Homogenisation of a system of Stokes flow and advection–reaction–diffusion transport in a porous medium with coupled evolving microstructure

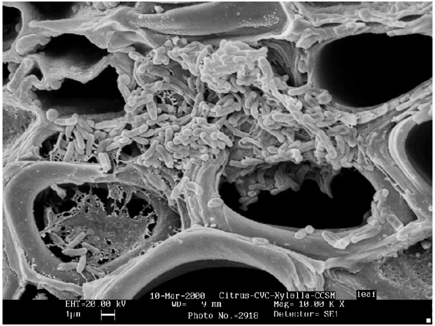
Markus Gahn, Malte A. Peter, Iulio Sorin Pop, David Wiedemann

KAAS

Dezember 13, 2023

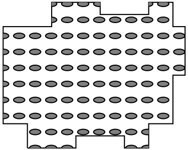
Motivation

By Xylella fastidiosa infected olive trees in Surano in Apulien, Italy.
via Wikimedia Commons

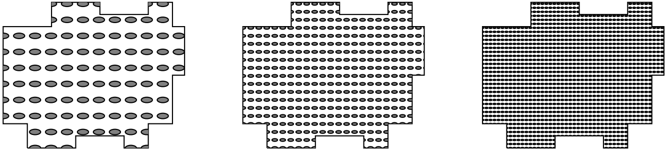


By Xylella Fastidiosa infected vessels.
via Xylella fastidiosa Genome Project

Outline

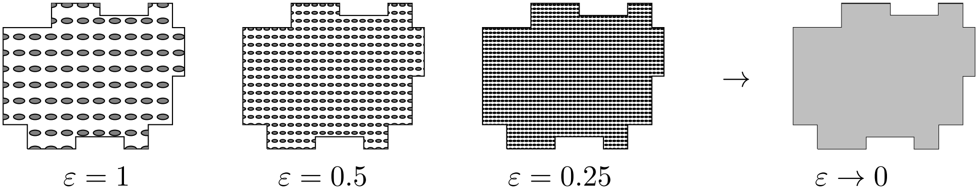


How to describe the effective processes in materials with microstructure?



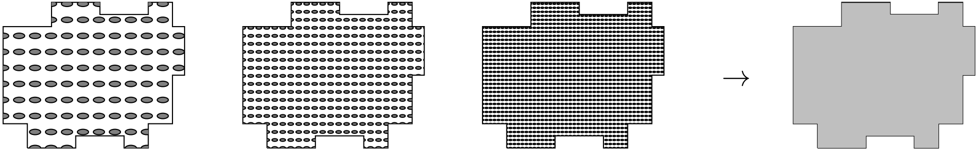
$\varepsilon = 1$ $\varepsilon = 0.5$ $\varepsilon = 0.25$

How to describe the effective processes in materials with microstructure?



How to describe the effective processes in materials with microstructure?

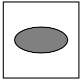
Homogenisation



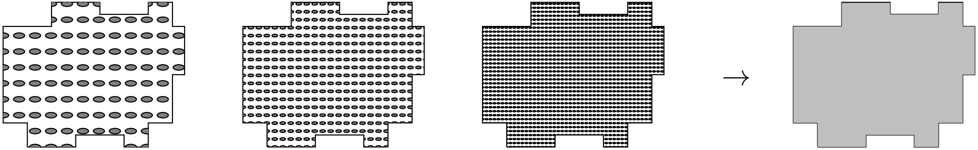
$\varepsilon = 1$ $\qquad$ $\varepsilon = 0.5$ $\qquad$ $\varepsilon = 0.25$ $\qquad$ $\varepsilon \to 0$

$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$$

$$\text{for } a(y) = \begin{cases} \gamma_1 \text{ if } y \in Y_1 \\ \gamma_2 \text{ if } y \in Y_2 \end{cases} \qquad \text{with } \overline{Y_1 \cup Y_2} = Y = [0,1]^n =$$



Homogenisation

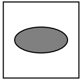


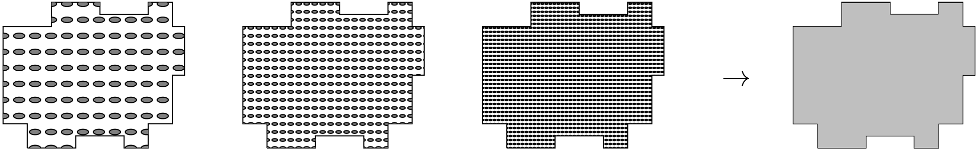
$\varepsilon = 1$ $\quad \varepsilon = 0.5$ $\quad \varepsilon = 0.25$ $\quad \rightarrow \quad \varepsilon \to 0$

$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$$

$$\updownarrow$$

$$u_\varepsilon$$

for $a(y) = \begin{cases} \gamma_1 \text{ if } y \in Y_1 \\ \gamma_2 \text{ if } y \in Y_2 \end{cases}$ with $\overline{Y_1 \cup Y_2} = Y = [0,1]^n =$ 

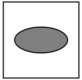


$\varepsilon = 1$ $\quad \varepsilon = 0.5$ $\quad \varepsilon = 0.25$ $\quad \to \quad \varepsilon \to 0$

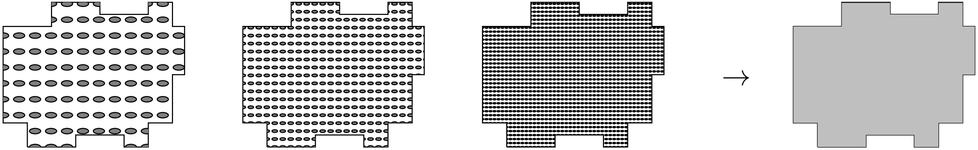
$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$$

$$\updownarrow$$

$$u_\varepsilon \qquad \to \qquad u_0$$

for $a(y) = \begin{cases} \gamma_1 \text{ if } y \in Y_1 \\ \gamma_2 \text{ if } y \in Y_2 \end{cases}$ with $\overline{Y_1 \cup Y_2} = Y = [0,1]^n =$ 

Homogenisation

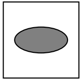


$\varepsilon = 1$ $\quad$ $\varepsilon = 0.5$ $\quad$ $\varepsilon = 0.25$ $\quad$ $\varepsilon \to 0$

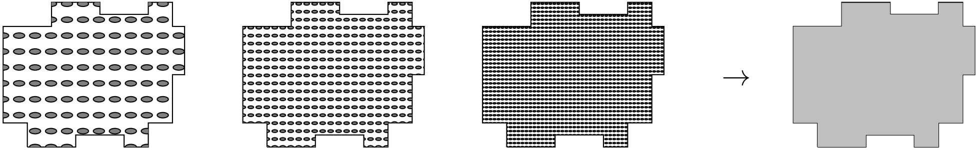
$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \quad \to \quad -\operatorname{div}(a^* \nabla u_0) = f$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$u_\varepsilon \quad \to \quad u_0$$

for $a(y) = \begin{cases} \gamma_1 \text{ if } y \in Y_1 \\ \gamma_2 \text{ if } y \in Y_2 \end{cases}$ with $\overline{Y_1 \cup Y_2} = Y = [0,1]^n =$ 

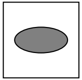
Porous media



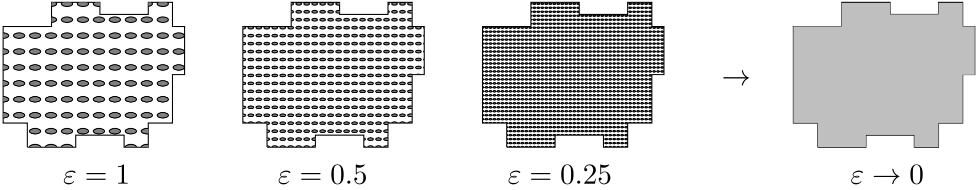
$\varepsilon = 1$ $\quad\varepsilon = 0.5$ $\quad\varepsilon = 0.25$ $\quad\varepsilon \to 0$

$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \qquad \text{in } \Omega$$

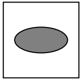
$$\text{for } a(y) = \begin{cases} \gamma_1 \text{ if } y \in Y_1 \\ \gamma_2 \text{ if } y \in Y_2 \end{cases} \qquad \text{with } \overline{Y_1 \cup Y_2} = \overline{Y} = [0,1]^n =$$



Porous media

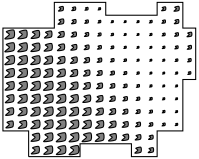


$$-\operatorname{div}\big(\quad \nabla u_\varepsilon\big) = f \qquad \text{in } \Omega_\varepsilon$$

with $\overline{Y_1 \cup Y_2} = \overline{Y} = [0,1]^n =$ 

$$\text{for } \Omega_\varepsilon = \operatorname{int}\Big(\Omega \cap \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k + Y^*)\Big) \qquad \text{with } Y^* = Y_1$$

Evolving microstructure



Evolving microstructure

Homogenisation in evolving porous media

D. Wiedemann

Homogenisation in evolving domains

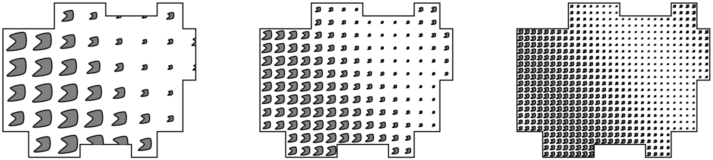
Stokes flow

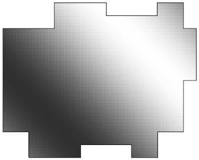
Coupled reaction–diffusion

Fully coupled model

Outlook

Conclusion





Analytical Background

$$-\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega,$$

Analytical Background

$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega,$$

$$a\left(\tfrac{x}{\varepsilon}\right) \rightharpoonup a_0 = \int_Y a(y)dy \qquad \text{weakly}^* \text{ in } L^\infty(\Omega),$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u_\varepsilon \qquad \text{weakly in } L^2(\Omega)$$

$$-\operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega,$$

$$a(\tfrac{x}{\varepsilon}) \rightharpoonup a_0 = \int_Y a(y)dy \qquad \text{weakly}^* \text{ in } L^\infty(\Omega),$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u_\varepsilon \qquad \text{weakly in } L^2(\Omega)$$

$$\operatorname{div}(a(\tfrac{x}{\varepsilon})\nabla u_\varepsilon(x)) \not\rightharpoonup \operatorname{div}(a_0 \nabla u_0(x))$$

Definition (Two-scale convergence)

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. *A sequence* $u_\varepsilon \in L^p(\Omega)$ *two-scale converges to* $u_0 \in L^p(\Omega \times Y)$ *with* $Y = (0,1)^n$ *if for every* $\varphi \in L^q(\Omega; C_\#(Y))$

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y) \varphi(x,y) \, \mathrm{d}y \, \mathrm{d}x.$$

We write $u_\varepsilon(x) \rightharpoonup\!\!\rightharpoonup u_0(x,y)$.

G. Allaire.
Homogenization and Two-Scale convergence.
SIAM J. MATH. ANAL., Vol. 23, 1992.

Definition (Two-scale convergence)

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. *A sequence* $u_\varepsilon \in L^p(\Omega)$ *two-scale converges to* $u_0 \in L^p(\Omega \times Y)$ *with* $Y = (0,1)^n$ *if for every* $\varphi \in L^q(\Omega; C_\#(Y))$

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\varphi(x,y) \, \mathrm{d}y \, \mathrm{d}x.$$

We write $u_\varepsilon(x) \rightharpoonup\rightharpoonup u_0(x,y)$.

Let $p \in (1,\infty)$. A sequence u_ε two-scale converges strongly to u_0 if $u_\varepsilon \rightharpoonup\rightharpoonup u_0$ and $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega \times Y)}$. We write $u_\varepsilon \twoheadrightarrow u_0$.

Properties of two-scale convergence

technische universität dortmund

Multiplication:
Let $p_1, p_2 \in (1,\infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ for $p \in [1,\infty)$:

$$a_\varepsilon \xrightarrow{p_1}\!\!\to a_0, \quad v_\varepsilon \overset{p_2}{\rightharpoonup\!\!\rightharpoonup} v_0 \quad \Rightarrow \quad a_\varepsilon v_\varepsilon \overset{p}{\rightharpoonup\!\!\rightharpoonup} u_0 v_0.$$

Multiplication:
Let $p_1, p_2 \in (1, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ for $p \in [1, \infty)$:

$$a_\varepsilon \xrightarrow{p_1} a_0, \quad v_\varepsilon \xrightarrow{p_2} v_0 \quad \Rightarrow \quad a_\varepsilon v_\varepsilon \xrightarrow{p} u_0 v_0.$$

Compactness:

$$\|u_\varepsilon\|_{L^2(\Omega)} \leq C \qquad \Rightarrow u_\varepsilon \rightharpoonup u_0(x, y),$$

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C \qquad \Rightarrow \nabla_x u_\varepsilon \rightharpoonup \nabla_x u_0(x) + \nabla_y u_1(x, y),$$

.

Multiplication:
Let $p_1, p_2 \in (1,\infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ for $p \in [1,\infty)$:

$$a_\varepsilon \xrightarrow{p_1} a_0, \quad v_\varepsilon \overset{p_2}{\rightharpoonup\rightharpoonup} v_0 \quad \Rightarrow \quad a_\varepsilon v_\varepsilon \overset{p}{\rightharpoonup\rightharpoonup} u_0 v_0.$$

Compactness:

$$\|u_\varepsilon\|_{L^2(\Omega)} \leq C \qquad \Rightarrow u_\varepsilon \rightharpoonup\rightharpoonup u_0(x,y),$$

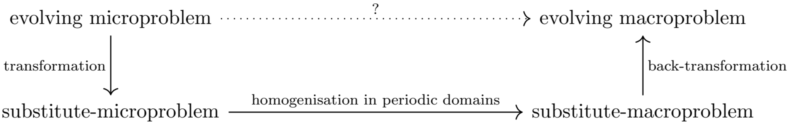
$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C \qquad \Rightarrow \nabla_x u_\varepsilon \rightharpoonup\rightharpoonup \nabla_x u_0(x) + \nabla_y u_1(x,y),$$

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \qquad \Rightarrow \nabla_x u_\varepsilon \rightharpoonup\rightharpoonup \chi_{Y^*}(y)\nabla_x u_0(x) + \nabla_y u_1(x,y)$$

for $\Omega_\varepsilon = \operatorname{int}(\Omega \cap \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k + Y^*))$.

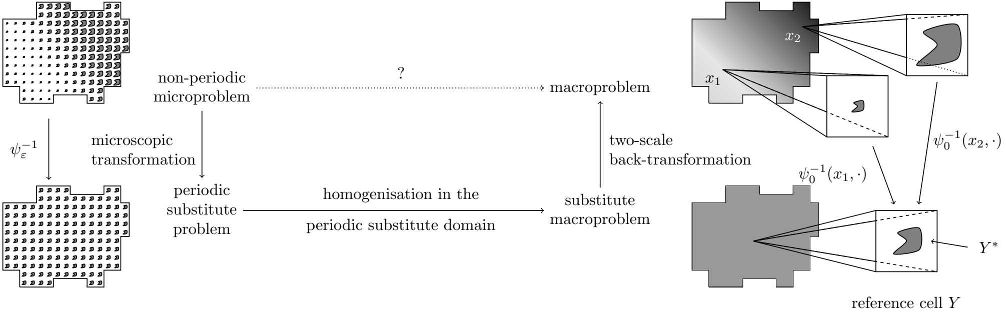
technische universität dortmund

Homogenisation for evolving microstructure



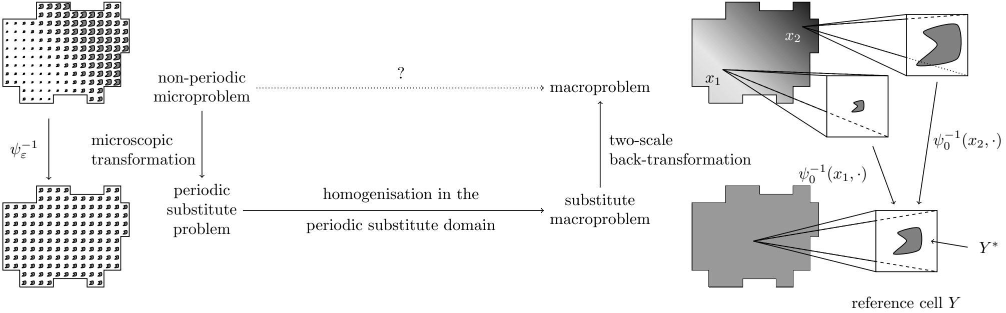
M. A. Peter.
Homogenisation in domains with evolving microstructure.
C. R. Mecanique, Vol. 335, 2007.

Homogenisation for evolving microstructure

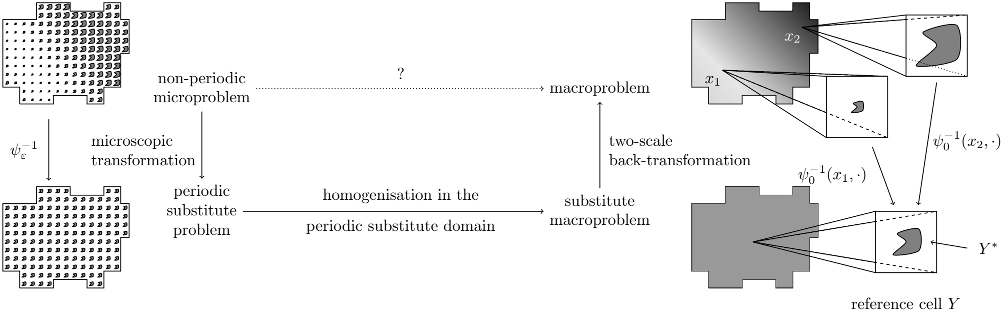


Homogenisation for evolving microstructure

$$-\operatorname{div}(\nabla u_\varepsilon) = f \text{ in } \Omega_\varepsilon(t)$$



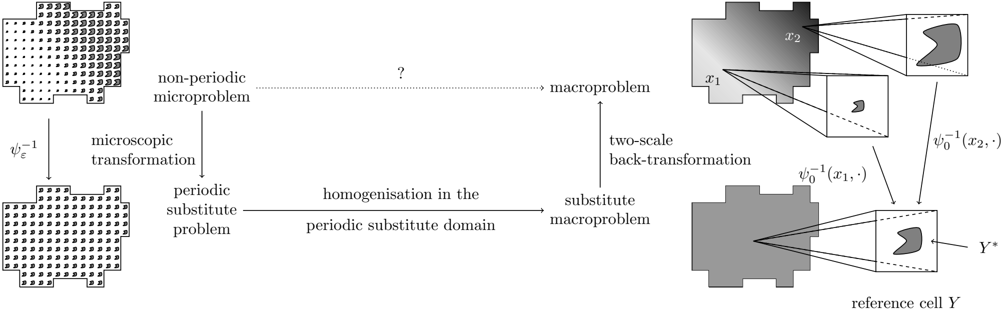
$$-\operatorname{div}(\nabla u_\varepsilon) = f \text{ in } \Omega_\varepsilon(t)$$



Let $\psi_\varepsilon : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ be a family of diffeomorphisms

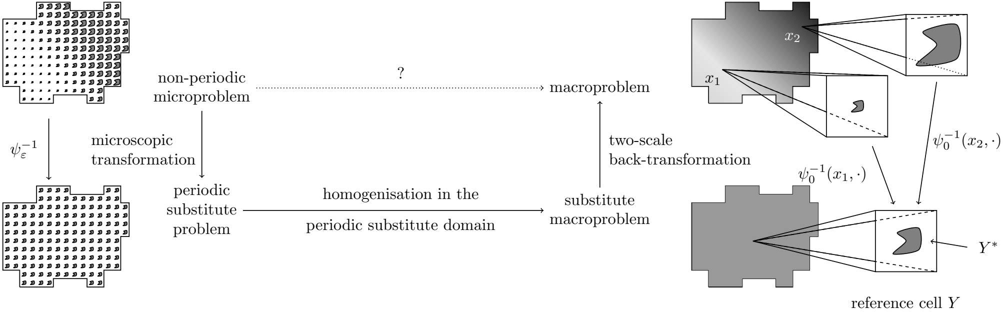
Homogenisation for evolving microstructure

$$-\operatorname{div}(\nabla u_\varepsilon) = f \text{ in } \Omega_\varepsilon(t)$$



Let $\psi_\varepsilon : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ be a family of diffeomorphisms and $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x)), \quad \hat{f}_\varepsilon(x) = f(\psi_\varepsilon(x))$.

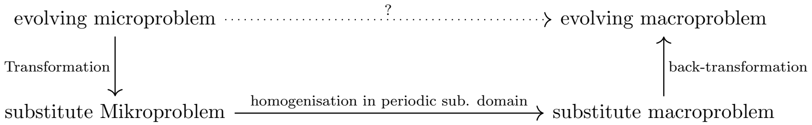
$$-\operatorname{div}(\nabla u_\varepsilon) = f \text{ in } \Omega_\varepsilon(t)$$



$$-\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \Psi_\varepsilon^{-\top} \nabla \hat{u}_\varepsilon) = J_\varepsilon \hat{f}_\varepsilon \text{ in } \Omega_\varepsilon$$

Let $\psi_\varepsilon : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ be a family of diffeomorphisms with $\Psi_\varepsilon := \partial_x \psi_\varepsilon,\ J_\varepsilon := \det(\Psi_\varepsilon)$ and $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x)), \quad \hat{f}_\varepsilon(x) = f(\psi_\varepsilon(x))$.

Homogenisation for evolving microstructure

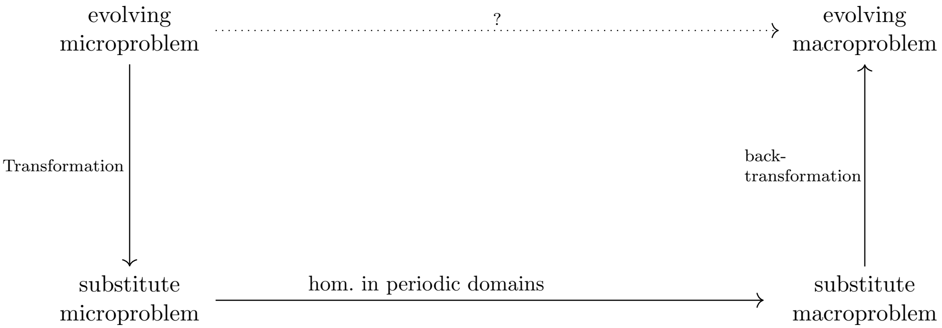


M. Gahn, M. Neuss-Radu, I. S. Pop.
Homogenization of a reaction–diffusion–advection problem in an evolving micro-domain and including nonlinear boundary conditions.
J. Differ. Equations, Vol. 289, 2021.

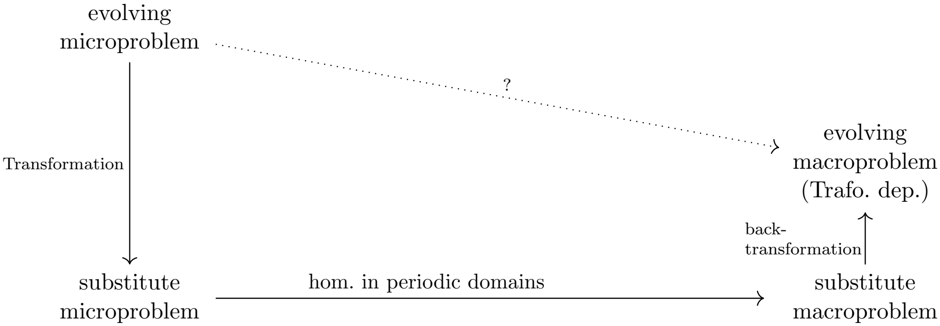
M. Eden, A. Muntean.
Homogenization of a fully coupled thermoelasticity problem for a highly heterogeneous medium with a priori known phase transformations.
Math. Methods Appl. Sci., Vol. 40, 2017.

M. A. Peter.
Coupled reaction–diffusion processes inducing an evolution of the microstructure: Analysis and homogenization.
Nonlinear Anal., Vol. 70, 2009.

The two-scale transformation method



The two-scale transformation method



Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms such that

Transformation and two-scale convergence

Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms such that

$$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C, \qquad \det(\nabla\psi_\varepsilon) \geq c_J > 0,$$

Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms and $\psi_0, \psi_0^{-1} \in L^\infty(\Omega; C^1(\overline{Y})^n)$ such that

$$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C, \qquad \det(\nabla\psi_\varepsilon) \geq c_J > 0,$$

Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms and $\psi_0, \psi_0^{-1} \in L^\infty(\Omega; C^1(\overline{Y})^n)$ such that

$$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C, \qquad \det(\nabla\psi_\varepsilon) \geq c_J > 0,$$

$$\varepsilon^{-1}(\psi_\varepsilon - x) \twoheadrightarrow \psi_0 - y, \qquad \nabla\psi_\varepsilon \twoheadrightarrow \nabla_y\psi_0.$$

Transformation and two-scale convergence

Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms and $\psi_0, \psi_0^{-1} \in L^\infty(\Omega; C^1(\overline{Y})^n)$ such that

$$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C, \qquad \det(\nabla\psi_\varepsilon) \geq c_J > 0,$$

$$\varepsilon^{-1}(\psi_\varepsilon - x) \longrightarrow\!\!\!\!\!\rightarrow \psi_0 - y, \qquad \nabla\psi_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \nabla_y\psi_0.$$

Let $\hat{u}_\varepsilon \in L^p(\Omega_\varepsilon)$ and $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x))$:

$$u_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup u_0(x,y) \quad \Longleftrightarrow \quad \hat{u}_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup \hat{u}_0(x,y)$$

for $\hat{u}_0(x,y) = u_0(x, \psi_0(x,y))$,

D. Wiedemann.
The two-scale-transformation method.
Asymptotic Analysis, 2023.

Let $\psi_\varepsilon : \overline{\Omega_\varepsilon} \to \overline{\Omega_\varepsilon}(t)$ C^1-diffeomorphisms and $\psi_0, \psi_0^{-1} \in L^\infty(\Omega; C^1(\overline{Y})^n)$ such that

$$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C, \qquad \det(\nabla\psi_\varepsilon) \geq c_J > 0,$$

$$\varepsilon^{-1}(\psi_\varepsilon - x) \longrightarrow\!\!\!\!\!\rightarrow \psi_0 - y, \qquad \nabla\psi_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \nabla_y\psi_0.$$

Let $\hat{u}_\varepsilon \in L^p(\Omega_\varepsilon)$ and $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x))$:

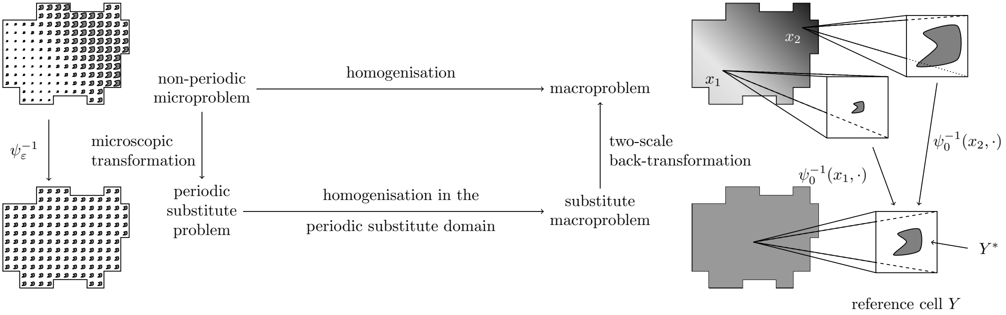
$$u_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup u_0(x,y) \iff \hat{u}_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup \hat{u}_0(x,y)$$

for $\hat{u}_0(x,y) = u_0(x, \psi_0(x,y))$,

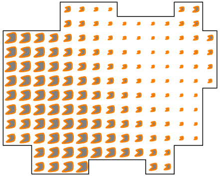
$$\nabla u_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup \nabla_x u_0(x) + \nabla_y u_1(x,y) \iff \nabla\hat{u}_\varepsilon(x) \rightharpoonup\!\!\!\!\rightharpoonup \nabla_x\hat{u}_0(x) + \nabla_y\hat{u}_1(x,y)$$

for $\hat{u}_0(x) = u_0(x)$ and

$$\hat{u}_1(x,y) = u_1(x, \psi_0(x,y)) + \nabla_x u_0(x) \cdot (\psi_0(x,y) - y).$$

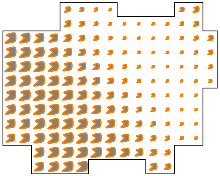

non-periodic
microproblem
homogenisation
macroproblem
x_2
x_1
ψ_ε^{-1}
microscopic
transformation
two-scale
back-transformation
$\psi_0^{-1}(x_2,\cdot)$
$\psi_0^{-1}(x_1,\cdot)$
periodic
substitute
problem
homogenisation in the
periodic substitute domain
substitute
macroproblem
Y^*
reference cell Y

Outline



Let $\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ be given.

Stokes flow for evolving microstructure



Let $\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ be given. We consider

$$
\begin{aligned}
-\operatorname{div}(\varepsilon^2 \nu (\nabla v_\varepsilon + \nabla v_\varepsilon^\top)) + \nabla p_\varepsilon &= f && \text{in } \Omega_\varepsilon(t),\\
\operatorname{div}(v_\varepsilon) &= 0 && \text{in } \Omega_\varepsilon(t),\\
v_\varepsilon &= v_{\Gamma_\varepsilon} && \text{on } \Gamma_\varepsilon(t),\\
-\varepsilon^2 \nu (\nabla v_\varepsilon + \nabla v_\varepsilon^\top)^\top n + p_\varepsilon\, n &= p_b\, n && \text{on } \partial\Omega_\varepsilon(t) \cap \partial\Omega.
\end{aligned}
\tag{S}
$$

Weak form of the Stokes equations

Substitute $w_\varepsilon(t) = v_\varepsilon(t) - v_{\Gamma_\varepsilon}(t)$ for $v_{\Gamma_\varepsilon} = \partial_t \psi_\varepsilon \circ \psi_\varepsilon^{-1}$

Weak form of the Stokes equations

Substitute $w_\varepsilon(t) = v_\varepsilon(t) - v_{\Gamma_\varepsilon}(t)$ for $v_{\Gamma_\varepsilon} = \partial_t \psi_\varepsilon \circ \psi_\varepsilon^{-1}$

Find $(w_\varepsilon(t), p_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t)) \times L^2(\Omega_\varepsilon(t))$, such that

$$\int_{\Omega_\varepsilon(t)} \nu \varepsilon^2 2e(w_\varepsilon) : \nabla \varphi + p_\varepsilon \operatorname{div}(\varphi) dx = \int_{\Omega_\varepsilon(t)} f \varphi dx + O(\varepsilon),$$

$$\operatorname{div}(w_\varepsilon) = -\operatorname{div}(\partial_t \psi_\varepsilon \circ \psi_\varepsilon^{-1})$$

for all $\varphi \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))$ with $e(w_\varepsilon) = (\nabla w_\varepsilon + \nabla w_\varepsilon^\top)/2$.

Transformation with ψ_ε, i.e.

$$\hat{w}_\varepsilon(t) = w_\varepsilon(t, \psi_\varepsilon(t,x)), \qquad \hat{p}_\varepsilon(t) = p_\varepsilon(t, \psi_\varepsilon(t,x)).$$

Transformation with ψ_ε, i.e.

$$\hat{w}_\varepsilon(t) = w_\varepsilon(t, \psi_\varepsilon(t,x)), \qquad \hat{p}_\varepsilon(t) = p_\varepsilon(t, \psi_\varepsilon(t,x)).$$

$$\int_{\Omega_\varepsilon(t)} \nu\varepsilon^2 \quad 2e \quad (w_\varepsilon) : \nabla\varphi + p_\varepsilon \operatorname{div}(\quad \varphi) dx = \int_{\Omega_\varepsilon(t)} f\varphi dx + O(\varepsilon),$$

$$\operatorname{div}(\quad w_\varepsilon) = -\operatorname{div}(\partial_t \psi_\varepsilon \circ \psi_\varepsilon^{-1})$$

Transformation with ψ_ε, i.e.

$$\hat{w}_\varepsilon(t) = w_\varepsilon(t, \psi_\varepsilon(t,x)), \qquad \hat{p}_\varepsilon(t) = p_\varepsilon(t, \psi_\varepsilon(t,x)).$$

$$\int_{\Omega_\varepsilon} \nu\varepsilon^2 J_\varepsilon \Psi_\varepsilon^{-1} 2 e_{\Psi_\varepsilon}(\hat{w}_\varepsilon) : \nabla\varphi + \hat{p}_\varepsilon \operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1}\varphi) dx = \int_{\Omega_\varepsilon} J_\varepsilon \hat{f}_\varepsilon \varphi dx + O(\varepsilon),$$

$$\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \hat{w}_\varepsilon) = -\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \partial_t \psi_\varepsilon)$$

with $e_{\Psi_\varepsilon}(\hat{w}_\varepsilon) := \left(\Psi_\varepsilon^{-\top} \nabla \hat{w}_\varepsilon + (\Psi_\varepsilon^{-\top}\nabla\hat{w}_\varepsilon)^\top\right)/2$

Transformation with ψ_ε, i.e.

$$\hat{w}_\varepsilon(t) = w_\varepsilon(t, \psi_\varepsilon(t,x)), \qquad \hat{p}_\varepsilon(t) = p_\varepsilon(t, \psi_\varepsilon(t,x)).$$

Find $(\hat{w}_\varepsilon(t), \hat{p}_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, such that

$$\int_{\Omega_\varepsilon} \nu\varepsilon^2 J_\varepsilon \Psi_\varepsilon^{-1} 2 e_{\Psi_\varepsilon}(\hat{w}_\varepsilon) : \nabla\varphi + \hat{p}_\varepsilon \operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1}\varphi) dx = \int_{\Omega_\varepsilon} J_\varepsilon \hat{f}_\varepsilon \varphi dx + O(\varepsilon),$$

$$\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \hat{w}_\varepsilon) = -\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \partial_t \psi_\varepsilon)$$

with $e_{\Psi_\varepsilon}(\hat{w}_\varepsilon) := \left(\Psi_\varepsilon^{-\top}\nabla\hat{w}_\varepsilon + (\Psi_\varepsilon^{-\top}\nabla\hat{w}_\varepsilon)^\top\right)/2$
for all $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$.

$$\int_{\Omega_\varepsilon} \nu\varepsilon^2 J_\varepsilon \Psi_\varepsilon^{-1} (e_{\Psi_\varepsilon}(\hat{w}_\varepsilon)) : \nabla\varphi + \hat{p}_\varepsilon \operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \hat{\varphi}) dx = \int_{\Omega_\varepsilon} f \varphi dx + O(\varepsilon),$$

$$\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \hat{w}_\varepsilon) = -\operatorname{div}(J_\varepsilon \Psi_\varepsilon^{-1} \partial_t \psi_\varepsilon(t))$$

with $e_{\Psi_\varepsilon}(\hat{w}_\varepsilon) := (\Psi_\varepsilon^{-\top} \nabla \hat{w}_\varepsilon + (\Psi_\varepsilon^{-\top} \nabla \hat{w}_\varepsilon)^\top)/2$.

There exists a constant $C > 0$, such that

$$\|\hat{w}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|\nabla \hat{w}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + \|\hat{p}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} \leq C.$$

Theorem (Korn-type inequality for two-scale transformation)

There exists a constant $\alpha > 0$ *such that*

$$\|\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top\|^2_{L^2(\Omega_\varepsilon)} \geq \alpha\|\nabla\varphi\|^2_{L^2(\Omega_\varepsilon)}$$

for all $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$.

D. Wiedemann, M. A. Peter.
Homogenisation of the Stokes equations for evolving microstructure.
arXiv:2109.05997.

Proof of the Korn-type inequality for two-scale transformations

$$||\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top||^2_{L^2(\Omega_\varepsilon)} \geq \alpha||\nabla\varphi||^2_{L^2(\Omega_\varepsilon)} \qquad \forall\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

Proof of the Korn-type inequality for two-scale transformations

$$||\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top||^2_{L^2(\Omega_\varepsilon)} \geq \alpha||\nabla\varphi||^2_{L^2(\Omega_\varepsilon)} \qquad \forall\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

Step 1: Upscale on reference cell: i.e. for all $k \in I_\varepsilon \subset \mathbb{Z}^n$

$$\|\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi + (\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha\|\nabla\varphi\|^2_{L^2(Y^*)} \qquad \forall\varphi \in H^1_\Gamma(Y^*)^n$$

Proof of the Korn-type inequality for two-scale transformations

$$||\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top||^2_{L^2(\Omega_\varepsilon)} \geq \alpha||\nabla\varphi||^2_{L^2(\Omega_\varepsilon)} \qquad \forall\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

Step 1: Upscale on reference cell: i.e. for all $k \in I_\varepsilon \subset \mathbb{Z}^n$

$$\|\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi + (\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha\|\nabla\varphi\|^2_{L^2(Y^*)} \qquad \forall\varphi \in H^1_\Gamma(Y^*)^n$$

Step 2: For every $A \in C(\overline{Y^*};\mathbb{R}^{n\times n})$ with $\det(A) \geq c_0 > 0$ there exists $\alpha(A) > 0$ such that

$$\|A\nabla\varphi + (A\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha\big(A\big)\|\nabla\varphi\|^2_{L^2(Y^*)}.$$

Proof of the Korn-type inequality for two-scale transformations

$$||\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top||^2_{L^2(\Omega_\varepsilon)} \geq \alpha||\nabla\varphi||^2_{L^2(\Omega_\varepsilon)} \qquad \forall\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

Step 1: Upscale on reference cell: i.e. for all $k \in I_\varepsilon \subset \mathbb{Z}^n$

$$\|\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi + (\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha\|\nabla\varphi\|^2_{L^2(Y^*)} \qquad \forall\varphi \in H^1_\Gamma(Y^*)^n$$

Step 2: For every $A \in C(\overline{Y^*};\mathbb{R}^{n\times n})$ with $\det(A) \geq c_0 > 0$ there exists $\alpha(A) > 0$ such that

$$\|A\nabla\varphi + (A\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha(A)\|\nabla\varphi\|^2_{L^2(Y^*)}.$$

Step 3: the optimal constant $\alpha(A)$ in Step 2 depends continuously on $A \in C(\overline{Y^*};\mathbb{R}^{n\times n})$.

Proof of the Korn-type inequality for two-scale transformations

$$||\Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top||^2_{L^2(\Omega_\varepsilon)} \geq \alpha||\nabla\varphi||^2_{L^2(\Omega_\varepsilon)} \qquad \forall\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

Step 1: Upscale on reference cell: i.e. for all $k \in I_\varepsilon \subset \mathbb{Z}^n$

$$\|\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi + (\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot))\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha\|\nabla\varphi\|^2_{L^2(Y^*)} \qquad \forall\varphi \in H^1_\Gamma(Y^*)^n$$

Step 2: For every $A \in C(\overline{Y^*};\mathbb{R}^{n\times n})$ with $\det(A) \geq c_0 > 0$ there exists $\alpha(A) > 0$ such that

$$\|A\nabla\varphi + (A\nabla\varphi)^\top\|^2_{L^2(Y^*)} \geq \alpha(A)\|\nabla\varphi\|^2_{L^2(Y^*)}.$$

Step 3: the optimal constant $\alpha(A)$ in Step 2 depends continuously on $A \in C(\overline{Y^*};\mathbb{R}^{n\times n})$.

Step 4: $\{\Psi_\varepsilon^{-\top}(\varepsilon(k+\cdot)) \in C(\overline{Y^*};\mathbb{R}^{n\times n}) \mid \varepsilon > 0, k \in I_\varepsilon\}$ is precompact.

Homogenisation of Stokes flow

Theorem

Let $(v_\varepsilon(t), p_\varepsilon(t)) \in H^1(\Omega_\varepsilon(t)) \times L^2(\Omega_\varepsilon(t))$ *be the solution of* (S) *and* $\tilde{v}_\varepsilon(t), \tilde{p}_\varepsilon(t)$ *some extension to* Ω*. Then*

$$\tilde{v}_\varepsilon(t) \rightharpoonup v(t) \qquad \text{in } L^2(\Omega),$$
$$\tilde{p}_\varepsilon(t) \to p(t) \qquad \text{in } L^2(\Omega),$$

where $(v(t), p(t))$ *is the unique solution of the Darcy law for evolving microtructure.*

D. Wiedemann, M. A. Peter.
Homogenisation of the Stokes equations for evolving microstructure.
arXiv:2109.05997.

Darcy's law for evolving microstructure

$$v = \frac{1}{\nu} K(t,x)(f - \nabla p) \qquad \text{in } \Omega,$$

D. Wiedemann, M. A. Peter.
Homogenisation of the Stokes equations for evolving microstructure.
arXiv:2109.05997.

Darcy's law for evolving microstructure

$$v = \frac{1}{\nu} K(t,x)(f - \nabla p) \qquad \text{in } \Omega,$$

for $(K(t,x))_{ij} := \int_{Y_x^*(t)} w_i(t,x,y) \cdot e_j \,\mathrm{d}y$ with

$$\begin{aligned} -\Delta w_i(t,x,y) + \nabla \pi_i(t,x,y) &= e_i && \text{in } Y_x^*(t), \\ \operatorname{div}(w_i(t,x,y)) &= 0 && \text{in } Y_x^*(t), \\ w_i(t,x,y) &= 0 && \text{on } \Gamma_x^*(t), \\ y \mapsto w_i(t,x,y) & && Y-\text{periodic.} \end{aligned}$$

D. Wiedemann, M. A. Peter.
Homogenisation of the Stokes equations for evolving microstructure.
arXiv:2109.05997.

Darcy's law for evolving microstructure

$$v = \frac{1}{\nu} K(t,x)(f - \nabla p) \qquad \text{in } \Omega,$$
$$\operatorname{div}(v(t)) = -\tfrac{d}{dt}|Y_x^*(t)| \qquad \text{in } \Omega,$$
$$p(t) = p_b(t) \qquad \text{on } \partial\Omega$$

for $(K(t,x))_{ij} := \int\limits_{Y_x^*(t)} w_i(t,x,y) \cdot e_j \, dy$ with

$$-\Delta w_i(t,x,y) + \nabla \pi_i(t,x,y) = e_i \qquad \text{in } Y_x^*(t),$$
$$\operatorname{div}(w_i(t,x,y)) = 0 \qquad \text{in } Y_x^*(t),$$
$$w_i(t,x,y) = 0 \qquad \text{on } \Gamma_x^*(t),$$
$$y \mapsto w_i(t,x,y) \qquad Y-\text{periodic}.$$

D. Wiedemann, M. A. Peter.
Homogenisation of the Stokes equations for evolving microstructure.
arXiv:2109.05997.

Homogenisation of the instationary Stokes equations

$$
\begin{aligned}
\partial_t v_\varepsilon - \operatorname{div}(\varepsilon^2 \nu (\nabla v_\varepsilon + \nabla v_\varepsilon^\top)) + \nabla p_\varepsilon &= f && \text{in } \Omega_\varepsilon(t), \\
\operatorname{div}(v_\varepsilon) &= 0 && \text{in } \Omega_\varepsilon(t), \\
v_\varepsilon &= v_{\Gamma_\varepsilon} && \text{on } \Gamma_\varepsilon(t), \\
-\varepsilon^2 \nu (\nabla v_\varepsilon + \nabla v_\varepsilon^\top)^\top n + p_\varepsilon \, n &= p_b \, n && \text{on } \partial\Omega \setminus \Gamma_\varepsilon(t), \\
v_\varepsilon(t=0) &= v_\varepsilon^{\text{in}} && \text{in } \Omega_\varepsilon(0).
\end{aligned}
$$

Darcys law with memory for evolving microstructure

$$
\begin{aligned}
v(t,x) &= \frac{1}{\nu}\int_0^t K(t,s,x)(f(s,x) - \nabla p(s,x))\,\mathrm{d}s && \text{in } (0,T)\times\Omega,\\
\operatorname{div}(v(t,x)) &= -\frac{d}{dt}|Y_x^*(t)| && \text{in } (0,T)\times\Omega,\\
p(t,x) &= p_b(t,x) && \text{on } (0,T)\times\partial\Omega
\end{aligned}
$$

Darcys law with memory for evolving microstructure

$$v(t,x) = \frac{1}{\nu}\int_0^t K(t,s,x)(f(s,x) - \nabla p(s,x))\,\mathrm{d}s \qquad \text{in } (0,T)\times\Omega,$$

$$\operatorname{div}(v(t,x)) = -\frac{d}{dt}|Y_x^*(t)| \qquad \text{in } (0,T)\times\Omega,$$

$$p(t,x) = p_b(t,x) \qquad \text{on } (0,T)\times\partial\Omega$$

D. Wiedemann.
Analytical homogenisation of transport processes in evolving porous media.
PhD Thesis, 2023.

Darcys law with memory for evolving microstructure

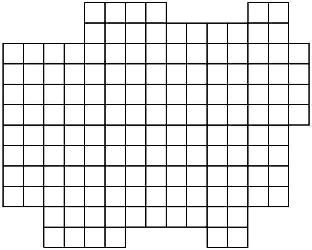
$$
\begin{aligned}
v(t,x) &= \frac{1}{\nu}\int_0^t K(t,s,x)(f(s,x) - \nabla p(s,x))\,\mathrm{d}s && \text{in } (0,T)\times\Omega,\\
\operatorname{div}(v(t,x)) &= -\frac{d}{dt}|Y_x^*(t)| && \text{in } (0,T)\times\Omega,\\
p(t,x) &= p_b(t,x) && \text{on } (0,T)\times\partial\Omega
\end{aligned}
$$

with $K(t,s,x)_{ij} = \int_{Y_x^*(t)} u_i(t,s,x,y)\cdot e_j\,\mathrm{d}y$ and u_i the solution of

$$
\begin{aligned}
\partial_t u_i - \Delta u_i + \nabla \pi_i &= 0 && \text{for } t\in(s,T) \text{ and } y \in Y_x^*(t),\\
\operatorname{div}(u_i) &= 0 && \text{for } t\in(s,T) \text{ and } y \in Y_x^*(t),\\
u_i &= 0 && \text{for } t\in(s,T) \text{ and } y \in \Gamma_x(t),\\
y &\mapsto u_i(y), \pi(y) && Y\text{-periodic},\\
u_i(s,s,x) &= e_i && \text{for } x \in Y_x^*(s).
\end{aligned}
$$

Outline

Microscopic domain

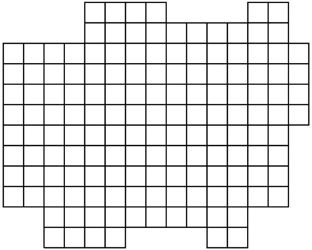


$$I_\varepsilon \subset \mathbb{Z}^n,$$

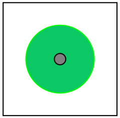
$$Y = (0,1)^n,$$

$$\Omega = \operatorname{int}\Big(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y} \Big),$$

Microscopic domain

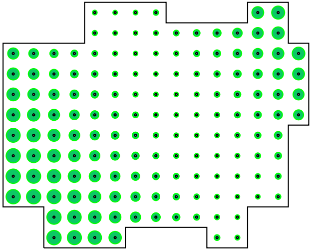


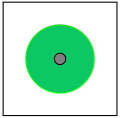
$$I_\varepsilon \subset \mathbb{Z}^n,$$
$$Y = (0,1)^n,$$
$$\Omega = \operatorname{int}\Big(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y} \Big),$$



$$Y_r^* = Y \setminus \overline{B_r(\mathfrak{m})},$$
$$\mathfrak{m} = (0.5, \dots, 0.5)^\top,$$
$$r_{\varepsilon,k} \in [r_{\min}, r_{\max}],$$
$$0 < r_{\min} < r_{\max} < 0.5.$$

Microscopic domain

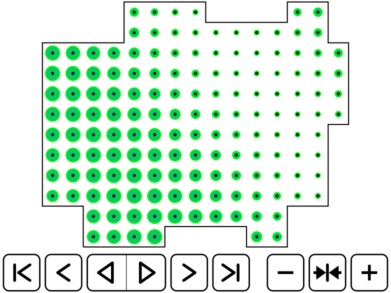


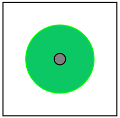


$$I_\varepsilon \subset \mathbb{Z}^n,$$
$$Y = (0,1)^n,$$
$$\Omega = \operatorname{int}\Big(\bigcup_{k\in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y}\Big),$$
$$\Omega_\varepsilon = \operatorname{int}\Big(\bigcup_{k\in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y^*_{r_{\varepsilon,k}}}\Big),$$
$$\Gamma_{\varepsilon,k} = \varepsilon \partial B_{r_{\varepsilon,k}}(k+\mathfrak{m}),$$

$$Y^*_r = Y \setminus \overline{B_r(\mathfrak{m})},$$
$$\mathfrak{m} = (0.5,\dots,0.5)^\top,$$
$$r_{\varepsilon,k} \in [r_{\min}, r_{\max}],$$
$$0 < r_{\min} < r_{\max} < 0.5.$$

Microscopic domain

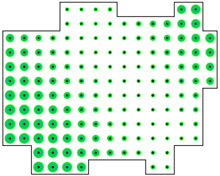




$$I_\varepsilon \subset \mathbb{Z}^n,$$
$$Y = (0,1)^n,$$
$$\Omega = \operatorname{int}\Big(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y} \Big),$$
$$\Omega_\varepsilon(t) = \operatorname{int}\Big(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y^*_{r_{\varepsilon,k}(t)}} \Big),$$
$$\Gamma_{\varepsilon,k}(t) = \varepsilon \partial B_{r_{\varepsilon,k}(t)}(k + \mathfrak{m}),$$

$$Y^*_r = Y \setminus \overline{B_r(\mathfrak{m})},$$
$$\mathfrak{m} = (0.5, \dots, 0.5)^\top,$$
$$r_{\varepsilon,k} \in C^{0,1}([0,T];[r_{\min}, r_{\max}]),$$
$$0 < r_{\min} < r_{\max} < 0.5.$$

Microscopic reaction–diffusion process with coupled microstructure evolution



$$\Omega_\varepsilon^T := \bigcup_{t\in(0,T)} \{t\} \times \Omega_\varepsilon(t) \subset (0,T) \times \Omega,$$

$$\Gamma_{\varepsilon,k}^T := \bigcup_{t\in(0,T)} \{t\} \times \Gamma_{\varepsilon,k}(t),$$

$$\begin{aligned}
\partial_t u_\varepsilon - \operatorname{div}(D\nabla u_\varepsilon) &= f && \text{in } \Omega_\varepsilon^T,\\
-D\nabla u_\varepsilon \cdot n + \varepsilon \partial_t r_{\varepsilon,k} u_\varepsilon &= \varepsilon g(u_\varepsilon, r_{\varepsilon,k}) && \text{on } \Gamma_{\varepsilon,k}^T \text{ für } k \in I_\varepsilon,\\
-D\nabla u_\varepsilon \cdot n &= 0 && \text{on } \partial\Omega,\\
\partial_t r_{\varepsilon,k} &= \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}(t)} g(u_\varepsilon, r_{\varepsilon,k}) d\sigma_x && \text{for } k \in I_\varepsilon,
\end{aligned}$$

Transformation of the microproblem

$$\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t), \qquad \psi_\varepsilon = \psi_\varepsilon(r_\varepsilon),$$

Transformation of the microproblem

$$\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t), \qquad \psi_\varepsilon = \psi_\varepsilon(r_\varepsilon),$$

$$\Psi_\varepsilon(t,x) = \partial_x \psi_\varepsilon(t,x), \qquad \Psi_\varepsilon = \Psi_\varepsilon(r_\varepsilon),$$

$$J_\varepsilon(t,x) = \det(\Psi_\varepsilon(t,x)), \qquad J_\varepsilon = J_\varepsilon(r_\varepsilon),$$

$$A_\varepsilon(t,x) = \operatorname{Adj}(\Psi_\varepsilon(t,x)) \ (= J_\varepsilon \Psi_\varepsilon^{-1}) \qquad A_\varepsilon = A_\varepsilon(r_\varepsilon),$$

$$\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t), \qquad \psi_\varepsilon = \psi_\varepsilon(r_\varepsilon),$$

$$\Psi_\varepsilon(t,x) = \partial_x \psi_\varepsilon(t,x), \qquad \Psi_\varepsilon = \Psi_\varepsilon(r_\varepsilon),$$

$$J_\varepsilon(t,x) = \det(\Psi_\varepsilon(t,x)), \qquad J_\varepsilon = J_\varepsilon(r_\varepsilon),$$

$$A_\varepsilon(t,x) = \mathrm{Adj}(\Psi_\varepsilon(t,x)) \ (= J_\varepsilon \Psi_\varepsilon^{-1}) \qquad A_\varepsilon = A_\varepsilon(r_\varepsilon),$$

$$\begin{aligned}
&\partial_t(J_\varepsilon \hat{u}_\varepsilon) - \mathrm{div}(A_\varepsilon D \Psi_\varepsilon^{-\top} \nabla \hat{u}_\varepsilon + A_\varepsilon \partial_t \psi_\varepsilon \hat{u}_\varepsilon) = J_\varepsilon \hat{f}_\varepsilon && \text{in } (0,T) \times \Omega_\varepsilon, \\
&- A_\varepsilon D \Psi_\varepsilon^{-\top} \nabla \hat{u}_\varepsilon \cdot n + \varepsilon \partial_t r_{\varepsilon,k} \hat{u}_\varepsilon = \varepsilon g(\hat{u}_\varepsilon, r_{\varepsilon,k}) && \text{on } (0,T) \times \Gamma_{\varepsilon,k}, \\
&- A_\varepsilon D \Psi_\varepsilon^{-\top} \nabla \hat{u}_\varepsilon \cdot n = 0 && \text{on } (0,T) \times \partial\Omega, \\
&\partial_t r_{\varepsilon,k} = \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} g(\hat{u}_\varepsilon, r_{\varepsilon,k}) d\sigma_x && \text{for } k \in I_\varepsilon.
\end{aligned} \tag{R-D}$$

Existence and uniform a priori estimates

Theorem

There exists a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ *of* (R-D)*, such that* $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega)')$*.*

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Theorem

There exists a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ *of* (R-D), *such that* $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega)')$. *Moreover,*

$$\begin{aligned} \|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leq C, \\ \|\partial_t(J_\varepsilon \hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} &\leq C. \end{aligned}$$

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Theorem

There exists a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ *of* (R-D), *such that* $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega)')$. *Moreover,*

$$\|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leq C,$$

$$\|\partial_t(J_\varepsilon \hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} \leq C.$$

But:

$$\|\partial_t \hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} \leq C\varepsilon^{-1}$$

Theorem

There exists a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ *of* (R-D), *such that* $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega)')$. *Moreover,*

$$\begin{aligned} \|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leq C, \\ \|\partial_t(J_\varepsilon \hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} &\leq C. \end{aligned}$$

Lemma

$$\|\hat{u}_\varepsilon(\cdot + h) - \hat{u}_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \xrightarrow{h\to 0} 0 \qquad \textit{uniformly with respect to } \varepsilon.$$

Lemma

Let

$$\|\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\nabla\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leq C,$$

$$\|\hat{u}_\varepsilon(\cdot + h) - \hat{u}_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \xrightarrow{h\to 0} 0 \qquad \textit{uniformly with respect to } \varepsilon.$$

Then, there exists $\hat{u}_0 \in L^2((0,T)\times\Omega)$ *and a subsequence* ε*, such that* $\hat{u}_\varepsilon \twoheadrightarrow \chi_{Y^*}\hat{u}_0$.

Lemma

Let

$$\|\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\nabla\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leq C,$$

$$\|\hat{u}_\varepsilon(\cdot + h) - \hat{u}_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \xrightarrow{h\to 0} 0 \qquad \textit{uniformly with respect to } \varepsilon.$$

Then, there exists $\hat{u}_0 \in L^2((0,T)\times\Omega)$ *and a subsequence* ε*, such that* $\hat{u}_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \chi_{Y^*}\hat{u}_0$*.*

Lemma

Let $\hat{u}_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \chi_{Y^*}\hat{u}_0$*, then* $r_\varepsilon \to r_0$ *and* $J_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow J_0$*,* $\Psi_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \Psi_0$*.*

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Step 1: Standard estimates

Step 2:

- $\left\|\delta_{h_t}\hat{u}_\varepsilon\right\|_{L^2((0,T)\times\Omega_\varepsilon)} \to 0 \Rightarrow \hat{u}_\varepsilon \to \hat{u}_0 \Rightarrow r_\varepsilon \to r_0.$

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

- $\left\|\delta_{h_{t,x}} r_\varepsilon\right\|_{L^2((0,T)\times\Omega_\varepsilon)} \to 0 \Rightarrow r_\varepsilon \to r_0, \qquad \cdots \Rightarrow u_\varepsilon \to u_0.$

M. Gahn, I. S. Pop.
Homogenization of a mineral dissolution and precipitation model involving free boundaries at the micro scale.
Journal of Differential Equations, 2023.

Find $(u_0, r) \in L^2(0,T;H^1(\Omega)) \times C^{0,1}([0,T];L^2(\Omega))$, such that

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Coupled effective reactive–diffusive transport

Find $(u_0, r) \in L^2(0,T;H^1(\Omega)) \times C^{0,1}([0,T];L^2(\Omega))$, such that

$$\partial_t(\theta(r)u_0) + \operatorname{div}(A^*(r)\nabla u_0) = \theta(r)f + \partial_t\theta(r)c_s$$
$$\partial_t r = c_s^{-1} g(u_0, r)$$

where $\theta(r) = |Y_r^*|$

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Find $(u_0, r) \in L^2(0,T;H^1(\Omega)) \times C^{0,1}([0,T];L^2(\Omega))$, such that

$$\begin{aligned} \partial_t(\theta(r)u_0) + \operatorname{div}(A^*(r)\nabla u_0) &= \theta(r)f + \partial_t\theta(r)c_s \\ \partial_t r &= c_s^{-1} g(u_0, r) \end{aligned}$$

where $\theta(r) = |Y_r^*|$ and A^* is given by

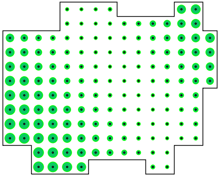
$$A^*_{ij}(r) = \int_{Y_r^*} \delta_{ij} + \partial_{y_i} w_j(r;y)\,\mathrm{d}y$$

$$\begin{aligned} -\operatorname{div}(\nabla w_j) &= 0, && \text{in } Y_r^*, \\ \nabla_y w_j \cdot n &= -e_j \cdot n, && \text{on } \partial B_r(\mathfrak{m}), \\ y \mapsto w_j &\text{ periodic.} \end{aligned}$$

D. Wiedemann, M. A. Peter.
Homogenisation of local colloid evolution induced by reaction and diffusion.
Nonlinear Analysis, 2023.

Outline

Coupled Stokes-advektions–reactions–diffusion process



$$\Omega_\varepsilon^T := \bigcup_{t\in[0,T]} \{t\} \times \Omega_\varepsilon(t) \subset [0,T] \times \Omega,$$

$$\Gamma_{\varepsilon,k}^T := \bigcup_{t\in[0,T]} \{t\} \times \Gamma_{\varepsilon,k}(t),$$

$$\begin{aligned}
\partial_t u_\varepsilon - \operatorname{div}(D\nabla u_\varepsilon - v_\varepsilon u_\varepsilon) &= f && \text{in } \Omega_\varepsilon^T,\\
-D\nabla u_\varepsilon \cdot n &= \varepsilon g(u_\varepsilon, r_{\varepsilon,k}) && \text{on } \Gamma_{\varepsilon,k}^T \text{ for } k \in I_\varepsilon,\\
u_\varepsilon &= 0 && \text{on } \partial\Omega,\\
\partial_t r_{\varepsilon,k} &= \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}(t)} g(u_\varepsilon, r_{\varepsilon,k}) d\sigma_x && \text{for } k \in I_\varepsilon,\\
-\varepsilon^2 \operatorname{div}(\nu(\nabla v_\varepsilon + \nabla v_\varepsilon^\top)) + \nabla p_\varepsilon &= h && \text{in } \Omega_\varepsilon^T,\\
\operatorname{div}(v_\varepsilon) &= 0 && \text{in } \Omega_\varepsilon^T,\\
v_\varepsilon = v_{\Gamma_\varepsilon} &= -\varepsilon \partial_t r_{\varepsilon,k} n && \text{on } \Gamma_\varepsilon^T,\\
-\varepsilon^2 \nu(\nabla v_\varepsilon + \nabla v_\varepsilon^\top)^\top n + p_\varepsilon n &= p_b n && \text{on } \partial\Omega.
\end{aligned}$$

Fluid equation:

$$\|\hat{v}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \hat{v}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|p_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C.$$

Advection–reaction–diffusion equation

$$\begin{aligned} \|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leq C, \\ \|\hat{u}_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} &\leq C, \\ \|\partial_t (J_\varepsilon \hat{u}_\varepsilon)\|_{L^2(0,T;H^1_{\partial\Omega}(\Omega_\varepsilon)')} &\leq C. \end{aligned}$$

M. Gahn, I. S. Pop, M. A. Peter, D. Wiedemann.
Homogenisation of a coupled advection–reaction–diffusion process for evolving microstructure.
in preparation.

Two-scale compactness result

Lemma

Let $u_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon)) \cap L^\infty(0,T;L^2(\Omega_\varepsilon))$ and $J_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)$ with $\partial_t(J_\varepsilon u_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)')$ such that $J_\varepsilon \geq c_J > 0$,

$$\|\partial_t(J_\varepsilon u_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} + \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla u_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leq C,$$

$$\|J_\varepsilon(\cdot+h) - J_\varepsilon\|_{L^\infty(\Omega_\varepsilon;L^2((0,T-h)))} \xrightarrow{h\to 0} 0 \text{ uniformly with respect to } \varepsilon.$$

Then, $\|u_\varepsilon(\cdot+h) - u_\varepsilon\|_{L^2((0,T-h);L^2(\Omega_\varepsilon))} \xrightarrow{h\to 0} 0$ uniformly with respect to ε.

In particular there exists $u_0 \in L^2((0,T)\times\Omega)$ such that $u_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \chi_{Y^*} u_0$.

Advection–reaction–diffusion equation

$$\partial_t(\theta(r)u_0) + \operatorname{div}(A_{\text{hom}}(r)\nabla u_0 - vu_0) = \theta(r)f^{\text{p}} + \partial_t\theta(r)c_s$$

Darcy equation

$$v = \frac{1}{\nu}K(r)\left(h - \nabla p\right)$$

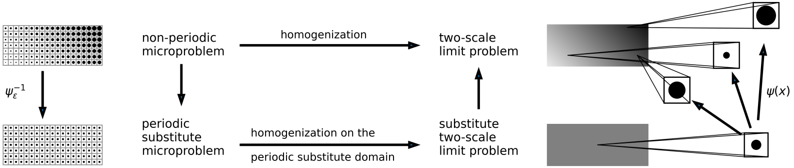
$$\operatorname{div}(v) = -\frac{d}{dt}\theta(r)$$

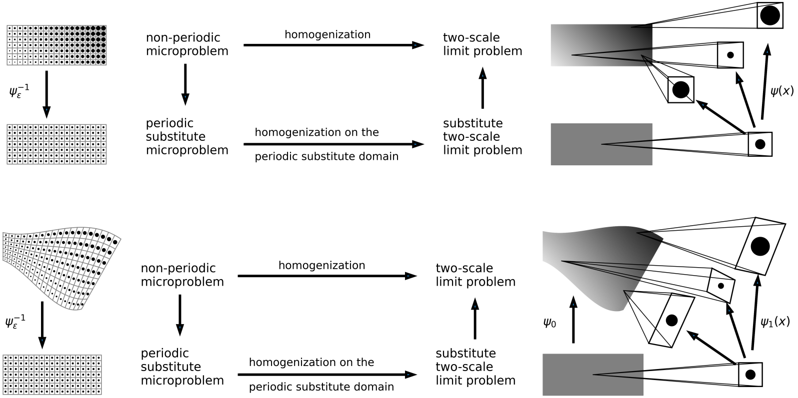
Microstructure evolution

$$\partial_t r = c_s^{-1} f(u_0, r)$$

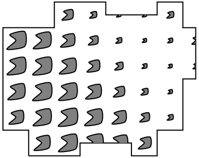
Outline

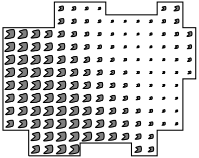
Macro- and micro-evolution

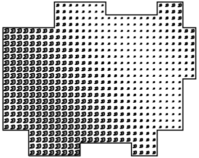


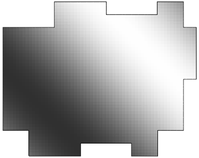

non-periodic microproblem
homogenization
two-scale limit problem
ψ_ε^{-1}
periodic substitute microproblem
homogenization on the periodic substitute domain
substitute two-scale limit problem
$\psi(x)$
non-periodic microproblem
homogenization
two-scale limit problem
ψ_ε^{-1}
periodic substitute microproblem
homogenization on the periodic substitute domain
substitute two-scale limit problem
ψ_0
$\psi_1(x)$

Pulsating microstructure

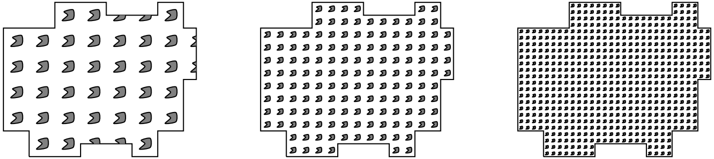








Pulsating microstructure



Conclusion

- Homogenization in non-periodic domains by the two-scale-transformation method
- Homogenization of Stokes flow for evolving microstructure
  - Korn-type inequality for two-scale transformations
  - Darcy's law (with memory) for evolving microstructure
- Homogenization of a reaction–diffusion process with coupled microscopic domain evolution
  - existence and compactness results
  - Limit problem: parabolic PDE on fixed domain coupled with family of ODEs
- Stokes- advection–reaction–diffusion process with free microscopic boundary

Thank you for your attention.

D. Wiedemann. The two-scale-transformation method, 2023.

D. Wiedemann, M. A. Peter. Homogenisation of the Stokes equations for evolving microstructure, 2021.

D. Wiedemann, M. A. Peter. Homogenisation of local colloid evolution induced by reaction and diffusion, 2023.

D. Wiedemann. Analytical homogenisation of transport processes in evolving porous media, 2023.

M. Gahn, I. S. Pop, M. A. Peter, D. Wiedemann. Homogenisation of a coupled advection–reaction–diffusion process for evolving microstructure, in preparation.