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This paper reports on a pilot study with the focus on (re)design of a digitized task environment utilizing two types of technology – a dynamic mathematics software and a computer-aided assessment system. The data consist of responses from 256 first year engineering students, taking their first Calculus course, on two different types of task. The results are discussed in relation to (re)design of tasks as well as possible feedback design options to enable a formative assessment approach.

Keywords: Asymptotes, dynamic mathematics software, computer-aided assessment, task design, feedback.

Introduction

Central in introductory Calculus courses – a critical course for first year engineering students – is the concept of functions. For example, Oehrtman et al. (2008) highlight a weak function understanding as one of the reasons why many students fail in their first undergraduate mathematics courses. Oehrtman et al. (2008), in discussing important features of students' function understanding, advocate a focus on promoting "…rich conceptions and powerful reasoning abilities…" (p. 27), and not merely "…symbolic manipulations and procedural techniques…" (p.28).

As one way to foster students' conceptual understanding in mathematics, the literature suggest utilizing dynamic mathematics software (DMS) environments for student collaboration on inquiry-based tasks, particularly in relation to functions (e.g. Brunström & Fahlgren, 2015; Jaworski & Matthews, 2011). Today many mathematics courses in higher education utilize computer-aided assessment (CAA) systems (e.g. Rønning, 2017), in which it is possible to embed DMS environments (Sangwin, 2013). However, there are few studies that have investigated the integration of these two types of technology (Luz & Yerushalmy, 2019). Indeed, designing tasks that utilize the affordances provided by a DMS environment and that also can be automatically assessed by a CAA system adds to the already established complexity of designing tasks for interactive learning environments that promote mathematical understanding (Joubert, 2017).

This paper reports some results from a pilot study conducted during autumn 2020 with the aim of trialling different types of task designed for a combined use of DMS and CAA, and to get a deeper understanding of student strategies when performing these tasks. Findings from the pilot study will inform the (re)design of tasks as well as the development of possible types of automated feedback to increase first year engineering students' engagement and conceptual understanding of functions.

Theoretical framing

Besides topic-specific theories, related to mathematical functions, the task design was guided by theories associated with student generated examples and theories on feedback.

Functions and graphs

It is well established that for students to comprehend the concept of functions it is essential to be able to move flexibly between different representations, such as formula and graph (Leinhardt et al., 1990). Moreover, according to Duval (1999), representation and visualization are at the core of mathematical understanding. He distinguishes between two cognitive operations; processing and conversion, where the former concerns mathematical processes made within the same representation, such as algebraic manipulations, and the latter means changing between different representations. While many students can learn processing, it is the conversion, e.g. to translate a formula into a graph or vice versa, that many students find challenging (Duval, 1999). In particular, it is the translation from graph to formula that is the most challenging (Leinhardt et al., 1990). Furthermore, this kind of task, referred to as 'translation task' in this paper, is suitable for a CAA system since it can recognize any correct form of the function formula.

Student generated examples (SGEs)

Prompting students to generate examples that fulfil certain conditions has been proposed as a way to engage students actively in their development of conceptual mathematical understanding (e.g. Watson & Mason, 2002). This idea has been adapted to CAA systems since it allows for automatic assessment of higher-order mathematical skills (Sangwin, 2003). In this paper we use the notion 'SGE task' when referring to tasks using this idea. To further challenge students' thinking Yerushalmy et al. (2017) suggest asking students for several examples, which differ as much as possible. Moreover, Yerushalmy et al. emphasize the importance of designing feedback on students' responses on such tasks to support their mathematical reasoning processes (2017).

Feedback

So far, CAA systems have mainly been used for assessing basic mathematical procedural skills. It is a challenge to design tasks for a CAA system that address higher-order skills in mathematics, and to design feedback that goes beyond categorizing a final answer as being right or wrong (Rønning, 2017). In the wider literature, this type of feedback is referred to as 'elaborated feedback' (e.g. Shute, 2008). In a meta study investigating the effects of computer-based feedback on students' learning outcomes, van der Kleij et al. (2015) found that elaborated feedback was more effective than verificative types of feedback, especially for higher-order skills. In particular, the eight studies related to mathematics pointed in this direction.

When interacting with a DMS environment, the other type of technology reported in this paper, students are provided instant feedback on their action. It is this feedback that makes it possible to use a DMS environment as an arena for exploration, conjecturing, verification, and reflection. However, this feedback does not provide explicit suggestions on how to proceed, and thus the benefit of the feedback depends on students themselves being able to interpret the results of their actions in the DMS environment (Olsson, 2018). By embedding DMS tasks in a CAA system and utilizing the affordances provided by the two types of technology, we endeavor to enhance the provision of feedback – the goal of our upcoming project.

Method

Research context

The pilot study took place at a Swedish university, involving 256 first year engineering students taking a first course in Calculus. The course assignment included small group activities, in the form of task sequences focusing on function understanding, designed for a combined use of a DMS environment (*GeoGebra*) and a CAA system (*Möbius*). However, there were also tasks with individual elements for each group member requiring an individual answer, to ensure active involvement by all students.

This paper will examine patterns of student response to two related tasks concerning rational functions, and specifically the relationship between asymptotes and function formula. In the light of these patterns, we will offer thoughts on the types of elaborated feedback that could be beneficial, and also provided by an automated assessment system.

The tasks

Task 5 (see Figure 1) is an example of a 'translation task' intended to be solved in groups. Students are expected to realize that it must be a rational function with one horizontal and two vertical asymptotes. Then, they are supposed to utilize the vertical asymptotes to construct the (factorized) denominator, and the horizontal asymptote to conclude that the numerator should be of degree two with the coefficient 2 in front of the x^2 term. However, there is a need for further information to arrive at a final formula, i.e. two points on the graph. The reason behind asking for an explanation (Task 5b) was twofold; both to promote student reasoning, and to provide insight into student strategies when solving the task.

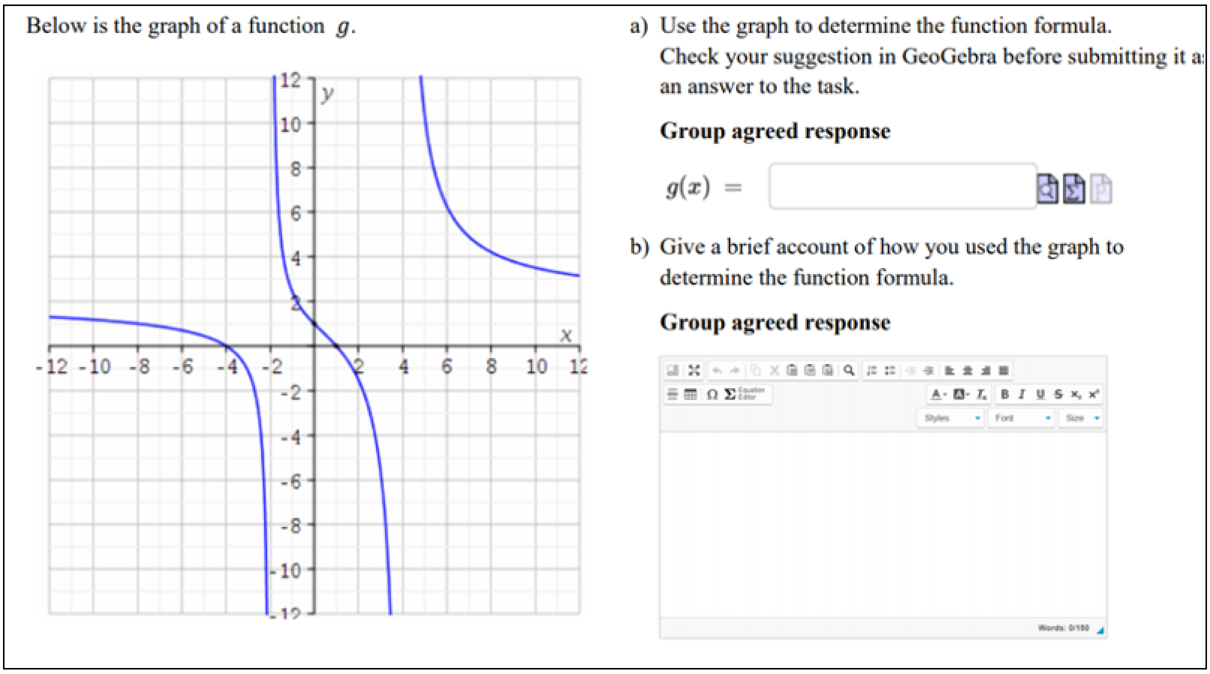
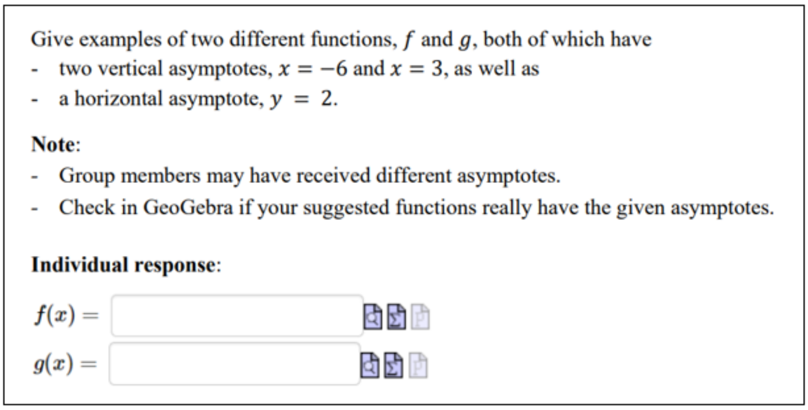


Figure 1: Task 5 as it is presented in Möbius

Task 7 (see Figure 2) is an example of a 'SGE task' in which students received different values of the asymptotes, and were supposed to provide individual answers. This task closely relates to Task 5 in that it involves two vertical asymptotes and one horizontal asymptote. In performing this task, students are supposed to consolidate the key ideas addressed in these tasks.


Give examples of two different functions, f and g, both of which have
- two vertical asymptotes, $x = -6$ and $x = 3$, as well as
- a horizontal asymptote, $y = 2$.

Note:
- Group members may have received different asymptotes.
- Check in GeoGebra if your suggested functions really have the given asymptotes.

Individual response:

$f(x) =$

$g(x) =$

Figure 2: Task 7 as it is presented in Möbius

Data collection and analysis

The data used to analyse student responses to these tasks consists of their answers submitted to the CAA system. The data analysis process involved several stages. To provide an overview of the data material, we made a preliminary analysis based on about a quarter of the submitted responses. This overview of students' various responses to the tasks gave insights into interesting instances. For example, when students provided unexpected responses, i.e. not in line with the expected solution strategy, or when it was a wide range of student responses. In addition, this preliminary analysis generated initial codes to be further developed and used in the next stage of the analysis process. At this stage, all responses to the tasks were analysed and coded. Next, the initial codes were organized into categories to discern general patterns in the data material (Saldaña, 2013). The categories obtained for these tasks are introduced in the tables in the Result Section.

Results

Table 1 provides an overview of the group responses, in terms of function formulas, to Task 5a.

Table 1: Overview of the responses provided in Task 5a[1]

Code	Description	Function formula	Frequence
Formula 1	Single quotient	$g(x) = \frac{2(x+4)(x-1)}{(x+2)(x-4)}$	47 (46,5%)
Formula 2	Partial fraction, reduced quotients, and the constant term 2 (i.e. the horizontal asymptote)	$g(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$	17 (16,8%)
Formula 3	Reduced quotient, and the constant term 2 (i.e. the horizontal asymptote)	$g(x) = \frac{10x+8}{(x+2)(x-4)} + 2$	22 (21,8%)
Other	Not categorized		9 (8,9%)
No answer			6 (5,9%)
Total			101 (100%)

[1] In some responses, the numerator and/or the denominator is/are not factorised.

Mainly three types of formula were observed. Almost half (47/101) of the groups used Formula 1. Notable is that as many as 39 of the groups gave a Formula 2 (17) or a Formula 3 (22) response. These types of formula have not been present either in the textbook or in lectures when treating asymptotes in the course. Examples presented at lectures concerning horizontal asymptotes has been in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions of the same degree (except when the horizontal asymptote is $y = 0$). Among the 9 responses not categorized, there are 4 incorrect answers.

The group responses to Task 5b provide some information about students' thinking behind their answer in Task 5a, since they were encouraged to explain how they arrived at a particular function formula. Group responses were inspected and compared to identify a set of elements of explanation which could be used to summarise the content of any response. This made it possible to code the responses in terms of explanation elements, i.e. descriptions of what students referred to.

In Table 2, an overview of the explanation elements provided by the student groups is given. The rightmost column shows the total number of each explanation element among the group responses. Almost all groups (82/86) explicitly refer to the vertical asymptotes in their explanation. However, all the categorized formulas (1 to 3) indicate that all 86 groups, that provided a function formula, utilized the vertical asymptotes. Similarly, the constant term in Formula 2 ($g(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$) and Formula 3 ($g(x) = \frac{10x+8}{(x+2)(x-4)} + 2$) indicate that these groups have utilized the horizontal asymptote, even though eight of these (39) groups did not mention this in their explanation.

Table 2: Overview of the explanation elements referred to in Task 5b

Type of formula / Explanation elements	Formula 1	Formula 2	Formula 3	Total
Vertical asymptotes	46 (97,9%)	16 (94,1%)	20 (90,9%)	82 (95,3%)
Horizontal asymptote	16 (34,0%)	16 (94,1%)	15 (68,2%)	47 (54,7%)
Zeros	44 (93,6%)	0	0	44 (51,2%)
One further point	22 (46,8%)	3 (17,6%)	0	25 (44,6%)
GeoGebra	5 (10,6%)	3 (17,6%)	6 (27,3%)	14 (16,3%)
Equation system	1 (2,1%)	7 (41,2%)	10 (45,5%)	18 (20,9%)
Total	47	17	22	

A closer look at the Formula 1 responses reveals that the predominant (46/47) characteristic of the associated explanations was to refer to the vertical asymptotes (to form the denominator of a single quotient expression as the product of the corresponding linear factors). The predominant (44/47) approach was then to refer to the zeros of the function (to form a numerator for the quotient expression as the product of the corresponding linear factors). One further step is needed to complete the quotient expression. Only 16 groups (out of 47) used the approach that the task design hoped to elicit, i.e. to use the horizontal asymptote to establish a limiting value for the quotient expression as $x \to \infty$. Thus, the remaining 31 groups did not refer to the horizontal asymptote as an explanation (see Table 2) for the factor 2 (of the numerator). Among these 31 groups, 22 refer to one further point, 5 to GeoGebra, and one group to a system of equations. Hence, we can conclude that at least 28 groups (out of 47) did not utilize the horizontal asymptote.

In Task 7, each student response consists of two examples of function formulas. However, since few students (6/256) provided a different type of formula in their second example, only the first example is reported in this paper. Table 3 provides an overview of the types of formula discerned (based on the numerical values in the example in Figure 2) as well as the total number of responses belonging to each category (the rightmost column). Moreover, Table 3 shows the correspondence between these answers and the group answer on Task 5a.

Table 3: Overview of each individual response to Task 7 in relation to their group answer in Task 5a[2]

Formula Task 7 \ Formula Task 5a	Formula 1	Formula 2	Formula 3	Not categorized	No answer	Total
$f(x) = \frac{2x^2+ax+b}{(x+6)(x-3)}$	26 (20,8%)	2 (4,3%)	3 (5,4%)	1 (5,6%)	5 (50,0%)	37 (14,5%)
$f(x) = \frac{a}{x+6} + \frac{b}{x-3} + 2$	16 (13,1%)	30 (63,8%)	8 (14,3%)	1 (5,6%)	2 (20,0%)	57 (22,3%)
$f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$	75 (61,5%)	14 (29,8%)	43 (76,8%)	15 (83,3%)	2 (20,0%)	149 (58,2%)
$f(x) = \frac{x^2+ax+b}{(x+6)(x-3)} + 1$	4 (3,3%)	1 (2,1%)	0	0	0	5 (2,0%)
No answer	4 (3,3%)	0	2 (3,6%)	1 (5,6%)	1 (10,0%)	8 (3,1%)
Total	125	47	56	18	10	256 (100%)

The predominant (149/256) type of formula used was a reduced quotient and the horizontal asymptote as a constant term ($f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$). Notably, quite a few students (37/256) expressed the formula as a single quotient, i.e. in the form $\frac{p(x)}{q(x)}$.

As there were no zeros given in Task 7, it is primarily the group answers Formula 2 and Formula 3 in Task 5a that closely relate to the answers in Task 7; Formula 2 ($f(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$) corresponds to the second type of formula (in the leftmost column) and Formula 3 ($f(x) = \frac{10x+8}{(x+2)(x-4)} + 2$) corresponds to the third type of formula. Most of these students use the same type of formula in their individual response, 63,8% and 76,8% respectively, as in their group answer. However, Table 3 shows that as many as 29,8% of the students who used Formula 2 in Task 5a switched to the third type of formula ($f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$) in Task 7. Moreover, few students that responded Formula 2 (4,3%) or 3 (5,4%) in Task 5a provided an answer in the form $\frac{p(x)}{q(x)}$ in Task 7. Overall, Task 7 worked

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[2] In the student responses, different numerical values are used instead of *a* and *b*.

well in that most of the students seemed to realize how they could use all asymptotes to produce a function formula.

Discussion

As stated in the introduction, this pilot study will inform the (re)design of tasks as well as the development of possible types of automated feedback. In this section, we elaborate on this issue in relation to the findings.

Since the intention with the 'translation task' (Task 5) was to encourage students to reflect on the relation between a function graph with asymptotes and its formula, the high number of Formula 1 responses (to Task 5a) without reference to the horizontal asymptote in the explanation was unexpected and undesirable. One way to tackle this issue might be to use a graph without evident zeros. However, the possibility to use different approaches based on various graph features may promote instructive student discussions. Another way could be to indicate the asymptotes in the graph. However, since the identification of asymptotic behavior in a graph is central in understanding rational functions, this might simplify the task too much.

Yet another way to tackle this issue is to develop automated and adapted feedback, which in turn require a redesign of Task 5b. Instead of asking for an explanation, ask students to declare the explanation elements used by choosing among various suggested options. Depending on their response, they will receive different elaborated feedback (Shute, 2008). For example, if they not have used the horizontal asymptote, they will be asked to solve a new task in which they are (explicitly) asked to utilize the horizontal asymptote.

Also, when considering that almost 10% of the student groups failed to provide a correct answer to Task 5a, there is a need for developing formative instant feedback for these students. One suggestion is to offer the students a second chance to solve the task after they have been watching a short video introducing the key ideas addressed in the task.

The 'SGE task' (Task 7) worked well, and revealed various student strategies. However, almost all students provided the same type of formula in both their examples. Since we think that it is instructive for students to realize that there are various ways of thinking, which results in different types of formula, it would have been great if the CAA system could recognize the type of formula used by a student. So, for example, if a student uses a formula of the following type: $f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$ (in both examples), the elaborated feedback (Shute, 2008) could be something like: "Great, the answers are correct. However, another correct answer could be: $f(x) = \frac{2x^2}{(x+6)(x-3)}$. How do you think a student who came up with this answer has been reasoning? Now, use this strategy to provide an example of a function with the following asymptotes…"

The study is limited by the lack of information on how students utilized the DMS environment to check their conjectured function formulas before submitting them as answers to the tasks. To be able to further develop the task design, particularly the formative feedback from the CAA system, we need to better understand the reasoning behind students' various responses. This requires empirical data in terms of screen recordings (including audio) from students 'ongoing work. Consequently, we suggest this as a natural progression of this pilot study.

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