Well-posedness of Unsteady Poroelastohydrodynamics in Solid Tumors

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7^{th} Feb 2024

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Karlstad Applied Analysis Seminar
Karlstad University, Karlstad, Sweden

Outline

What we model?

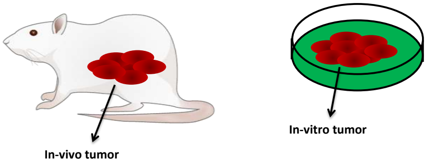

Figure: Cartoon of an in-vivo and in-vitro tumor

- An abnormal mass of tissue that forms when cells grow and divide more than they should or do not die when they should. Tumors may be benign (not cancer) or malignant (cancer).
- Typical approaches: (a) Long time scale - growth model (b) Short time scale - no growth/avascular phase of tumor growth - transport scale.

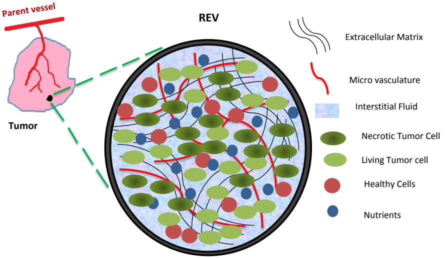

Figure: Anatomy of tumor within a representative elementary volume (REV)

- Deformable solid phase: cell population, fibrous matrix (ECM) and vascular space.
- The cell population which is the major part of the solid phase is constituted by single type of cells.
- Fluid phase: blood flow through blood vessels and interstitial/extracellular fluid.
- Homogenized model: biphasic mixture approach assuming tumor as a homogeneous deformable porous medium

Motivation and Aim

- Motivation to the present work comes from a study of fluid and nutrient transport inside soft biological tissues, in particular through solid tumors.
- Necessity: To describe the mechanics of the tumor growth one needs to know about fluid flow and nutrient distribution inside the tumor.
- We focus on the mathematical modeling and analysis of the coupled phenomena of fluid flow and solid phase deformation (poroelastohydrodynamics) inside soft biomaterials, such as a tumor.

Biphasic mixture Theory

Following [1, 3, 4, 5]

$$\frac{\partial(\tilde{\rho}_f \varphi_f)}{\partial t} + \nabla \cdot [(\tilde{\rho}_f \varphi_f)\mathbf{V}_f] = \tilde{\rho}_f S_f(x,t) \text{ in } \Omega_T = \Omega \times (0,T) \tag{1}$$

$$\frac{\partial(\tilde{\rho}_s \varphi_s)}{\partial t} + \nabla \cdot [(\tilde{\rho}_s \varphi_s)\mathbf{V}_s] = \tilde{\rho}_s S_s(x,t), \text{ in } \Omega_T = \Omega \times (0,T) \tag{2}$$

For the saturated mixture,

$$\varphi_f + \varphi_s = 1, \tag{3}$$

when $\tilde{\rho}_f$, $\tilde{\rho}_f$ is constant and equal

$$\nabla \cdot (\varphi_f \mathbf{V}_f + \varphi_s \mathbf{V}_s) = S_f(x,t) + S_s(x,t). \tag{4}$$

$\mathbf{V}_{com} = \varphi_f \mathbf{V}_f + \varphi_s \mathbf{V}_s$ composite velocity.

Note: $\Omega \subset \mathbb{R}^d$ and $\Gamma_T = \partial\Omega \times (0,T)$.

If the mixture is closed so that mass exchange occurs only between the constituents taken into consideration, i.e., $S_f(x,t) + S_s(x,t) = 0$, then conservation of mass

$$\nabla \cdot (\varphi_f \mathbf{V}_f + \varphi_s \mathbf{V}_s) = 0. \tag{5}$$

Another case is when there is no generation of new tumor cells and fibrous skeleton during perfusion of solutes i.e., conservation of mass when the mixture is not a closed mass

$$\nabla \cdot (\varphi_f \mathbf{V}_f + \varphi_s \mathbf{V}_s) = S_f. \tag{6}$$

continue...

Momentum balance equation for each of the constituent phases (solid and fluid) in the binary mixture of cellular phase (solid) and extracellular fluid are given by [2, 5]

$$\rho_f \left(\frac{\partial \mathbf{V}_f}{\partial t} + (\mathbf{V}_f \cdot \nabla)\mathbf{V}_f \right) = \nabla \cdot \mathbf{T}_f + \mathbf{\Pi}_f + \mathbf{b}_f, \text{ in } \Omega_T \tag{7}$$

Fluid stress:

$$\mathbf{T}_f = -[\varphi_f P - \lambda_f \nabla \cdot \mathbf{V}_f]\mathbf{I} + \mu_f (\nabla \mathbf{V}_f + (\nabla \mathbf{V}_f)^{tr}), \tag{8}$$

$$\rho_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + (\mathbf{V}_s \cdot \nabla)\mathbf{V}_s \right) = \nabla \cdot \mathbf{T}_s + \mathbf{\Pi}_s + \mathbf{b}_s, \text{ in } \Omega_T \tag{9}$$

Solid stress:

$$\mathbf{T}_s = -[(\varphi_s P) - \chi_s(\varphi_s)(\nabla \cdot \mathbf{U}_s)]\mathbf{I} + \mu_s(\varphi_s)(\nabla \mathbf{U}_s + (\nabla \mathbf{U}_s)^{tr}). \tag{10}$$

where $\mathbf{V}_s = \frac{\partial \mathbf{U}_s}{\partial t}, \quad \frac{\partial \mathbf{V}_f}{\partial t} = \frac{\partial^2 \mathbf{U}_s}{\partial t^2}$ and $\rho_j = \tilde{\rho}_j \varphi_j$ for $j = f, \ s$.

continue...

Interaction force:

$$-\mathbf{\Pi}_s = \mathbf{\Pi}_f = \mathbf{K}(\mathbf{V}_s - \mathbf{V}_f). \tag{11}$$

$$\mathbf{K} = \mu_f \mathbf{k}^{-1}, \quad \chi_s = \nu_p \mathcal{Y}/(1+\nu_p)(1-2\nu_p), \quad \mu_s = \mathcal{Y}/2(1+\nu_p).$$

The drag coefficient/hydraulic resistivity

- The supply of fluids and macromolecules within a tumor is quite heterogeneous owing to the heterogeneous blood vessel distribution. As a consequence, the physiological transport parameters should depend on space.
- Various experimental and theoretical investigations indicate clearly that for the deformable porous medium (or soft biological tissue such as articular cartilage, arterial tissue, and tumor), the permeability also called in this context the drag coefficient/hydraulic resistivity depends on stress, dilatation, volume fractions (porosity), etc, [10, 11].

Modeling Assumptions

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- Nutrient proliferation rate is much faster than the tumor cell growth (static perfused model i.e., φ_s is constant and $\tilde{S}_s = 0$). Further, as $\varphi_f = 1 - \varphi_s$, thus φ_f is also a constant.
- IFV and SPD are slow (i.e. we can neglect inertial terms compared to viscous stress terms).

With these assumptions, biphasic mixture equations reduce to:
Find $(\mathbf{V}_f, \mathbf{U}_s, P)$ such that

$$\rho_f \dot{\mathbf{V}}_f - \nabla \cdot [2\mu_f D(\mathbf{V}_f) + \lambda_f (\nabla \cdot \mathbf{V}_f)\mathbf{I} - \varphi_f P \mathbf{I}] + \frac{\mu_f}{k}\left(\mathbf{V}_f - \dot{\mathbf{U}}_s\right) = \mathbf{b}_f, \qquad \text{in} \quad \Omega_T \tag{12}$$

$$\rho_s \ddot{\mathbf{U}}_s - \nabla \cdot [2\mu_s D(\mathbf{U}_s) + \chi (\nabla \cdot \mathbf{U}_s)\mathbf{I} - \varphi_s P \mathbf{I}] + \frac{\mu_f}{k}\left(\dot{\mathbf{U}}_s - \mathbf{V}_f\right) = \mathbf{b}_s, \qquad \text{in} \quad \Omega_T \tag{13}$$

$$\nabla \cdot \left(\varphi_f \mathbf{V}_f + \varphi_s \dot{\mathbf{U}}_s\right) = S_f, \qquad \text{in} \quad \Omega_T, \tag{14}$$

where $\dot{\mathbf{V}}_f = \frac{\partial \mathbf{V}_f}{\partial t}$, $\mathbf{V}_s = \dot{\mathbf{U}}_s = \frac{\partial \mathbf{U}_s}{\partial t}$, $\ddot{\mathbf{U}}_s = \frac{\partial^2 \mathbf{U}_s}{\partial t^2}$. $D(\mathbf{U}_s) = \frac{1}{2}(\nabla \mathbf{U}_s + (\nabla \mathbf{U}_s)^T)$ and $D(\mathbf{V}_f) = \frac{1}{2}(\nabla \mathbf{V}_f + (\nabla \mathbf{V}_f)^T)$ denote the deformation and rate of deformation tensors, respectively.

Boundary conditions:

$$\mathbf{T}_f \cdot \mathbf{n} = \mathbf{T}_\infty^f, \qquad \mathbf{T}_s \cdot \mathbf{n} = 0, \qquad \text{in} \qquad \Gamma_T, \tag{15}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary $\partial\Omega$. We propose the following initial conditions.
Initial Conditions:

$$\mathbf{V}_f(x,0) = \mathbf{V}_0, \ \mathbf{U}_s(x,0) = \mathbf{U}_0, \ \dot{\mathbf{U}}_s(x,0) = \mathbf{U}_1. \tag{16}$$

Note: Fluid source is assumed to be driven by the average transmural pressure and is given by [5, 6]

$$S_f = -a_0(P - P_F).$$

Dimensionless Governing equations

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Using following transformations

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{R_T}, \quad \hat{P} = \frac{P}{P_F}, \quad \hat{\mathbf{V}}_f = \frac{\mathbf{V}_f}{\frac{R_T P_F}{\mu_f}}, \quad \hat{\mathbf{U}}_s = \frac{\mathbf{U}_s}{\frac{R_T^3 P_F}{\mu_f \nu}}, \quad \hat{t} = \frac{t\mu_f}{R_T^2 \rho_f},$$

we get the following dimensionless form of the governing equations

$$\frac{\partial \mathbf{V}_f}{\partial t} - \nabla \cdot (2D(\mathbf{V}_f) + \lambda(\nabla \cdot \mathbf{V}_f)\mathbf{I} - \varphi_f P\mathbf{I}) + \frac{1}{Da}(\mathbf{V}_f - \dot{\mathbf{U}}_s) = \mathbf{b}_f, \tag{17}$$

$$\rho_r \frac{\partial \mathbf{V}_s}{\partial t} - \nabla \cdot (2\alpha_1 D(\mathbf{U}_s) + \alpha_2(\nabla \cdot \mathbf{U}_s)\mathbf{I} - \varphi_s P\mathbf{I}) + \frac{1}{Da}(\dot{\mathbf{U}}_s - \mathbf{V}_f) = \mathbf{b}_s, \tag{18}$$

$$\nabla \cdot (\varphi_f \mathbf{V}_f + \varphi_s \dot{\mathbf{U}}_s) + a_0 P = a_0, \tag{19}$$

Boundary conditions:

$$(2D(\mathbf{V}_f) + \lambda(\nabla \cdot \mathbf{V}_f)\mathbf{I} - \varphi_f P\mathbf{I}) \cdot \mathbf{n} = \mathbf{T}_\infty^f, \qquad \text{on} \qquad \Gamma_T \tag{20}$$

and

$$(2\alpha_1 D(\mathbf{U}_s) + \alpha_2(\nabla \cdot \mathbf{U}_s)\mathbf{I} - \varphi_s P\mathbf{I}) \cdot \mathbf{n} = 0, \quad \text{on} \quad \Gamma_T. \tag{21}$$

Initial conditions:

$$\mathbf{V}_f(x,0) = \mathbf{V}_0,\ \mathbf{U}_s(x,0) = \mathbf{U}_0,\ \dot{\mathbf{U}}_s(x,0) = \mathbf{U}_1. \tag{22}$$

Non-dimensionalization parameters

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- $Da = k/R_{\mathrm{T}}^2$: is the Darcy number (Permeability parameter).
- $\lambda = \frac{\lambda_f}{\mu_f}$: is the ratio of the two viscosity coefficients.
- $L_r A_r = L_{\mathsf{p_L}} A_{\mathsf{L}} / L_{\mathsf{p}} A$: is the ratio of the hydraulic conductivities of blood and lymph vessels.
- $\alpha_{\mathrm{T}}^2 = L_{\mathsf{p}} \mu_f \, (A/V)$: is the strength of solute source.
- $\varrho_T = \mathcal{Y} R_{\mathrm{T}}^2 \rho_f / \mu_f^2$: represents response of solid phase (cellular phase + extracellular matrix or ECM) towards viscous drag force due to interstitial fluid movement.
- $\rho_r = \frac{\rho_s}{\rho_f}$: density ratio.
- $|\Omega|$: volume in $3d$ and area in $2d$ of the domain Ω.
- $\alpha_1 = \frac{\varrho_{\mathrm{T}}}{2(1+\nu_p)}$, $\alpha_2 = \frac{\nu_p \varrho_{\mathrm{T}}}{(1+\nu_p)(1-2\nu_p)}$, and $a_0 = \alpha_{\mathrm{T}}^2 (1 + L_r A_r)$.

We define a function of 't'

$$[G_1(t)]^2 = ||\mathbf{b}_f(t)||^2_{0,\Omega} + \frac{c_k c_t}{3}||\mathbf{T}^f_\infty(t)||^2_{0,\partial\Omega} + ||\mathbf{b}_s(t)||^2_{0,\Omega} + a_0|\Omega| \tag{23}$$

for a.e. t in $(0, T)$, and the following constants

$$(G_2)^2 = 4\alpha_1||\mathbf{U}_0||^2_{0,\Omega} + ||\mathbf{V}_0||^2_{0,\Omega} + \rho_r||\mathbf{U}_1||^2_{0,\Omega} + 2\alpha_1||D(\mathbf{U}_0)||^2_{0,\Omega} + \alpha_2||\nabla\cdot\mathbf{U}_0||^2_{0,\Omega}, \tag{24}$$

$$(G_3)^2 = \frac{1}{\alpha_3}[\alpha_3 + 2T(3 + 2\alpha_1 T)]\left[||G_1||^2_{L^2(0,T)} + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}}. \tag{25}$$

Here

$$\alpha_3 = \min\left\{1, \rho_r, \frac{2\alpha_1}{c_k}\right\}, \quad \alpha_4 = \max\left\{5, 1 + 4\alpha_1 T, \frac{2\alpha_1}{c_k}\right\}, \tag{26}$$

where c_k, c_t are constants that appear in the Korn and trace inequalities, respectively.

Weak Formulation

A triplet $(\mathbf{V}_f, \mathbf{U}_s, P) \in L^2(0,T;H^1(\Omega)^d) \times L^2(0,T;H^1(\Omega)^d) \times L^2(0,T;L^2(\Omega))$, is called a weak solution of the system of equations (17)-(19) with respect to initial and boundary conditions (20)-(22) if $\dot{\mathbf{V}}_f \in L^2(0,T;(H^1(\Omega)^d)^*)$, and $\dot{\mathbf{U}}_s \in L^2(0,T;L^2(\Omega)^d)$, $\ddot{\mathbf{U}}_s \in L^2(0,T;(H^1(\Omega)^d)^*)$ such that for every $(\mathbf{W},\mathbf{W},q) \in H^1(\Omega)^d \times H^1(\Omega)^d \times L^2(\Omega)$ and for a.e. $t \in (0,T)$,

$$
(A_w)\begin{cases}
\langle \dot{\mathbf{V}}_f(t), \mathbf{W}\rangle_* + 2(D(\mathbf{V}_f(t)), \nabla \mathbf{W})_\Omega + \lambda(\nabla\cdot\mathbf{V}_f(t), \nabla\cdot\mathbf{W})_\Omega \\
-\varphi_f(P(t), \nabla\cdot\mathbf{W})_\Omega + \frac{1}{Da}(\mathbf{V}_f(t),\mathbf{W})_\Omega - \frac{1}{Da}(\dot{\mathbf{U}}_s(t),\mathbf{W})_\Omega \\
= (\mathbf{b}_f(t),\mathbf{W})_\Omega + (\mathbf{T}^f_\infty(t),\mathbf{W})_{\partial\Omega}, \\
\rho_r\langle \ddot{\mathbf{U}}_s(t),\mathbf{W}\rangle_* + 2\alpha_1(D(\mathbf{U}_s(t)),\nabla\mathbf{W})_\Omega + \alpha_2(\nabla\cdot\mathbf{U}_s(t),\nabla\cdot\mathbf{W})_\Omega \\
-\varphi_s(P(t),\nabla\cdot\mathbf{W})_\Omega + \frac{1}{Da}(\dot{\mathbf{U}}_s(t),\mathbf{W})_\Omega - \frac{1}{Da}(\mathbf{V}_f(t),\mathbf{W})_\Omega = (\mathbf{b}_s(t),\mathbf{W})_\Omega, \\
\varphi_f(\nabla\cdot\mathbf{V}_f(t),q)_\Omega + \varphi_s(\nabla\cdot\dot{\mathbf{U}}_s(t),q)_\Omega + a_0(P(t),q)_\Omega = (a_0,q)_\Omega, \\
(\mathbf{V}_f(0),\mathbf{W})_\Omega = (\mathbf{V}_0,\mathbf{W})_\Omega,\ (\mathbf{U}_s(0),\mathbf{W})_\Omega = (\mathbf{U}_0,\mathbf{W})_\Omega, \\
(\dot{\mathbf{U}}_s(0),\mathbf{W})_\Omega = (\mathbf{U}_1,\mathbf{W})_\Omega.
\end{cases}
$$

The spaces $H^1(\Omega)^d$ and $L^2(\Omega)$ are separable Hilbert, thus one can find a basis consisting of smooth functions $\{\mathbf{W}_i, \mathbf{W}_i, r_i\}$ of $\mathbf{Y} = H^1(\Omega)^d \times H^1(\Omega)^d \times L^2(\Omega)$. Define the finite dimensional subspaces of $\mathbf{Y}$ as $\mathbf{Y}_m = \text{span}\{(\mathbf{W}_i, \mathbf{W}_i, r_i),\ i = 1, \cdots, m\}$. Denote π_m ($\boldsymbol{\pi}_m$) the projection of $L^2(\Omega)$ ($H^1(\Omega)^d$) onto $M_m = \text{span}\{r_i,\ i = 1, \cdots, m\}$ ($\mathbf{X}_m = \text{span}\{\mathbf{W}_i,\ i = 1, \cdots, m\}$) then a Galerkin approximation to problem (A_w) is the finite dimensional problem (A_m) defined as:

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Find $(\mathbf{V}_f^m, \mathbf{U}_s^m, P^m) \in L^2(0,T;\mathbf{Y}_m)$, with $\mathbf{V}_f^m \in H^1(0,T;L^2(\Omega)^d)$ and $\mathbf{U}_s^m \in H^2(0,T;L^2(\Omega)^d)$ such that

$$(A_m)\begin{cases} (\dot{\mathbf{V}}_f^m(t), \mathbf{W}_i)_\Omega + 2(D(\mathbf{V}_f^m(t)), \nabla \mathbf{W}_i)_\Omega + \lambda(\nabla\cdot\mathbf{V}_f^m(t), \nabla\cdot\mathbf{W}_i)_\Omega \\ -\varphi_f(P^m(t), \nabla\cdot\mathbf{W}_i)_\Omega + \frac{1}{Da}(\mathbf{V}_f^m(t), \mathbf{W}_i)_\Omega - \frac{1}{Da}(\dot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega \\ = (\mathbf{b}_f(t), \mathbf{W}_i)_\Omega + (\mathbf{T}_\infty^f(t), \mathbf{W}_i)_{\partial\Omega}, \\ \rho_r(\ddot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega + 2\alpha_1(D(\mathbf{U}_s^m(t)), \nabla\mathbf{W}_i)_\Omega + \alpha_2(\nabla\cdot\mathbf{U}_s^m(t), \nabla\cdot\mathbf{W}_i)_\Omega \\ -\varphi_s(P^m(t), \nabla\cdot\mathbf{W}_i)_\Omega + \frac{1}{Da}(\dot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega - \frac{1}{Da}(\mathbf{V}_f^m(t), \mathbf{W}_i)_\Omega = (\mathbf{b}_s(t), \mathbf{W}_i)_\Omega, \\ \varphi_f(\nabla\cdot\mathbf{V}_f^m(t), r_i)_\Omega + \varphi_s(\nabla\cdot\dot{\mathbf{U}}_s^m(t), r_i)_\Omega + a_0(P^m(t), r_i)_\Omega = (a_0, r_i)_\Omega, \\ (\mathbf{V}_f^m(0), \mathbf{W}_i)_\Omega = (\boldsymbol{\pi}_m\mathbf{V}_0, \mathbf{W}_i)_\Omega,\ (\mathbf{U}_s^m(0), \mathbf{W}_i)_\Omega = (\boldsymbol{\pi}_m\mathbf{U}_0, \mathbf{W}_i)_\Omega, \\ (\dot{\mathbf{U}}_s^m(0), \mathbf{W}_i)_\Omega = (\boldsymbol{\pi}_m\mathbf{U}_1, \mathbf{W}_i)_\Omega. \end{cases}$$

We now show that the problem (A_m) has a unique solution.

Galerkin Procedure

We look for an approximation of the solution $(\mathbf{V}_f^m, \mathbf{U}_s^m, P^m)$ in the following form

$$\mathbf{V}_f^m(x,t) = \sum_{j=1}^{m} a_j^m(t)\mathbf{W}_j(x),\;\; \mathbf{U}_s^m(x,t) = \sum_{j=1}^{m} b_j^m(t)\mathbf{W}_j(x),\;\; P^m(x,t) = \sum_{j=1}^{m} c_j^m(t) r_j(x),$$

where the coefficients a_j^m, b_j^m, and c_j^m are to be determined. With this form of approximate solution, problem (A_m) leads to a system of ordinary differential equations (ODEs) for the coefficients a_j^m, b_j^m, and c_j^m as

$$\mathbf{A}_1\frac{d\mathbf{a}}{dt} - \frac{1}{Da}\mathbf{A}_1\frac{d\mathbf{b}}{dt} + \left(2\mathbf{A}_2 + \lambda\mathbf{A}_3 + \frac{1}{Da}\mathbf{A}_1\right)\mathbf{a} - \varphi_f\mathbf{A}_4\mathbf{c} = \mathbf{F}_1, \tag{27}$$

$$\rho_r\mathbf{A}_1\frac{d^2\mathbf{b}}{dt^2} + \frac{1}{Da}\mathbf{A}_1\frac{d\mathbf{b}}{dt} + (2\alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_3)\mathbf{b} - \frac{1}{Da}\mathbf{A}_1\mathbf{a} - \varphi_s\mathbf{A}_4\mathbf{c} = \mathbf{F}_2, \tag{28}$$

$$\varphi_s\mathbf{A}_1\frac{d\mathbf{b}}{dt} + \varphi_f\mathbf{A}_1\mathbf{a} + \mathbf{A}_5\mathbf{c} = \mathbf{F}_3, \tag{29}$$

$$\mathbf{A}_1\mathbf{a}(0) = \mathbf{\Lambda}_1,\;\; \mathbf{A}_1\mathbf{b}(0) = \mathbf{\Lambda}_2,\;\; \mathbf{A}_1\frac{d\mathbf{b}(0)}{dt} = \mathbf{\Lambda}_3, \tag{30}$$

Coefficients

where the coefficient matrices are

$$\mathbf{A}_1 = ((\mathbf{W}_j, \mathbf{W}_i)_\Omega)_{1\le i,j\le m},\ \mathbf{A}_2 = ((D(\mathbf{W}_j), D(\mathbf{W}_i))_\Omega)_{1\le i,j\le m},$$

$$\mathbf{A}_3 = ((\nabla\cdot\mathbf{W}_j, \nabla\cdot\mathbf{W}_i)_\Omega)_{1\le i,j\le m},\ \mathbf{A}_4 = ((r_j, \nabla\cdot\mathbf{W}_i)_\Omega)_{1\le i,j\le m},\ \mathbf{A}_5 = a_0((r_j, r_i)_\Omega)_{1\le i,j\le m}$$

and the unknown coefficients are given by:

$$\mathbf{a} = \begin{pmatrix} a_1^m(t) \\ \vdots \\ a_m^m(t) \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} b_1^m(t) \\ \vdots \\ b_m^m(t) \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} c_1^m(t) \\ \vdots \\ c_m^m(t) \end{pmatrix},$$

together with the functions $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ on the right hand side are given by:

$$\mathbf{F}_1 = \begin{pmatrix} (\mathbf{b}_f(t), \mathbf{W}_1)_\Omega + (\mathbf{T}_\infty^f(t), \mathbf{W}_1)_{\partial\Omega} \\ \vdots \\ (\mathbf{b}_f(t), \mathbf{W}_m)_\Omega + (\mathbf{T}_\infty^f(t), \mathbf{W}_m)_{\partial\Omega} \end{pmatrix},\ \mathbf{F}_2 = \begin{pmatrix} (\mathbf{b}_s(t), \mathbf{W}_1)_\Omega \\ \vdots \\ (\mathbf{b}_s(t), \mathbf{W}_m)_\Omega \end{pmatrix},$$

$$\mathbf{F}_3 = \begin{pmatrix} (a_0, r_1)_\Omega \\ \vdots \\ (a_0, r_m)_\Omega \end{pmatrix}$$

Coefficients

$$\boldsymbol{\Lambda}_1 = \begin{pmatrix} (\boldsymbol{\pi}_m \mathbf{V}_0, \mathbf{W}_1)_\Omega \\ \vdots \\ (\boldsymbol{\pi}_m \mathbf{V}_0, \mathbf{W}_m)_\Omega \end{pmatrix}, \ \boldsymbol{\Lambda}_2 = \begin{pmatrix} (\boldsymbol{\pi}_m \mathbf{U}_0, \mathbf{W}_1)_\Omega \\ \vdots \\ (\boldsymbol{\pi}_m \mathbf{U}_0, \mathbf{W}_m)_\Omega \end{pmatrix}, \ \boldsymbol{\Lambda}_3 = \begin{pmatrix} (\boldsymbol{\pi}_m \mathbf{U}_1, \mathbf{W}_1)_\Omega \\ \vdots \\ (\boldsymbol{\pi}_m \mathbf{U}_1, \mathbf{W}_m)_\Omega \end{pmatrix}.$$

We introduce the following vector

$$\mathbf{e} = \dot{\mathbf{b}}, \tag{31}$$

into Eq. (29) to obtain

$$\mathbf{c} = \mathbf{A}_5^{-1}(\mathbf{F}_3 - \varphi_s \mathbf{A}_1 \mathbf{e} - \varphi_f \mathbf{A}_1 \mathbf{a}). \tag{32}$$

Using equation (31) and (32) into the system of equations (27)-(30), we get the following autonomous system of first order ODE in $\mathbf{B}$:

$$\dot{\mathbf{B}} = \mathbb{M}^{-1}(\mathbb{N}\mathbf{B} + \mathbb{F}) \tag{33}$$
$$\mathbf{B}(0) = \mathbf{B}_0 \ \text{(given)}$$

where

$$\mathbb{M} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_r \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{pmatrix},$$

$$\mathbb{N} = \begin{pmatrix} \mathbf{A}_0 & \left(\frac{1}{Da}\mathbf{A}_1 + \varphi_s\varphi_f \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{A}_1\right) & \mathbf{0} \\ \left(\frac{1}{Da}\mathbf{A}_1 - \varphi_s\varphi_f \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{A}_1\right) & -\left(\frac{1}{Da}\mathbf{A}_1 + \varphi_s^2 \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{A}_1\right) & -\left(2\beta_1\mathbf{A}_2 + \beta_2\mathbf{A}_3\right) \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{A}_0 = -\left(2\mathbf{A}_2 + \lambda\mathbf{A}_3 + \frac{1}{Da}\mathbf{A}_1 + \varphi_f^2 \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{A}_1\right)$, and

$$\mathbf{B} = \begin{pmatrix} \mathbf{a} \\ \mathbf{e} \\ \mathbf{b} \end{pmatrix}, \quad \mathbb{F} \begin{pmatrix} \mathbf{F}_1 + \varphi_f \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{F}_3 \\ \mathbf{F}_2 + \varphi_s \mathbf{A}_4\mathbf{A}_5^{-1}\mathbf{F}_3 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} \mathbf{A}_1^{-1}\mathbf{\Lambda}_1 \\ \mathbf{A}_1^{-1}\mathbf{\Lambda}_2 \\ \mathbf{A}_1^{-1}\mathbf{\Lambda}_3 \end{pmatrix}.$$

- The right hand side of the system of ODEs (33) depend continuously (even Lipschitz) on the variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{e}$ and on time,
- Hence from the theory of ordinary differential equations [9] the system of ODE (33) has a unique solution $(\mathbf{a}, \mathbf{b}, \mathbf{e}) \in H^1(0,T;\mathbb{R}^m) \times H^1(0,T;\mathbb{R}^m) \times H^1(0,T;\mathbb{R}^m)$, and $\mathbf{c} \in H^1(0,T;\mathbb{R}^m)$.
- The finite dimensional problem (A_m) has a unique solution $(\mathbf{V}_f^m, \mathbf{U}_s^m, P^m) \in L^2(0,T;\mathbf{Y}_m)$, with $\mathbf{V}_f^m \in H^1(0,T;L^2(\Omega)^d)$ and $\mathbf{U}_s^m \in H^2(0,T;L^2(\Omega)^d)$.
- Next, we find a priori estimates on the finite dimensional solution $(\mathbf{V}_f^m, \mathbf{U}_s^m, P^m)$, which are known as energy estimates.

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To find the energy estimates, we multiply a_i^m, $\dot{b}_i^m$, and c_i^m respectively, with each of the equations of the system (A_m) and take the summation from $i = 1 \dots m$. Further, adding all the equations, we get

$$
(E_1) \begin{cases}
\frac{d}{dt}||\mathbf{V}_f^m(t)||_{0,\Omega}^2 + 4||D(\mathbf{V}_f^m(t))||_{0,\Omega}^2 + 2\lambda||\nabla\cdot\mathbf{V}_f^m(t)||_{0,\Omega}^2 + \frac{2}{Da}||\mathbf{V}_f^m(t)||_{0,\Omega}^2 \\
+\rho_r\frac{d}{dt}||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + 2\alpha_1\frac{d}{dt}||D(\mathbf{U}_s^m(t))||_{0,\Omega}^2 + \alpha_2\frac{d}{dt}||\nabla\cdot\mathbf{U}_s^m(t)||_{0,\Omega}^2 \\
+\frac{2}{Da}||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + 2a_0||P^m(t)||_{0,\Omega}^2 = 2(\mathbf{b}_f(t), \mathbf{V}_f^m(t))_\Omega + 2(\mathbf{T}_\infty^f(t), \mathbf{V}_f^m(t))_{\partial\Omega} \\
+2(\mathbf{b}_s(t), \dot{\mathbf{U}}_s^m(t))_\Omega + 2(a_0, P^m(t))_\Omega + \frac{4}{Da}(\dot{\mathbf{U}}_s^m(t), \mathbf{V}_f^m(t))_\Omega.
\end{cases}
$$

Using Cauchy-Schwarz, Young's, Korn's, and trace inequalities, in (E_1), and integrating over $(0,t)$ gives

$$
(E_2) \begin{cases}
||\mathbf{V}_f^m(t)||_{0,\Omega}^2 + \frac{1}{c_k}\int_0^t||\mathbf{V}_f^m(\xi)||_{1,\Omega}^2\,d\xi + 2\lambda\int_0^t||\nabla\cdot\mathbf{V}_f^m(\xi)||_{0,\Omega}^2\,d\xi + \rho_r||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \\
+\frac{2\beta}{c_k}||\mathbf{U}_s^m(t)||_{1,\Omega}^2 + \alpha_2||\nabla\cdot\mathbf{U}_s^m(t)||_{0,\Omega}^2 + a_0\int_0^t||P^m(\xi)||_{0,\Omega}^2\,d\xi \le \int_0^t[G_1(\xi)]^2\,d\xi \\
+(G_2)^2 + 5\int_0^t||\mathbf{V}_f^m(\xi)||_{0,\Omega}^2\,d\xi + (1+4\alpha_1 T)\int_0^t||\dot{\mathbf{U}}_s^m(\xi)||_{0,\Omega}^2\,d\xi
\end{cases}
$$

Further,

$$(E_3)\begin{cases} ||\mathbf{V}_f^m(t)||_{0,\Omega}^2 + ||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + ||\mathbf{U}_s^m(t)||_{1,\Omega}^2 \leq \frac{1}{\alpha_3}\left[\int_0^t [G_1(\xi)]^2\, d\xi + (G_2)^2\right] \\ +\frac{\alpha_4}{\alpha_3}\left(\int_0^t ||\mathbf{V}_f^m(\xi)||_{0,\Omega}^2\, d\xi + \int_0^t ||\dot{\mathbf{U}}_s^m(\xi)||_{0,\Omega}^2\, d\xi + \int_0^t ||\mathbf{U}_s^m(\xi)||_{1,\Omega}^2\, d\xi\right) \end{cases}$$

Define

$$\Psi(t) = ||\mathbf{V}_f^m(t)||_{0,\Omega}^2 + ||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + ||\mathbf{U}_s^m(t)||_{1,\Omega}^2.$$

(E_3) implies

$$\Psi(t) \leq \frac{1}{\alpha_3}\left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] + \frac{\alpha_4}{\alpha_3}\int_0^t \Psi(s)\, ds,$$

and Gronwall's inequality gives,

$$\Psi(t) \leq \frac{1}{\alpha_3}\left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}},$$

i.e.,

$$(E_4)\left\{ ||\mathbf{V}_f^m(t)||_{0,\Omega}^2 + ||\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + ||\mathbf{U}_s^m(t)||_{1,\Omega}^2 \leq \frac{1}{\alpha_3}\left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}}. \right.$$

(E_4) implies $\mathbf{V}_f^m$, $\dot{\mathbf{U}}_s^m$, and $\mathbf{U}_s^m$ are bounded sequences in $L^\infty(0,T,L^2(\Omega)^d)$, $L^\infty(0,T,L^2(\Omega)^d)$, and $L^\infty(0,T,H^1(\Omega)^d))$, respectively.

Similarly, using (E_4), we get

$$\int_0^t ||\mathbf{V}_f^m(\xi)||_{1,\Omega}^2 \, d\xi \leq \frac{c_k}{\alpha_3}[\alpha_3 + 2T(3 + 2\alpha_1 T)] \left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}}. \quad (34)$$

Similarly, with the help of (E_2), and (E_4), we have

$$\int_0^t ||P^m(\xi)||_{0,\Omega}^2 \, d\xi \leq \frac{1}{\alpha_3 a_0}[\alpha_3 + 2T(3 + 2\alpha_1 T)] \left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}}. \quad (35)$$

Eqs. (34) and (35) indicate that $\mathbf{V}_f^m$, P^m are bounded sequences in $L^2(0, T, H^1(\Omega)^d)$ and $L^2(0, T, L^2(\Omega))$, respectively.

Similarly, we have shown that $\dot{\mathbf{V}}_f^m$, and $\ddot{\mathbf{U}}_s^m$ are bounded sequences in $L^2(0,T,(H^1(\Omega)^d)^*)$. They satisfy

$$\int_0^t ||\dot{\mathbf{V}}_f^m(\xi)||^2_{(H^1(\Omega)^d)^*}\, d\xi \leq 2\Bigg[\left(4c_k + \frac{\lambda}{2} + \frac{\varphi_f^2}{a_0} + \frac{1+c_k}{(Da)^2}\right)(G_3)^2 + \int_0^T ||\mathbf{b}_f(\xi)||^2_{0,\Omega}\, d\xi + c_t^2 \int_0^T ||\mathbf{T}_\infty^f(\xi)||^2_{0,\partial\Omega}\, d\xi\Bigg], \tag{36}$$

and

$$\int_0^t ||\ddot{\mathbf{U}}_s^m(\xi)||^2_{(H^1(\Omega)^d)^*}\, d\xi \leq \frac{2}{\rho_r^2}\left[\left(4\alpha_1^2 + \alpha_2^2 + \frac{\varphi_s^2}{a_0} + \frac{1+c_k}{(Da)^2}\right)(G_3)^2 + \int_0^T ||\mathbf{b}_s(\xi)||^2_{0,\Omega}\, d\xi\right] \tag{37}$$

Weak convergence

The above energy estimates gives the following weak convergence results up to a subsequence (as $m \to \infty$)

$$\int_0^T (\mathbf{V}_f^m(t), \mathbf{W}(t))_{1,\Omega}\, dt \to \int_0^T (\mathbf{V}_f(t), \mathbf{W}(t))_{1,\Omega}\, dt \;\forall\; \mathbf{W} \in L^2(0,T; H^1(\Omega)^d),$$

$$\int_0^T \langle \dot{\mathbf{V}}_f^m(t), \mathbf{W}(t) \rangle_*\, dt \to \int_0^T \langle \dot{\mathbf{V}}_f(t), \mathbf{W}(t) \rangle_*\, dt, \;\forall\; \mathbf{W} \in L^2(0,T; H^1(\Omega)^d),$$

where $(\cdot,\cdot)_{1,\Omega}$ denotes the inner product in the space $H^1(\Omega)^d$.

$$\int_0^T (\mathbf{U}_s^m(t), \mathbf{W}(t))_{1,\Omega}\, dt \to \int_0^T (\mathbf{U}_s(t), \mathbf{W}(t))_{1,\Omega}\, dt \;\forall\; \mathbf{W} \in L^2(0,T; H^1(\Omega)^d),$$

$$\int_0^T (\dot{\mathbf{U}}_s^m(t), \mathbf{W}(t))_{\Omega}\, dt \to \int_0^T (\dot{\mathbf{U}}_s(t), \mathbf{W}(t))_{\Omega}\, dt, \;\forall\; \mathbf{W} \in L^2(0,T; L^2(\Omega)^d),$$

$$\int_0^T \langle \ddot{\mathbf{U}}_s^m(t), \mathbf{W}(t) \rangle_*\, dt \to \int_0^T \langle \ddot{\mathbf{U}}_s(t), \mathbf{W}(t) \rangle_*\, dt, \;\forall\; \mathbf{W} \in L^2(0,T; H^1(\Omega)^d),$$

$$\int_0^T (P^m(t), q(t))_{\Omega}\, dt \to \int_0^T (P(t), q(t))_{\Omega}\, dt, \;\forall\; q \in L^2(0,T; L^2(\Omega)).$$

Passing to the limit

Following some standard arguments of weak convergence, we pass to the limits in weak formulation (A_m) as $m \to \infty$ and recover original weak formulation (A_w).

Lemma 4.1

Assume that the given data $\mathbf{b}_f,\ \mathbf{b}_s \in H^1(0,T,L^2(\Omega)^d)$ *and* $\mathbf{V}_0 = 0,\ \mathbf{U}_0 = 0,\ \mathbf{U}_1 = 0$ *and* $\mathbf{T}^f_\infty = 0$. *Then*

$$\dot{\mathbf{U}}_s \in L^\infty(0,T;H^1(\Omega)^d) \tag{38}$$

for any weak solution $\mathbf{U}_s$ *of the system of equations (17)-(19) and*

$$||\dot{\mathbf{U}}_s(t)||^2_{1,\Omega} \leq \frac{c_k}{2\alpha_1}\left[(G_5)^2 + ||G_4||^2_{L^2(0,T)}\right]\left(1 + \frac{\beta^* T}{\alpha^*} e^{\frac{\beta^* T}{\alpha^*}}\right), \tag{39}$$

Regularity result proof

Proof: In order to establish the proof of Lemma - 4.1, we follow a method shown in [8]. Moreover, we need following result [7]: For any $\eta \in H^1(0,T,L^2(\Omega))$, we have

$$\eta(t) = \eta(0) + \int_0^t \eta_t(s)\, ds \tag{40}$$

it implies

$$||\eta(t)||^2_{0,\Omega} \leq 2||\eta(0)||^2_{0,\Omega} + 2T \int_0^t ||\eta_t(s)||^2_{0,\Omega}\, ds. \tag{41}$$

Differentiating finite dimensional weak formulation (A_m) with respect to 't', we get

$$(E_5) \begin{cases} (\ddot{\mathbf{V}}_f^m(t), \mathbf{W}_i)_\Omega + 2(D(\dot{\mathbf{V}}_f^m(t)), \nabla \mathbf{W}_i)_\Omega + \lambda(\nabla \cdot \dot{\mathbf{V}}_f^m(t), \nabla \cdot \mathbf{W}_i)_\Omega \\ -\varphi_f(\dot{P}^m(t), \nabla \cdot \mathbf{W}_i)_\Omega + \frac{1}{Da}(\dot{\mathbf{V}}_f^m(t), \mathbf{W}_i)_\Omega - \frac{1}{Da}(\ddot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega \\ = (\dot{\mathbf{b}}_f(t), \mathbf{W}_i)_\Omega, \\ \rho_r(\dddot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega + 2\alpha_1(D(\dot{\mathbf{U}}_s^m(t)), \nabla \mathbf{W}_i)_\Omega + \alpha_2(\nabla \cdot \dot{\mathbf{U}}_s^m(t), \nabla \cdot \mathbf{W}_i)_\Omega \\ -\varphi_s(\dot{P}^m(t), \nabla \cdot \mathbf{W}_i)_\Omega + \frac{1}{Da}(\ddot{\mathbf{U}}_s^m(t), \mathbf{W}_i)_\Omega - \frac{1}{Da}(\dot{\mathbf{V}}_f^m(t), \mathbf{W}_i)_\Omega = (\dot{\mathbf{b}}_s(t), \mathbf{W}_i)_\Omega, \\ \varphi_f(\nabla \cdot \dot{\mathbf{V}}_f^m(t), r_i)_\Omega + \varphi_s(\nabla \cdot \ddot{\mathbf{U}}_s^m(t), r_i)_\Omega + a_0(\dot{P}^m(t), r_i)_\Omega = 0. \end{cases}$$

Multiply $\dot{a}_i^m(t)$, $\ddot{b}_i^m(t)$, and $\dot{c}_i^m(t)$ respectively, with each of the equations of the system (E_5) and take the summation from $i = 1 \ldots m$.

$$(E_6)\begin{cases} \frac{d}{dt}||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + 4||D(\dot{\mathbf{V}}_f^m(t))||_{0,\Omega}^2 + 2\lambda||\nabla\cdot\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + \frac{2}{Da}||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 \\ +\rho_r\frac{d}{dt}||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + 2\alpha_1\frac{d}{dt}||D(\dot{\mathbf{U}}_s^m(t))||_{0,\Omega}^2 + \alpha_2\frac{d}{dt}||\nabla\cdot\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \\ +\frac{2}{Da}||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + 2a_0||\dot{P}(t)||_{0,\Omega}^2 = 2(\dot{\mathbf{b}}_f(t),\dot{\mathbf{V}}_f^m(t))_\Omega + 2(\dot{\mathbf{b}}_s(t),\ddot{\mathbf{U}}_s^m(t))_\Omega \\ +\frac{4}{Da}(\ddot{\mathbf{U}}_s^m(t),\dot{\mathbf{V}}_f^m(t))_\Omega. \end{cases}$$

Cauchy-Schwarz and Young's inequalities help to get

$$(E_7)\begin{cases} \frac{d}{dt}||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + 4||D(\dot{\mathbf{V}}_f^m(t))||_{0,\Omega}^2 + 2\lambda||\nabla\cdot\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 \\ +\rho_r\frac{d}{dt}||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 + 2\alpha_1\frac{d}{dt}||D(\dot{\mathbf{U}}_s^m(t))||_{0,\Omega}^2 + \alpha_2\frac{d}{dt}||\nabla\cdot\dot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \\ +2a_0||\dot{P}(t)||_{0,\Omega}^2 \leq ||\dot{\mathbf{b}}_f(t)||_{0,\Omega}^2 + ||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + ||\dot{\mathbf{b}}_s(t)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \end{cases}$$

continue...

adding both sides $4||\dot{\mathbf{V}}_f^m(t)||^2_{0,\Omega}$ and $2\alpha_1||\dot{\mathbf{U}}_s^m(t)||^2_{0,\Omega}$ and using (41) and Korn's inequality. Further, by integrating over (0,t), we obtain (using zero initial conditions)

$$(E_8)\begin{cases} ||\dot{\mathbf{V}}_f^m(t)||^2_{0,\Omega} + \frac{1}{c_k}\int_0^t ||\dot{\mathbf{V}}_f^m(\xi)||^2_{1,\Omega}\,d\xi + 2\lambda\int_0^t ||\nabla\cdot\dot{\mathbf{V}}_f^m(\xi)||^2_{0,\Omega}\,d\xi \\ +\rho_r||\ddot{\mathbf{U}}_s^m(t)||^2_{0,\Omega} + \frac{2\alpha_1}{c_k}||\dot{\mathbf{U}}_s^m(t)||^2_{1,\Omega} + \alpha_2||\nabla\cdot\dot{\mathbf{U}}_s^m(t)||^2_{0,\Omega} + 2a_0\int_0^t ||\dot{P}(\xi)||^2_{0,\Omega}\,d\xi \\ \leq ||\dot{\mathbf{V}}_f^m(0)||^2_{0,\Omega} + \rho_r||\ddot{\mathbf{U}}_s^m(0)||^2_{0,\Omega} + ||G_4||^2_{L^2(0,T)} + (1+4\alpha_1 T)\int_0^t ||\ddot{\mathbf{U}}_s^m(\xi)||^2_{0,\Omega}\,d\xi \\ +5\int_0^t ||\dot{\mathbf{V}}_f^m(\xi)||^2_{0,\Omega}\,d\xi, \end{cases}$$

where

$$[G_4(t)]^2 = ||\dot{\mathbf{b}}_f(t)||^2_{0,\Omega} + ||\dot{\mathbf{b}}_s(t)||^2_{0,\Omega}$$

for a.e. 't' in (0,T). In order to apply Gronwall's inequality in (E_8), we need to find the bound on $||\dot{\mathbf{V}}_f^m(0)||^2_{0,\Omega}$ and $||\ddot{\mathbf{U}}_s^m(0)||^2_{0,\Omega}$. Multiply by $\dot{\mathbf{a}}_i^m(0)$, $\ddot{\mathbf{b}}_i^m(0)$, and $c_i^m(0)$ to finite dimensional formulation (A_m) again and sum over $i = 1,\ldots,m$ and $t = 0$, we obtain (using zero initial conditions)

$$\begin{aligned}(\dot{\mathbf{V}}_f^m(0),\dot{\mathbf{V}}_f^m(0))_\Omega + \rho_r(\ddot{\mathbf{U}}_s^m(0),\ddot{\mathbf{U}}_s^m(0))_\Omega + a_0(P^m(0),P^m(0))_\Omega \\ = (\mathbf{b}_f(0),\dot{\mathbf{V}}_f^m(0))_\Omega + (\mathbf{b}_s(0),\ddot{\mathbf{U}}_s^m(0))_\Omega + (a_0,P^m(0))_\Omega\end{aligned} \quad (42)$$

continue...

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$$||\dot{\mathbf{V}}_f^m(0)||_{0,\Omega}^2 + \rho_r||\ddot{\mathbf{U}}_s^m(0)||_{0,\Omega}^2 + a_0||P^m(0)||_{0,\Omega}^2$$
$$= (\mathbf{b}_f(0), \dot{\mathbf{V}}_f^m(0))_\Omega + (\mathbf{b}_s(0), \ddot{\mathbf{U}}_s^m(0))_\Omega + (a_0, P^m(0))_\Omega.$$

Using Cauchy-Schwarz inequality, we get

$$\alpha\left(||\dot{\mathbf{V}}_f^m(0)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(0)||_{0,\Omega}^2 + ||P^m(0)||_{0,\Omega}^2\right)$$
$$\leq \left(||\mathbf{b}_f(0)||_{0,\Omega}^2 + ||\mathbf{b}_s(0)||_{0,\Omega}^2 + a_0^2|\Omega|\right)^{1/2}\left(||\dot{\mathbf{V}}_f^m(0)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(0)||_{0,\Omega}^2 + ||P^m(0)||_{0,\Omega}^2\right)^{1/2} \quad (43)$$

or,

$$||\dot{\mathbf{V}}_f^m(0)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(0)||_{0,\Omega}^2 + ||P^m(0)||_{0,\Omega}^2 \leq \frac{1}{\alpha^2}\left(||\mathbf{b}_f(0)||_{0,\Omega}^2 + ||\mathbf{b}_s(0)||_{0,\Omega}^2 + a_0^2|\Omega|\right).$$

(E_8) implies

$$(E_9)\begin{cases} ||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \leq \frac{1}{\alpha^*}\left[(G_5)^2 + ||G_4||_{L^2(0,T)}^2\right] \\ +\frac{\beta^*}{\alpha^*}\left[\int_0^t [||\ddot{\mathbf{U}}_s^m(\xi)||_{0,\Omega}^2 + ||\dot{\mathbf{V}}_f^m(\xi)||_{0,\Omega}^2]\, d\xi\right], \end{cases}$$

where $\alpha = \min\{1, \rho_r, a_0\}$, $\alpha^* = \min\{1, \rho_r\}$, $\beta = \max\{1, \rho_r\}$, $\beta^* = \max\{5, 1 + 4\alpha_1 T\}$ and

$$(G_5)^2 = \frac{\beta}{\alpha^2}\left(||\mathbf{b}_f(0)||_{0,\Omega}^2 + ||\mathbf{b}_s(0)||_{0,\Omega}^2 + a_0^2|\Omega|\right).$$

continue...

Define

$$\Upsilon(t) = ||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2,$$

(E_9) gives

$$\Upsilon(t) \leq \frac{1}{\alpha^*}\left[(G_5)^2 + ||G_4||_{L^2(0,T)}^2\right] + \frac{\beta^*}{\alpha^*}\int_0^t \Upsilon(\xi)\,d\xi, \tag{44}$$

using Gronwall's inequality in (44), we find

$$||\dot{\mathbf{V}}_f^m(t)||_{0,\Omega}^2 + ||\ddot{\mathbf{U}}_s^m(t)||_{0,\Omega}^2 \leq \frac{1}{\alpha^*}\left[(G_5)^2 + ||G_4||_{L^2(0,T)}^2\right] e^{\frac{\beta^* T}{\alpha^*}}. \tag{45}$$

From (E_8), we have

$$\frac{2\alpha_1}{c_k}||\dot{\mathbf{U}}_s^m(t)||_{1,\Omega}^2 \leq \left[(G_5)^2 + ||G_4||_{L^2(0,T)}^2\right] + \beta^*\left[\int_0^t [||\ddot{\mathbf{U}}_s^m(\xi)||_{0,\Omega}^2 + ||\dot{\mathbf{V}}_f^m(\xi)||_{0,\Omega}^2]\,d\xi\right].$$

Further, using (45), we obtain

$$||\dot{\mathbf{U}}_s^m(t)||_{1,\Omega}^2 \leq \frac{c_k}{2\alpha_1}\left[(G_5)^2 + ||G_4||^2_{L^2(0,T)}\right]\left(1 + \frac{\beta^* T}{\alpha^*} e^{\frac{\beta^* T}{\alpha^*}}\right), \tag{46}$$

that is $\dot{\mathbf{U}}_s^m \in L^\infty(0,T;H^1(\Omega)^d) \subset L^2(0,T;H^1(\Omega)^d)$. Inequality (46) implies $\dot{\mathbf{U}}_s^m$ has a subsequence (we denote it by the same symbol) and there exists a function $\dot{\mathbf{U}}_s \in L^2(0,T;H^1(\Omega)^d)$, such that $\dot{\mathbf{U}}_s^m$ weakly converges to $\dot{\mathbf{U}}_s$ in $L^2(0,T;H^1(\Omega)^d)$. Weak lower semi-continuity property of norm and (46) implies

$$||\dot{\mathbf{U}}_s(t)||_{1,\Omega}^2 \leq \frac{c_k}{2\alpha_1}\left[(G_5)^2 + ||G_4||^2_{L^2(0,T)}\right]\left(1 + \frac{\beta^* T}{\alpha^*} e^{\frac{\beta^* T}{\alpha^*}}\right), \tag{47}$$

(47) implies that $\dot{\mathbf{U}}_s \in L^\infty(0,T;H^1(\Omega)^d)$. This completes the proof of the Lemma - 4.1.

Uniqueness

To establish the uniqueness of the weak solution, let us consider $(\mathbf{V}_f^1, \mathbf{U}_s^1, P^1)$ and $(\mathbf{V}_f^2, \mathbf{U}_s^2, P^2)$ as two solutions of the weak formulation (A_w). Further, we use $(\mathbf{W}, \mathbf{Z}, q)$ as the triplet of test functions. Then the differences $\mathbf{V}_f(t) = \mathbf{V}_f^1(t) - \mathbf{V}_f^2(t)$, $\mathbf{U}_s(t) = \mathbf{U}_s^1(t) - \mathbf{U}_s^2(t)$, and $P(t) = P^1(t) - P^2(t)$ satisfy

$$(E_{10}) \begin{cases} \langle \dot{\mathbf{V}}_f(t), \mathbf{W} \rangle_* + 2(D(\mathbf{V}_f(t)), \nabla \mathbf{W})_\Omega + \lambda(\nabla \cdot \mathbf{V}_f(t), \nabla \cdot \mathbf{W})_\Omega \\ + \frac{1}{Da}(\mathbf{V}_f(t), \mathbf{W})_\Omega + \rho_r \langle \ddot{\mathbf{U}}_s(t), \mathbf{Z} \rangle_* + 2\alpha_1 (D(\mathbf{U}_s(t)), \nabla \mathbf{Z})_\Omega \\ + \alpha_2 (\nabla \cdot \mathbf{U}_s(t), \nabla \cdot \mathbf{Z})_\Omega + \frac{1}{Da}(\dot{\mathbf{U}}_s(t), \mathbf{Z})_\Omega + a_0 (P(t), q)_\Omega \\ = \frac{1}{Da}(\mathbf{V}_f(t), \mathbf{Z})_\Omega + \frac{1}{Da}(\dot{\mathbf{U}}_s(t), \mathbf{W})_\Omega. \end{cases}$$

In (E_{10}), we substitute $\mathbf{W} = \mathbf{V}_f(t)$, $\mathbf{Z} = \dot{\mathbf{U}}_s(t)$ and $q = P(t)$. By using Cauchy-Schwarz and Young's inequalities, we have

$$(E_{11}) \begin{cases} \frac{d}{dt}||\mathbf{V}_f(t)||_{0,\Omega}^2 + 4||D(\mathbf{V}_f(t))||_{0,\Omega}^2 + 2\lambda||\nabla \cdot \mathbf{V}_f(t)||_{0,\Omega}^2 + \rho_r \frac{d}{dt}||\dot{\mathbf{U}}_s(t)||_{0,\Omega}^2 \\ + 2\alpha_1 \frac{d}{dt}||D(\mathbf{U}_s(t))||_{0,\Omega}^2 + \alpha_2 \frac{d}{dt}||\nabla \cdot \mathbf{U}_s(t)||_{0,\Omega}^2 + 2a_0||P(t)||_{0,\Omega}^2 \leq 0, \end{cases}$$

integrating (E_{11}) over $(0,t)$ and using initial conditions, we get

$$(E_{12}) \begin{cases} ||\mathbf{V}_f(t)||^2_{0,\Omega} + 4\int_0^t ||D(\mathbf{V}_f(t))||^2_{0,\Omega}\,dt + 2\lambda\int_0^t ||\nabla\cdot\mathbf{V}_f(t)||^2_{0,\Omega}\,dt \\ +\rho_r||\dot{\mathbf{U}}_s(t)||^2_{0,\Omega} + 2\alpha_1||D(\mathbf{U}_s(t))||^2_{0,\Omega} + \alpha_2||\nabla\cdot\mathbf{U}_s(t)||^2_{0,\Omega} \\ +2a_0\int_0^t ||P(t)||^2_{0,\Omega}\,dt \leq 0, \end{cases}$$

This gives $\mathbf{V}_f(t) = 0$, $\mathbf{U}_s(t) = 0$, and $P(t) = 0$ for all $t \in (0,T)$. That is, the system of equations (17)-(19) with respect to zero initial and boundary conditions (20)-(22) with $\mathbf{T}^f_\infty = 0$ has a unique weak solution.

Remark 4.1

In general

$$\mathbf{Z} = \dot{\mathbf{U}}_s \notin L^2(0,T;H^1(\Omega)^d).$$

But for any solution $\mathbf{U}_s(t)$, we have shown that

$$\dot{\mathbf{U}}_s(t) \in L^2(\Omega)^d, \quad \text{and} \quad \nabla \cdot \dot{\mathbf{U}}_s(t) \in L^2(\Omega). \tag{48}$$

In order to show uniqueness, the following additional assumption (which is proved in Lemma 4.1) is needed

$$\nabla \dot{\mathbf{U}}_s(t) \in (L^2(\Omega))^{d\times d}. \tag{49}$$

Then (48) and (49) imply

$$\dot{\mathbf{U}}_s(t) \in H^1(\Omega)^d \quad \text{or,} \quad \dot{\mathbf{U}}_s \in L^2(0,T;H^1(\Omega)^d). \tag{50}$$

This proves that the choice of test function $\mathbf{Z} = \dot{\mathbf{U}}_s(t)$ in (E_{10}) is valid.

Continuous Dependence

Assume that $(\mathbf{V}_{f,1}, \mathbf{U}_{s,1}, P_1)$ and $(\mathbf{V}_{f,2}, \mathbf{U}_{s,2}, P_2)$ are two weak solutions of the system of equations (17)-(19) with respect to zero initial and boundary conditions (20)-(22) with $\mathbf{T}^f_\infty = 0$ corresponding to two sets of data $\{\mathbf{b}_{f,1}, \mathbf{b}_{s,1}, a_{0,1}\}$ and $\{\mathbf{b}_{f,2}, \mathbf{b}_{s,2}, a_{0,2}\}$. Then, we have

$$\begin{cases} ||\mathbf{V}_{f,1} - \mathbf{V}_{f,2}||^2_{L^\infty(0,T;L^2(\Omega)^d)} + ||\dot{\mathbf{U}}_{s,1} - \dot{\mathbf{U}}_{s,2}||^2_{L^\infty(0,T;L^2(\Omega)^d)} \\ +||\mathbf{U}_{s,1} - \mathbf{U}_{s,2}||^2_{L^\infty(0,T;H^1(\Omega)^d)} + \frac{1}{c_k}||\mathbf{V}_{f,1} - \mathbf{V}_{f,2}||^2_{L^2(0,T;H^1(\Omega)^d)} \\ +a_0||P_1 - P_2||^2_{L^2(0,T;L^2(\Omega))} \leq C_0 \Bigg[||\mathbf{b}_{f,1} - \mathbf{b}_{f,2}||^2_{L^2(0,T;L^2(\Omega)^d)} \\ +||\mathbf{b}_{s,1} - \mathbf{b}_{s,2}||^2_{L^2(0,T;L^2(\Omega)^d)} + \frac{1}{a_0}||a_{0,1} - a_{0,2}||^2_{L^2(0,T;L^2(\Omega))} \Bigg], \end{cases}$$

where $C_0 = \frac{1}{\alpha_3} e^{\frac{\alpha_4}{\alpha_3}T} \left[\alpha_3 + 2T(3 + 2\alpha_1 T)\right].$

Main Theorem

Theorem: Assume that $\mathbf{b}_i \in L^2(0,T;L^2(\Omega)^d)$, $\mathbf{T}_\infty^f \in L^2(0,T;L^2(\partial\Omega)^d)$, $\mathbf{V}_0 \in H^1(\Omega)^d$, $\mathbf{U}_0 \in H^1(\Omega)^d$, $\mathbf{U}_1 \in L^2(\Omega)^d$. Then the system of equations (17)-(19) with respect to initial and boundary conditions (20)-(22) has a weak solution $(\mathbf{V}_f, \mathbf{U}_s, P) \in L^2(0,T;H^1(\Omega)^d) \times L^2(0,T;H^1(\Omega)^d) \times L^2(0,T;L^2(\Omega))$, with $\dot{\mathbf{V}}_f \in L^2(0,T;(H^1(\Omega)^d)^*)$, and $\dot{\mathbf{U}}_s \in L^2(0,T;L^2(\Omega)^d)$, $\ddot{\mathbf{U}}_s \in L^2(0,T;(H^1(\Omega)^d)^*)$ and $\nabla \cdot \dot{\mathbf{U}}_s \in L^2(0,T;L^2(\Omega))$. Further the following stability bounds hold

$$||\mathbf{V}_f(t)||_{0,\Omega}^2 + ||\dot{\mathbf{U}}_s(t)||_{0,\Omega}^2 + ||\mathbf{U}_s(t)||_{1,\Omega}^2 \leq \frac{1}{\alpha_3}\left[||G_1||_{L^2(0,T)}^2 + (G_2)^2\right] e^{\frac{\alpha_4 T}{\alpha_3}}, \tag{51}$$

$$\int_0^t ||\mathbf{V}_f(\xi)||_{1,\Omega}^2\, d\xi \leq c_k (G_3)^2, \qquad \int_0^t ||P(\xi)||_{0,\Omega}^2\, d\xi \leq \frac{1}{a_0}(G_3)^2, \tag{52}$$

$$\int_0^t ||\dot{\mathbf{V}}_f(\xi)||_{(H^1(\Omega)^d)^*}^2\, d\xi \leq 2\left(4c_k + \frac{\lambda}{2} + \frac{\varphi_f^2}{a_0} + \frac{1+c_k}{(Da)^2}\right)(G_3)^2$$
$$+2\int_0^T \left(||\mathbf{b}_f(\xi)||_{0,\Omega}^2\, d\xi + \alpha_2^2 ||\mathbf{T}_\infty^f(\xi)||_{L^2(\partial\Omega)^d}^2\right) d\xi, \tag{53}$$

and

$$\int_0^t ||\ddot{\mathbf{U}}_s(\xi)||^2_{(H^1(\Omega)^d)^*}\, d\xi \leq \frac{2}{\rho_r^2}\left[\left(4\alpha_1^2 + \alpha_2^2 + \frac{\varphi_s^2}{a_0} + \frac{1+c_k}{(Da)^2}\right)(G_3)^2 + \int_0^T ||\mathbf{b}_s(\xi)||^2_{0,\Omega}\, d\xi\right]. \tag{54}$$

Moreover, under higher regularity assumptions on the data i.e., $\mathbf{b}_i \in H^1(0,T;L^2(\Omega)^d)$, $(i = f,\ s)$ and for zero initial data i.e. $\mathbf{V}_0 = 0$, $\mathbf{U}_0 = 0$, $\mathbf{U}_1 = 0$, and $\mathbf{T}^f_\infty = 0$, we obtain the following regularity result

$$\dot{\mathbf{U}}_s \in L^\infty(0,T;H^1(\Omega)^d). \tag{55}$$

The above regularity result (55) ensures that the system of equations (17)-(19) has a unique weak solution which continuously depends on the given data.

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- We considered the mathematical model that describes the unsteady poroelastohydrodynamics inside an arbitrary solid tumor. The biphasic mixture theory has been applied for modeling purposes.
- Introducing a variational formulation and using semi-discrete Galerkin method and weak convergence, we have shown the existence of a solution in a weak sense of the corresponding mathematical model.
- Further, by proving some regularity results, the uniqueness and continuous dependence on given data have been established.
- Currently, I are working on the nonlinear model of this problem which occurs while considering hydraulic resistivity changes with deformation.
- Future, we want to develop numerical simulation to get more realistic results.

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https://arxiv.org/abs/2202.06059v1

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Thank you for listening!

Any questions or comments?