\xi_1))ds. \end{aligned} \tag{29}$$

TUDelft

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5. *The following error estimations hold:*

$$\begin{pmatrix} |x_\infty(t,\xi_0,\xi_1) - x_m(t,\xi_0,\xi_1)| \\ |y_\infty(t,\xi_0,\xi_1) - y_m(t,\xi_0,\xi_1)| \end{pmatrix} \le \Gamma_{pq} Q^m (I-Q)^{-1} \begin{pmatrix} M_f, \\ M_g \end{pmatrix}, \tag{30}$$

where I *is the identity matrix.*

TUDelft

Main result:

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Then $x_\infty(\cdot, {\xi_0}^*, {\xi_1}^*)$, $y_\infty(\cdot, {\xi_0}^*, {\xi_1}^*)$ *are unique solutions of* (20)-(21) *iff a pair* $({\xi_0}^*, {\xi_1}^*)$ *are is a solution of the determining system:*

$$\Delta(\xi_0, \xi_1) := (\Delta^p(\xi_0, \xi_1), \Delta^q(\xi_0, \xi_1)) = 0, \qquad (31)$$

where Δ^p, Δ^q *are given by* (29).

Fractional differential equations

Fractional differential equations: applications

Step-by-step application of the technique:

S.1: Check conditions of the type **(A.1)**-**(A.4)** in the given domain D.

S.2: If **S.1** holds, then construct parameter-dependent sequences $\{\mathbf{x}_m(\cdot, \xi)\}$.

S.3: For each m solve the determining system (numerically)

$$\boldsymbol{\Delta}_m^{\mathbf{p}}(\xi) = 0,$$

and find approximate values of $\xi^{(m)}$.

S.4: Substitute values $\xi^{(m)}$ into $\{\mathbf{x}_m(\cdot, \xi)\}$ to find the m-th approximation to the exact solution of the given system:

$$\mathbf{X}_m(t) = \mathbf{x}_m\left(t, \xi^{(m)}\right).$$

S.5: Compare results (via plotting, error functions calculation, etc.).

Numerical example

Fractional Duffing oscillator

Let us find approximate solutions of a mixed-order Duffing system [4]

$$\begin{cases} {}_0^C D_t^{1/2} u(t) = v(t) \ (:= f(t, u(t), v(t))), \\ v'(t) = -u^3(t) + 0.6 \sin(1.2t) \ (:= g(t, u(t), v(t))), \end{cases} \quad t \in (0, 1/4) \tag{32}$$

subject to periodic boundary conditions:

$$u(0) = u(1/4), \quad v(0) = v(1/4) \tag{33}$$

in the domain

$$D_f \times D_g = [-1, 1] \times [-1, 1].$$

[4] Z. Li, D. Chen, J. Zhu, Y. Liu, Nonlinear dynamics of fractional order Duffing system, *Chaos Solitons Fractals* (2015) 81, 11-116, doi:10.1016/j.chaos.2015.09.012

We first check if assumptions **(A.1)**–**(A.4)** are satisfied.

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Indeed, we find that for

1). $M_f = 1,\ M_g \approx 1.18$ and $K = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \Longrightarrow$ **(A.1)**–**(A.2)** hold;

2) $D_{\beta_f} \times D_{\beta_g} = [-0.44, 0.44] \times [-0.85, 0.85] \Longrightarrow$ **(A.3)** holds;

3) $r(Q) = 0.98 < 1 \Longrightarrow$ **(A.4)** holds.

Constructed approximations:

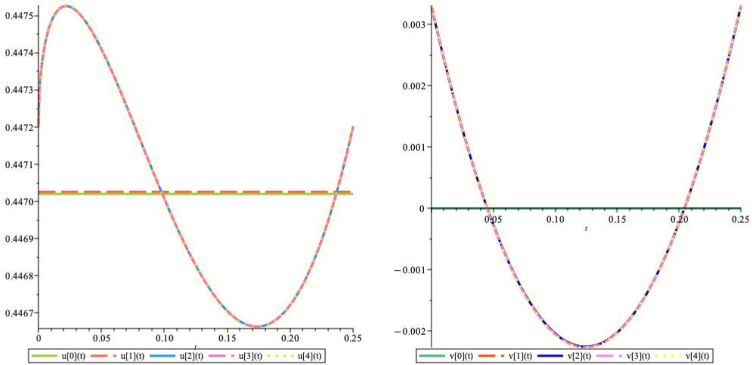

Figure: **Five approximations to the exact solution** $(u^*(t), v^*(t))$ **of the BVP** (32), (33): $u_m(t)$ − **left plot,** $v_m(t)$ − **right plot**

Errors of approximation for $u(t)$:

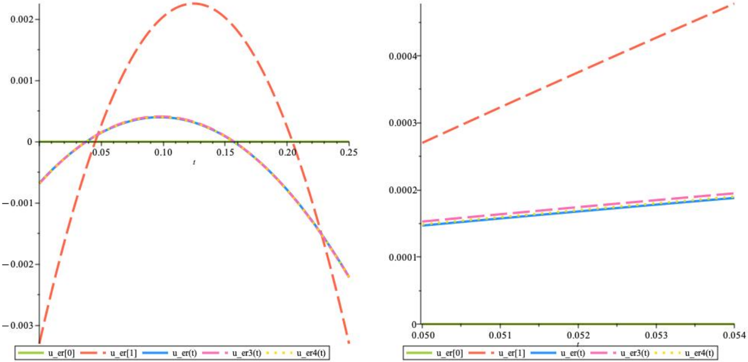

Figure: **Error of approximation of the first component of solution (left plot) and its zoomed version (right plot):** ${}_0^C D_t^{1/2} u_m(t) - f(t, u_m(t), v_m(t))$

Errors of approximation for $v(t)$:

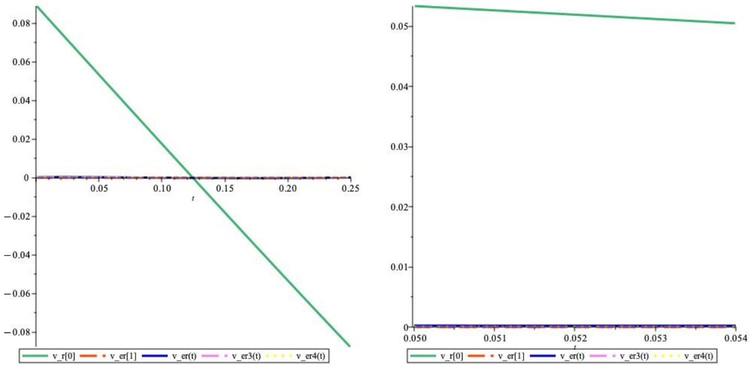

Figure: **Error of approximation of the second component of solution (left plot) and its zoomed version (right plot):** $v'_m(t) - g(t, u_m(t), v_m(t))$

Work in progress and open questions

1. **Work in progress:**

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- **(non-)periodicity under action of a fractional differential operator:**

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Do you want to explore possibilities for collaboration?
Just send me an email to K.Marynets@tudelft.nl

Thank you for your attention!