A multi-physics system for magneto-rheological suspensions

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Magnetorheological fluids

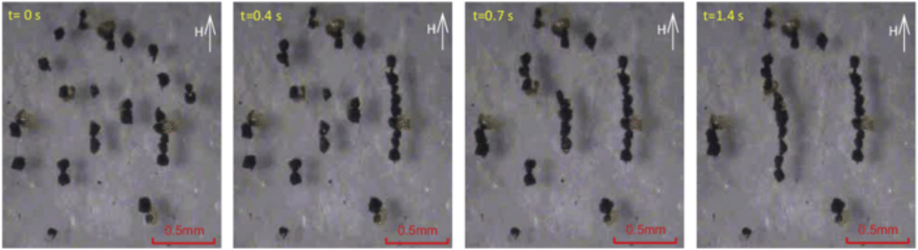

Figure: Magnetite particles aggregating into chains. Image from K. Jiangang et al (Miner. Enginrg. '15)

- Suspension of non-colloidal ferromagnetic particles in a non-magnetizable fluid
  - → Brownian motion effects are neglected
- .05-10 μm size particles
  - → Volume fractions of $\sim$ 10% to $\sim$ 50%
- Once a magnetic field is applied, the particles organize in chain structures
- Millisecond transformation form fluid to semi-solid state

Typical modeling approaches

- Phenomenological approach
  - Jacob Rabinow (AIEE Trans., '48)
  - Basic mathematical model by Rosensweig & Neuringer (Phys. Fluids, '64)
    - ⋆ Shliomis (Sov. Phys. JETP, '72) improves model by allowing "internal rotations"
  - Classical thermodynamics approach
    - ⋆ Brigadnov & Dorfmann (Cont. Mechanics Thermod., '05)
- Homogenization approach
  - First attempt using homogenization was in Lévy (J. Méc, Théor. App., '85)
  - Lévy & Hsieh (Int. J. Engng. Sci., '88) extended the work of Lévy
  - Perlak & Vernescu (Rev. Roumaine Math. Pures Appl., '00)
  - Gorb, Maris, Vernescu (J. Math. Anal. Appl., '14)
  - N. & Vernescu (ZAMP, '20, Emerg. Problems Homogen. PDE, '21)
  - Tang, Gorb, & Jimenez-Bolaños (SIAP, '21, SIMA, '23)

Cauchy stress

- Magnetorheological fluids exhibit non-Newtonian behavior
- In shear experiments the Bingham constitutive law models response of magnetotheological fluids

- Newtonian incompressible fluids

$$\sigma = -p\,\mathrm{I} + 2\,\nu\, e(\boldsymbol{v}), \quad e(\boldsymbol{v}) = \frac{1}{2}\,(\nabla \boldsymbol{v} + \nabla^t \boldsymbol{v})$$

$$\sigma' = 2\,\nu\, e'(\boldsymbol{v}), \quad A'(\boldsymbol{v}) = A - \frac{1}{n}\, tr(A)$$

- Bingham incompressible fluids

$$\begin{cases} \text{if } |\sigma| \geq \sigma_y, \text{ then } \sigma = 2\,\nu\, e(\boldsymbol{v}) + \sigma_y \frac{e(\boldsymbol{v})}{|e(\boldsymbol{v})|} \\ \text{if } |\sigma| \leq \sigma_y, \text{ then } e(\boldsymbol{v}) = 0 \end{cases}$$

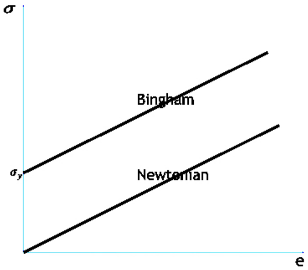

Figure: stress versus strain rate

Governing equations

$$-\mathrm{div}\sigma = \mathbf{0},\ \ \sigma = 2\,e(\boldsymbol{v}) - \mathrm{p}\,\mathrm{I} \text{ in } \Omega_{\mathrm{F}},$$
$$\mathrm{div}\boldsymbol{v} = 0 \text{ in } \Omega_{\mathrm{F}},$$
$$\boldsymbol{v} = \boldsymbol{v}^{(k)} + \boldsymbol{\omega}^{(k)} \times \left(\boldsymbol{x} - \boldsymbol{x}^{(k)}\right) \text{ on } \partial\mathrm{P}^{(k)},\ k = 1, \dots K,$$
$$\mathrm{div}\boldsymbol{B} = 0 \text{ in } \Omega,$$
$$\mathrm{curl}\boldsymbol{H} = \mathrm{R_m}\,\boldsymbol{v} \times \boldsymbol{B}\chi_{\Omega_{\mathrm{P}}},\ \text{ in } \Omega,$$
$$\underbrace{\mathrm{div}\left(R_m\,\boldsymbol{v}\times\boldsymbol{B}\,\chi_{\Omega_{\mathrm{P}}}\right)=0 \text{ in } \Omega,\ \left\langle \mathrm{R_m}\,\boldsymbol{v}\times\boldsymbol{B}\cdot\boldsymbol{n}^{(k)}, 1\right\rangle_{\mathrm{H}^{1/2}(\partial\mathrm{P}^{(k)}),\mathrm{H}^{1/2}(\partial\mathrm{P}^{(k)})}=0}_{\text{compatibility conditions}}$$

- Magnetic permeability,

$$\boldsymbol{H} = \hat{\mu}\boldsymbol{B},\ \ \mu := \begin{cases} \mu_{\mathrm{F}} \text{ in } \Omega_{\mathrm{F}}, \\ \mu_{\mathrm{P}} \text{ in } \Omega_{\mathrm{P}}, \end{cases} \quad (\hat{\mu} := 1/\mu > 0)$$

- Interface and exterior boundary conditions,

$$[\![\boldsymbol{v}]\!] = \mathbf{0} \text{ on } \partial\mathrm{P}^{(\kappa)}, \quad \boldsymbol{v} = \mathbf{0}, \quad \boldsymbol{B}\cdot\boldsymbol{n} = \boldsymbol{b}^0\cdot\boldsymbol{n} \text{ on } \Gamma_0,$$

- $[\,\mathrm{R_m}\,] = \left[\overline{\eta}\,\overline{\mu}\,\overline{L}\,\overline{V}\right]$ is the magnetic Reynolds number

Balance of forces and torques

- The force can be written in terms of the magnetic Maxwell stress,

$$\boldsymbol{F} := -\frac{1}{2}|\boldsymbol{H}|^2 \nabla\mu \iff \boldsymbol{F} = \operatorname{div}(\tau^{\text{mag}}) - \boldsymbol{B} \times \operatorname{curl}(\hat{\mu}\,\boldsymbol{B}),$$

$$\tau^{\text{mag}} := \hat{\mu}\,\boldsymbol{B}\otimes\boldsymbol{B} - \frac{1}{2}\hat{\mu}\,|\boldsymbol{B}|^2 \mathrm{I} \implies \operatorname{div}(\tau^{\text{mag}}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \Omega_{\mathrm{F}}, \\ \boldsymbol{B} \times \operatorname{curl}(\hat{\mu}\,\boldsymbol{B}) & \text{if } \boldsymbol{x} \in \Omega_{\mathrm{P}}. \end{cases}$$

- Hence, we can write the balance of forces and torques on each particle as,

$$0 = \int_{\partial \mathrm{P}^{(\kappa)}} \sigma \boldsymbol{n}^{(\kappa)}\, ds + \alpha \int_{\partial \mathrm{P}^{(\kappa)}} [\![\tau^{\text{mag}} \boldsymbol{n}^{(\kappa)}]\!]\, ds - \alpha \int_{\mathrm{P}^{(\kappa)}} \boldsymbol{B} \times \operatorname{curl}(\hat{\mu}\,\boldsymbol{B})\, d\boldsymbol{x},$$

$$0 = \int_{\partial \mathrm{P}^{(\kappa)}} \sigma \boldsymbol{n}^{(\kappa)} \times \left(\boldsymbol{x} - \boldsymbol{x}^{(\kappa)}\right) ds + \alpha \int_{\partial \mathrm{P}^{(\kappa)}} [\![\tau^{\text{mag}} \boldsymbol{n}^{(\kappa)}]\!] \times \left(\boldsymbol{x} - \boldsymbol{x}^{(\kappa)}\right) ds$$
$$- \alpha \int_{\mathrm{P}^{(\kappa)}} (\boldsymbol{B} \times \operatorname{curl}(\hat{\mu}\,\boldsymbol{B})) \times \left(\boldsymbol{x} - \boldsymbol{x}^{(\kappa)}\right) d\boldsymbol{x}.$$

- $[\alpha] = [\overline{\mu}\,\overline{H}\,\overline{L}/\overline{\nu}\,\overline{V}]$ is the *Alfven* number

Some results regarding function spaces

Proposition

Let $\mathcal{O} \subset \mathbb{R}^d$ be any open, bounded, multiply connected set with boundary $\Gamma := \partial\mathcal{O}$ of class C^2. The exterior boundary will be denoted by Γ_0 and by Γ_j, $j = 1, \ldots, \kappa - 1$, the other components of Γ. Define $\mathcal{Y}$ to be the Hilbert space of vector fields,

$$\mathcal{Y} := \left\{ \boldsymbol{u} \in \mathrm{L}^2(\mathcal{O};\mathbb{R}^d) \mid \operatorname{div}\boldsymbol{u} \in \mathrm{L}^2(\mathcal{O}),\ \operatorname{curl}(\hat{\mu}\,\boldsymbol{u}) \in \mathrm{L}^2(\mathcal{O};\mathbb{R}^d), \boldsymbol{u}\cdot\boldsymbol{n} \in \mathrm{H}^{1/2}(\Gamma_0) \right\},$$

for the norm,

$$\|\boldsymbol{w}\|_{\mathcal{Y}} := \|\boldsymbol{w}\|_{\mathrm{L}^2(\mathcal{O};\mathbb{R}^d)} + \|\operatorname{div}\boldsymbol{w}\|_{\mathrm{L}^2(\mathcal{O})} + \left\|\operatorname{curl}(\hat{\mu}\,\boldsymbol{w})\right\|_{\mathrm{L}^2(\mathcal{O};\mathbb{R}^d)} + \|\boldsymbol{w}\cdot\boldsymbol{n}\|_{\mathrm{H}^{1/2}(\Gamma_0)},$$

then for all $\boldsymbol{w} \in \mathcal{Y}$ we have, $\boldsymbol{w}_{|\mathcal{O}_i} \in \mathrm{H}^1(\mathcal{O}_i;\mathbb{R}^d)$ for $i = 1, \ldots, \kappa$ and

$$\left\|\boldsymbol{w}_{|\mathcal{O}_i}\right\|_{\mathrm{H}^1(\mathcal{O}_i;\mathbb{R}^d)} \leq C_{\mathcal{O}_i}\,\|\boldsymbol{w}\|_{\mathcal{Y}}.$$

- (small) extension of Prop. 3.1 in Foias & Temam (Ann. Sc. norm. super. Pisa, '78)

Some results regarding function spaces

Proposition

Define a new norm on $\mathcal{Y}$ by

$$[\boldsymbol{w}]_{\mathcal{Y}} := \|\operatorname{div} \boldsymbol{w}\|_{\mathrm{L}^2(\mathcal{O})} + \left\|\operatorname{curl}(\hat{\mu}\, \boldsymbol{w})\right\|_{\mathrm{L}^2(\mathcal{O};\mathbb{R}^d)} + \|\boldsymbol{w}\cdot\boldsymbol{n}\|_{\mathrm{H}^{1/2}(\Gamma_0)},$$

then $\mathcal{Y}$ is also a Hilbert space with norm $[\cdot]_{\mathcal{Y}}$.

Theorem (Poincaré type inequality for $(\mathcal{Y}, [\cdot]_{\mathcal{Y}})$)

There exists a constant, $c := c(\mathcal{O})$, such that

$$\|\boldsymbol{w}\|_{\mathrm{L}^2(\mathcal{O};\mathbb{R}^d)} \leq c\,[\boldsymbol{w}]_{\mathcal{Y}},$$

for all $\boldsymbol{w} \in \mathcal{Y}$.

- Proof by contradiction
- Use the positivity of $\hat{\mu}_0 > 0$ ($\hat{\mu}_0 := \min_i \hat{\mu}_i$)
- Global **Div-Curl** lemma of L. Tartar

The function spaces

- Inner product space for the velocity,

$$\mathcal{V} = \left\{ \boldsymbol{v} \in \mathrm{H}^1_{\Gamma_0}(\Omega_\mathrm{F}; \mathbb{R}^d) \mid \mathrm{div}\boldsymbol{v}{=}0 \text{ in } \Omega_\mathrm{F}, \boldsymbol{v}{=}\boldsymbol{v}^{(\kappa)}{+}\boldsymbol{\omega}^{(\kappa)}{\times}(\boldsymbol{x}-\boldsymbol{x}^{(\kappa)}) \text{ on } \partial \mathrm{P}^{(\kappa)} \right\}.$$

$$(\boldsymbol{v} \mid \boldsymbol{\phi})_{\mathcal{V}} = \int_{\Omega_\mathrm{F}} 2\, e(\boldsymbol{v}){:}e(\boldsymbol{\phi})\, d\boldsymbol{x}.$$

- Inner product space for the magnetic induction,

$$\mathcal{Y} = \left\{ \boldsymbol{w} \in \mathrm{L}^2(\Omega; \mathbb{R}^d) \mid \mathrm{div}(\boldsymbol{w}) \in \mathrm{L}^2(\Omega),\ \mathrm{curl}(\hat{\mu}\boldsymbol{w}) \in \mathrm{L}^2(\Omega; \mathbb{R}^d), \right.$$
$$\left. \boldsymbol{w} \cdot \boldsymbol{n} \in \mathrm{H}^{1/2}(\Gamma_0) \right\},$$

$$(\boldsymbol{h} \mid \boldsymbol{\psi})_{\mathcal{Y}} = \int_{\Omega} \mathrm{div}(\boldsymbol{h})\, \mathrm{div}(\boldsymbol{\psi}) d\boldsymbol{x} + \int_{\Omega} \mathrm{curl}(\hat{\mu}\, \boldsymbol{h}) \cdot \mathrm{curl}(\hat{\mu}\boldsymbol{\psi}) d\boldsymbol{x}$$
$$+ \int_{\Gamma_0} (\boldsymbol{h} \cdot \boldsymbol{n})(\boldsymbol{\psi} \cdot \boldsymbol{n}) ds.$$

Variational formulation of Stokes' equation

- Multiply with am appropriate test function,

$$-\int_{\cup_{\kappa=1}^{K}\partial P^{(\kappa)}} \sigma \boldsymbol{n}^{(\kappa)}.\boldsymbol{\phi}\, ds + \int_{\Omega_{\mathrm{F}}} 2e(\boldsymbol{v}):e(\boldsymbol{\phi})\, d\boldsymbol{x} = 0.$$

- Use balance of forces and torques,

$$\alpha \int_{\cup_{\kappa=1}^{K}\partial P^{(\kappa)}} [\![\tau^{\mathrm{mag}} \boldsymbol{n}^{(\kappa)}]\!].\boldsymbol{\phi}\, ds - \alpha \int_{\Omega_{\mathrm{P}}} \left[\boldsymbol{B} \times \mathrm{curl}(\hat{\mu}\boldsymbol{B})\right].\boldsymbol{\phi}\, d\boldsymbol{x} + \int_{\Omega_{\mathrm{F}}} 2\, e(\boldsymbol{v}):e(\boldsymbol{\phi})\, d\boldsymbol{x} = 0.$$

- Find $\boldsymbol{v} \in \mathcal{V}$,

$$(\boldsymbol{v} \mid \boldsymbol{\phi})_{\mathcal{V}} + \alpha \int_{\Omega_{\mathrm{F}}} \tau^{\mathrm{mag}}:e(\boldsymbol{\phi})\, d\boldsymbol{x} = \mathbf{0} \quad \text{for all } \boldsymbol{\phi} \in \mathcal{V}.$$

Augmented variational formulation of Maxwell's equations

For an appropriate test function,

- multiply the divergence part by $\frac{\alpha}{\mathrm{R_m}}\,\mathrm{div}(\boldsymbol{\psi})$
- multiply the rotational part by $\frac{\alpha}{\mathrm{R_m}}\,\mathrm{curl}\,(\hat{\mu}\boldsymbol{\psi})$
- multiply the exterior boundary condition by $\frac{\alpha}{\mathrm{R_m}}\,\boldsymbol{\psi}\cdot\boldsymbol{n}$

$$(\boldsymbol{h}\mid\boldsymbol{\psi})_{\mathcal{Y}} = \int_{\Omega}\mathrm{div}(\boldsymbol{h})\,\mathrm{div}(\boldsymbol{\psi})d\boldsymbol{x} + \int_{\Omega}\mathrm{curl}(\hat{\mu}\,\boldsymbol{h})\cdot\mathrm{curl}(\hat{\mu}\boldsymbol{\psi})d\boldsymbol{x} + \int_{\Gamma_0}(\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{\psi}\cdot\boldsymbol{n})ds.$$

Find $\boldsymbol{B}\in\mathcal{Y}$ such that,

$$\boxed{\frac{\alpha}{\mathrm{R_m}}(\boldsymbol{B}\mid\boldsymbol{\psi})_{\mathcal{Y}} = \alpha\int_{\Omega_{\mathrm{P}}}[\boldsymbol{v}\times\boldsymbol{B}]\cdot\mathrm{curl}\,(\hat{\mu}\,\boldsymbol{\psi})\;d\boldsymbol{x} + \frac{\alpha}{\mathrm{R_m}}\int_{\Gamma_0}(\boldsymbol{b}^0\cdot\boldsymbol{n})(\boldsymbol{\psi}\cdot\boldsymbol{n})\,ds,}$$

for all $\boldsymbol{\psi}\in\mathcal{Y}$.

Variational formulation of the problem

Find $(\boldsymbol{v}, \boldsymbol{B}) \in \mathcal{V} \times \mathcal{Y}$ such that,

$$(\boldsymbol{v} \mid \boldsymbol{\phi})_{\mathcal{V}} + \frac{\alpha}{\mathrm{R_m}} (\boldsymbol{B} \mid \boldsymbol{\psi})_{\mathcal{Y}} = -\alpha \int_{\Omega_\mathrm{F}} \tau^{\mathrm{mag}} : e(\boldsymbol{\phi})\, d\boldsymbol{x} + \alpha \int_{\Omega_\mathrm{P}} [\boldsymbol{v} \times \boldsymbol{B}].\mathrm{curl}\,(\hat{\mu}\,\boldsymbol{\psi})\; d\boldsymbol{x} + \frac{\alpha}{\mathrm{R_m}} \int_{\Gamma_0} (\boldsymbol{b}^0.\boldsymbol{n})(\boldsymbol{\psi}.\boldsymbol{n})\, ds,$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V} \times \mathcal{Y}$. Naturally, a norm is associated to the above inner (cross-) product space denoted by $|||(-,\cdot)||| := \|-\|_{\mathcal{V}} + \frac{\alpha}{\mathrm{R_m}}[\cdot]_{\mathcal{Y}}$.

Theorem (N., Vernescu (Banach J. Math. Anal., '24))

The pair $(\boldsymbol{v}, \boldsymbol{B})$ satisfies the strong form of Maxwell's and Stokes' equations as well as their BC if and only if it is a solution to the above weak formulation.

Equivalence between weak and strong form of the problem

- One direction is clear
- Recover Maxwell's equations: introduce $\zeta^\delta : \mathbb{R}^d \to [0,1]$

$$\zeta^\delta(\boldsymbol{x}) = \begin{cases} 1 & \text{if } d(\boldsymbol{x}, \Gamma_0) < \delta, \\ 0 & \text{if } d(\boldsymbol{x}, \Gamma_0) > 2\,\delta \end{cases}$$

- Using the approach of Ledyzhenskaya, ('63), define $\boldsymbol{\theta}(\boldsymbol{x}) := (b_2^0 x_3, b_3^0 x_1, b_1^0 x_2)$. Set $\boldsymbol{a}^\delta(\boldsymbol{x}) := \mathrm{curl}(\zeta^\delta(\boldsymbol{x})\,\boldsymbol{\theta}(\boldsymbol{x}))$. Then $\boldsymbol{a}^\delta(\boldsymbol{x})$ is a divergence free and equals $\boldsymbol{b}^0$ in the δ neighbourhood of Γ_0
- Using Lemma 3.5 in Amrouche et al., (Math. Meth. Applied Sci., '98) there exists a vector field $\boldsymbol{V} \in \mathrm{H}^1(\Omega, \mathbb{R}^3)$ such that $\mathrm{div}(\boldsymbol{V}) = 0$ and $\mathrm{curl}(\boldsymbol{V}) = \mathrm{R_m}\boldsymbol{v} \times \boldsymbol{B}\chi_{\Omega_\mathrm{P}}$.
- Use approach of P.-E. Druet, (Discrete Contin. Dyn. Syst. Ser. A, '15)

$$\begin{aligned} \mathrm{div}(\mu \nabla p) &= \mathrm{div}(\mu \boldsymbol{V}) \text{ in } \Omega_\mathrm{P} \cup \Omega_\mathrm{F}, \\ [\![\mu \partial_{\boldsymbol{n}^{(\kappa)}} p]\!] &= [\![\mu \boldsymbol{V}.\boldsymbol{n}^{(\kappa)}]\!] \text{ on } \partial P^{(\kappa)},\ \kappa = 1, \dots, K, \\ \partial_{\boldsymbol{n}} p &= 0 \text{ on } \Gamma_0 \end{aligned}$$

The test function

- Construct $\boldsymbol{W} := \mu \boldsymbol{V} - \mu \nabla p$ and verify,

$$\mathrm{div}(\boldsymbol{W}) = 0, \quad \mathrm{curl}(\hat{\mu}\boldsymbol{W}) = \mathrm{R_m} \boldsymbol{v} \times \boldsymbol{B} \chi_{\Omega_\mathrm{P}}, \quad \boldsymbol{W}.\boldsymbol{n} = 0$$

- Construct $\boldsymbol{\psi} := \boldsymbol{B} - \boldsymbol{W} - \boldsymbol{a}^\delta \in \mathcal{Y}$,

$$\begin{aligned} &\int_\Omega \mathrm{div}\,\boldsymbol{B}\,\mathrm{div}(\boldsymbol{B} - \boldsymbol{W} - \boldsymbol{a}^\delta)\,d\boldsymbol{x} \\ &+ \int_\Omega [\mathrm{curl}\,(\hat{\mu}\,\boldsymbol{B}) - \mathrm{R_m} \boldsymbol{v} \times \boldsymbol{B} \chi_{\Omega_\mathrm{P}}].\mathrm{curl}\,(\hat{\mu}\,(\boldsymbol{B} - \boldsymbol{W} - \boldsymbol{a}^\delta))\,d\boldsymbol{x} \\ &+ \int_{\Gamma_0} [(\boldsymbol{B} - \boldsymbol{b}^0).\boldsymbol{n}][(\boldsymbol{B} - \boldsymbol{W} - \boldsymbol{a}^\delta).\boldsymbol{n}]\,ds = 0 \end{aligned}$$

- Using the properties of the vector fields $\boldsymbol{W}$ and $\boldsymbol{a}^\delta$ we obtain:

$$\int_\Omega |\mathrm{div}\boldsymbol{B}|^2\,d\boldsymbol{x} + \int_\Omega |\mathrm{curl}\,(\hat{\mu}\,\boldsymbol{B}) - \mathrm{R_m} \boldsymbol{v} \times \boldsymbol{B} \chi_{\Omega_\mathrm{P}}|^2\,d\boldsymbol{x} + \int_{\Gamma_0} \left|(\boldsymbol{B} - \boldsymbol{b}^0).\boldsymbol{n}\right|^2 ds = 0$$

The Altman-Shinbrot fixed point theorem

Let $\mathcal{H}$ denote a real or complex Hilbert space, and $\mathcal{S}_r$ and $\mathcal{B}_r$ denote the sphere and the closed ball of radius r centered at zero, respectively:

$$\mathcal{S}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} = r\}, \quad \mathcal{B}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} \leq r\}.$$

Theorem (Altman, Bull. Acad. Polon. Sci. '57; Shinbrot, ARMA '64)

Let H be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. If there is a positive constant r such that $\Re(Hx, x) \leq \|x\|_{\mathcal{H}}^2$ for all $x \in \mathcal{B}_r$, then H has a fixed point in $\mathcal{B}_r$.

Corollary: Let G be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. Let y be an element of $\mathcal{H}$. If there exists a positive r such that either $\Re(Gx - y, x) \geq 0$ for all $x \in \mathcal{S}_r$ OR $\Re(Gx - y, x) \leq 0$ for all $x \in \mathcal{S}_r$ then y is in the range of G.

Corollary: Let G be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. Then, zero is in the range of G if (Gx, x) is of one sign on some sphere $\mathcal{S}_r$.

Existence

- For all $\boldsymbol{v}, \boldsymbol{B}, \boldsymbol{\phi}, \boldsymbol{\psi}$ we define the following expression $\mathcal{Q}$ by,

$$\mathcal{Q}[(\boldsymbol{v},\boldsymbol{B});(\boldsymbol{\phi},\boldsymbol{\psi})] := -\alpha \int_{\Omega_\mathrm{F}} \hat{\mu}_\mathrm{F}\, \boldsymbol{B}\otimes\boldsymbol{B} : e(\boldsymbol{\phi})\, d\boldsymbol{x} + \alpha \int_{\Omega_\mathrm{P}} \boldsymbol{v}\times\boldsymbol{B}\cdot\mathrm{curl}(\hat{\mu}_\mathrm{P}\,\boldsymbol{\psi})\, d\boldsymbol{x}.$$

- We have the following bound on $\mathcal{Q}$ in terms of the product space norm:

$$|\mathcal{Q}[(\boldsymbol{v},\boldsymbol{B});(\boldsymbol{\phi},\boldsymbol{\psi})]| \le \mathrm{C}(\alpha,\hat{\mu}_\mathrm{F},\Omega_\mathrm{F},\Omega_\mathrm{P})\, |||(\boldsymbol{v},\boldsymbol{B})|||^2\ |||(\boldsymbol{\phi},\boldsymbol{\psi})|||\,,$$

- Cauchy-Schwartz and Riesz's theorem allows us to write the variational formulation as,

$$(\mathcal{F}(\boldsymbol{v},\boldsymbol{B});(\boldsymbol{\phi},\boldsymbol{\psi})) = ((\boldsymbol{f},\boldsymbol{g});(\boldsymbol{\phi},\boldsymbol{\psi})) \text{ for all } (\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathcal{V}\times\mathcal{Y}\,,$$

  - $$(\mathcal{F}(\boldsymbol{v},\boldsymbol{B});(\boldsymbol{\phi},\boldsymbol{\psi})) := (\boldsymbol{v}\mid\boldsymbol{\phi})_\mathcal{V} + \frac{\alpha}{\mathrm{R_m}}(\boldsymbol{B}\mid\boldsymbol{\psi})_\mathcal{Y} - \mathcal{Q}[(\boldsymbol{v},\boldsymbol{B});(\boldsymbol{\phi},\boldsymbol{\psi})]\,,$$
  - $$((\boldsymbol{f},\boldsymbol{g});(\boldsymbol{\phi},\boldsymbol{\psi})) := \frac{\alpha}{\mathrm{R_m}} \int_{\Gamma_0} (\boldsymbol{b}^0\cdot\boldsymbol{n})(\boldsymbol{\phi}\cdot\boldsymbol{n})\, ds.$$

Existence

Lemma

The nonlinear operator $\mathcal{F} : (\boldsymbol{v}, \boldsymbol{B}) \mapsto \mathcal{F}(\boldsymbol{v}, \boldsymbol{B})$ is continuous in the weak topology of the product space $\mathcal{V} \times \mathcal{Y}$.

Lemma

If the magnetic Reynolds number, $\mathrm{R_m}$, is small then

$$(\mathcal{F}(\boldsymbol{v}, \boldsymbol{B}); (\boldsymbol{v}, \boldsymbol{B})) \geq \frac{1}{2} |||(\boldsymbol{v}, \boldsymbol{B})|||^2 \text{ for all } (\boldsymbol{v}, \boldsymbol{B}) \in \mathcal{V} \times \mathcal{Y}.$$

Theorem

If the magnetic Reynolds number, $\mathrm{R_m}$ satisfies,

$$1 - 2\frac{C_{\mathrm{FP}} |\boldsymbol{b}^0| \mathrm{mes}_{d-1}(\Gamma_0)}{1 - C_{\mathrm{FP}} \mathfrak{r}(\Omega_{\mathrm{P}}) \mathrm{R_m}} \geq \frac{1}{2},$$

then problem $(\mathcal{F}(\boldsymbol{v}, \boldsymbol{B}); (\boldsymbol{\phi}, \boldsymbol{\psi})) = ((\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{\phi}, \boldsymbol{\psi}))$ *for all* $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V} \times \mathcal{Y}$ *admits at least one weak solution.*

Existence and comments

- Apply Altman-Shinbrot theorem to the operator equation
- Show there exists r such that

$$(\mathcal{F}(\boldsymbol{v}, \boldsymbol{B}) - (\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{v}, \boldsymbol{B})) \geq 0$$

  for all $(\boldsymbol{v}, \boldsymbol{B})$ with $|||(\boldsymbol{v}, \boldsymbol{B})||| = r$
- Select $r = 2\ |||(\boldsymbol{f}, \boldsymbol{g})|||$
- The case of $\mathrm{R_m} \equiv 0$ can be thought off as a limit case of the above model.
- $\mathrm{R_m} \equiv 0$ the system becomes weakly coupled and, existence and uniqueness follow by invoking the Lax-Milgram lemma, once higher integrability of the magnetic induction is established
- In two spatial dimensions system can also be solved analytically. Resulting behavior is of a Bingham type fluid.