Three-dimensional steady water waves with vorticity

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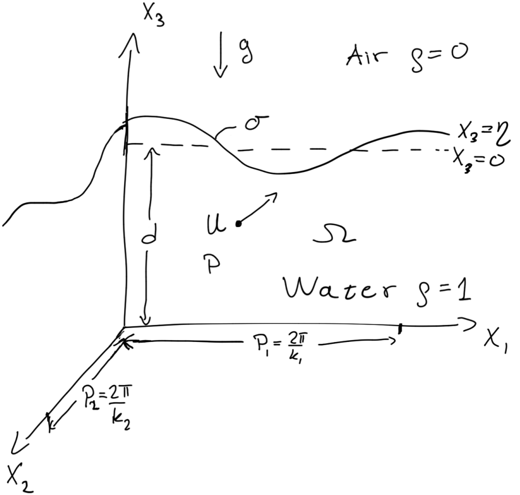


Ω - Domain
η - Surface
u - Velocity field
p - Pressure
g - Gravity
σ - Surface tension
ρ - Density
d - Depth
P_i - x_i-period

We study the problem under the traveling wave assumption, i.e. that the waves move with some constant horizontal velocity ν.

That is, we assume $\eta(x', t) = \eta(x' - \nu t)$ and $u(x, t) = u(x - \nu t)$ and move the problem to the frame traveling with constant speed ν to obtain the steady equations.

The velocity and pressure satisfy the steady Euler equations

$$\begin{aligned} (u \cdot \nabla)u + \nabla p + g e_3 &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

in the domain Ω. If no liquid leaves or enters the domain we also have the kinematic boundary condition

$$u \cdot n = 0$$

on $\partial\Omega$.

for the 2 or 3-dimensional water wave problem we let

$$\Omega = \{x \in \mathbb{R}^n : -d < x_3 < \eta(x')\}$$

for some unknown function η, which makes this a free boundary problem. To solve this requires an additional boundary condition; the dynamic boundary condition

$$p|_{x_n=\eta} + \sigma\nabla\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) = 0.$$

We want periodic waves, so we also require η to be periodic (with respect to some lattice in three dimensions).

To get a well posed problem we also add some form of integral condition depending on the constant(s) c_1 or (c_1, c_2) specifying the (average) flow in either one or both of the periodic directions.

$$\frac{1}{|\hat{\Omega}|}\int_{\hat{\Omega}} u_i \, dx = c_i, \qquad i = 1, 2.$$

Here $\hat{\Omega}$ is one ‘period’ of Ω. Let $c = c_1$ in 2D and $c = (c_1, c_2)$ in 3D.

We briefly consider a flow without vorticity, i.e. $\omega \coloneqq \nabla \times u = 0$. In this case we can introduce a potential ϕ satisfying $\nabla\phi = u$. Furthermore the vector calculus identity

$$\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \times (\nabla \times u)$$

means that in this case the first of the Euler equations is satisfied if $p = -\frac{1}{2}|u|^2 - gx_3 + Q$ and the second one is satisfied if $\Delta\phi = 0$.

This means we can first treat the problem

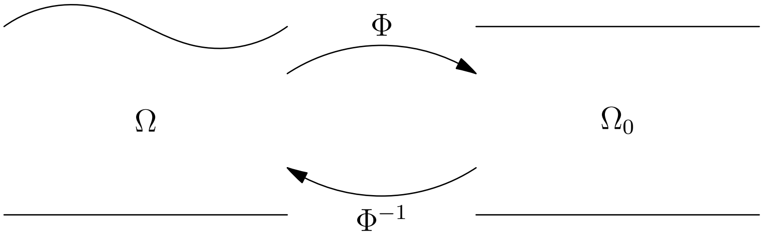
$$\begin{aligned} \Delta\phi &= 0 && \text{in } \Omega \\ n \cdot \nabla\phi &= 0 && \text{on } \partial\Omega \\ &+\text{integral conditions} \end{aligned}$$

for any given surface η. Then plug the solution $\phi[\eta, c]$ into the dynamic boundary condition at the free surface to get

$$\frac{1}{2}|\nabla\phi[\eta, c]|^2|_{x_n=\eta} + g\eta - \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right) = Q(c).$$

For simplicity we now look at the two dimensional problem. Note that we can simply assume the functions to be independent of x_2 to obtain the two dimensional problem.

To be find the η dependence of $\phi = \phi[\eta, c]$ we solve the problem in the flattened domain. For small amplitude waves we can easily find some Φ flattening the domain.



With $\phi[\eta, c]$ in hand we can solve the dynamic boundary condition as an equation

$$F(\eta, c) = 0,$$

using the Crandall-Rabinowitz bifurcation theorem. To apply this the requirements are the following:

$$\begin{aligned} F(0,c) &= 0 \text{ for all } c \\ L :&= D_\eta F(0,c^*) \text{ is a Fredholm operator} \\ &\qquad\qquad\qquad \text{with index } 0 \\ \ker L &= \mathsf{span}\{\eta_1\} \\ (I-P)D_\eta D_c F(0,c^*)[\eta_1, 1] &\neq 0. \text{ (transversality condition)} \end{aligned}$$

This means we can use a Lyapunov-Schmidt reduction to solve the problem. Considering

$$
\begin{aligned}
PF(s\eta_1 + \tilde{\eta}, c) &= 0 \\
(I - P)F(s\eta_1 + \tilde{\eta}, c) &= 0,
\end{aligned}
$$

where P is the projection onto $\mathsf{im} L$ and

$$\eta = (I - P)\eta + P\eta = s\eta_1 + \tilde{\eta}.$$

We can find $\tilde{\eta}(s, c)$ solving the first problem and plugging this into the second problem to find $c(s)$. This gives a solution $(\eta(s), c(s))$ in some neighbourhood of $(0, c^*)$ for all s close enough to 0.

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This means we have to study the linearization of

$$F(\eta, c) = \frac{1}{2}|\nabla\phi[\eta, c]|^2|_{x_3=\eta} + g\eta - \sigma\nabla\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) - Q(c).$$

which is given by

$$D_\eta F(0, c)[\eta] = c\partial_x D_\eta\phi(0, c)[\eta] + g\eta - \sigma\partial_x^2\eta$$

Using the periodicity we express the surface as a Fourier series

$$\eta = \sum_m \hat{\eta}_m e^{imkx_1}.$$

In the linearization it is sufficient to study the Fourier modes independently and we obtain

$$D_\eta F(0,c)[\hat{\eta}_m e^{imkx_1}] = \big(g + \sigma(mk)^2 - c^2 mk \coth(mkd)\big)\, \hat{\eta}_m e^{imkx_1}.$$

It is not difficult to find c^* such that $m = \pm 1$ are the only solutions to

$$\big(g + \sigma(mk)^2 - (c^*)^2 mk \coth(mkd)\big) = 0$$

Requiring η to be real and even gives that $D_\eta F(0, c^*)$ is a Fredholm operator with index 0 and kernel spanned by $\cos(kx_1)$.
It also clear that

$$D_c D_\eta F(0, c^*)[\cos(kx_1), 1] = -2c^* \coth(kd) \neq 0,$$

that is, the transversality condition is satisfied.

The same procedure in three dimensions gives the following linearization

$$D_\eta F(0,c)[\hat{\eta}_{m,n} e^{i(mkx_1+nlx_2)}] =$$
$$\left(g + \sigma |\boldsymbol{k}_{m,n}|^2 - \frac{(\boldsymbol{c}\cdot\boldsymbol{k}_{m,n})^2}{|\boldsymbol{k}_{m,n}|} \coth(|\boldsymbol{k}_{m,n}|d) \right) \hat{\eta}_{m,n} e^{i(mkx_1+nlx_2)}.$$

where $\boldsymbol{c} = (c_1, c_2)$ and $\boldsymbol{k}_{m,n} = (mk, nl)$.

- If we only have $(m,n) = \pm(1,0)$ in the kernel we will only recover the two dimensional solutions.
- $\sigma = 0$ gives a small divisor problem.

Some previous results in 3D

- The first rigorous existence result is by Reeder & Shinbrot '81. Gravity-capillary waves on a diamond lattice.
- Extended to general lattices by Craig & Nicholls '00.
- Spacial dynamics approach by Groves & Mielke '01, Haragus & Kirchgässner '01, and Groves & Haragus '03.
- Existence of gravity waves on diamond lattice by Iooss & Plotnikov '09, which they later extended to general lattices.
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Returning to the vector calculus identity from before

$$\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \times (\nabla \times u)$$

we see that the first Euler equation can be satisfied with the same expression for the pressure as long as $u \times (\nabla \times u) = 0$. Not only if $\nabla \times u = 0$. This is true in the case of a Beltrami flow, i.e.

$$\nabla \times u = \alpha u.$$

We reduce this problem to the surface by solving

$$
\begin{aligned}
\nabla \times u &= \alpha u && \text{in } \Omega \\
\nabla \cdot u &= 0 && \text{in } \Omega \\
u \cdot n &= 0 && \text{on } \partial\Omega \\
&\quad + \text{integral conditions}
\end{aligned}
$$

Solving the remaining equations requires us to find Fréchet derivatives of u.

Flattening and solving around a 'base flow' turns the problem into

$$
\begin{aligned}
\nabla \times u - \alpha u &= G(u, \eta) && \text{in } \Omega_0 \\
\nabla \cdot u &= 0 && \text{in } \Omega_0 \\
u_3 &= c_1 \partial_{x_1} \eta + c_2 \partial_{x_2} \eta && \text{on } x_3 = 0 \\
u_3 &= 0 && \text{on } x_3 = -d \\
&\text{+integral conditions}
\end{aligned}
$$

which has a unique solution $u(\eta, c)$ that can be plugged into the dynamic boundary condition

$$(c_1 u_1 + c_2 u_2)|_{x_3=0} + g\eta - \sigma \Delta \eta = R(u, \eta).$$

This equation can be solved by applying a Lyapunov-Schmidt reduction just as in the irrotational case. The difference here is that we have a two-dimensional kernel and two bifurcation parameters c_1 and c_2. This ensures that we find truly three-dimensional flows.

Theorem (Lokharu, Wahlén, S. '20)

There exists a sheet of solutions to the three dimensional water wave problem under the traveling wave assumption, where the surface is doubly periodic with respect to some lattice and the water have nonzero vorticity.

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For a third time we return to the vector calculus identity

$$\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \times (\nabla \times u)$$

Now we use an ansatz by Lortz from magnetohydrodynamics. Let

$$\nabla \times u = \beta \nabla h(q) \times \nabla \tau,$$

where

$$u \cdot \nabla \tau = 1$$
$$\tau|_{x_1=0} = 0$$

and

$$q = \tau(\cdot + L_1 e_1) - \tau$$

Then we get

$$u \times (\nabla \times u) = \beta \nabla h(q),$$

which means the first euler equation is solved if we set

$$p = \beta h(q) - \frac{1}{2}|u|^2 - g x_3 + Q.$$

The equations that we have to solve to reduce the problem to the boundary is

$$\begin{aligned} \nabla \times u &= \beta \nabla h(q) \times \nabla \tau && \text{in } \Omega \\ \nabla \cdot u &= 0 && \text{in } \Omega \\ n \cdot u &= 0 && \text{on } \partial\Omega \\ &\quad + \text{integral conditions} \end{aligned}$$

We find a solution again using the ideas of Lortz. As long as $u_1 > \delta > 0$ we can find τ and q for any given u. Then we can find v solving the equations above for any given τ and q. This gives us an operator $T(u) = v$. A fixed point of T is the solution we seek.

- One problem that arises is to preserve the periodicity. This can be resolved by imposing certain symmetries on u, which makes $\nabla h(q) \times \nabla \tau$ periodic. Additionally, $\nabla h(q) \times \nabla \tau$ will satisfy the symmetries required for v to satisfy the same symmetries as u.
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This analysis is performed in flattened coordinates to obtain $u(\eta, c)$ to plug into the dynamic boundary condition as in the Beltrami case. However in this case there is a complication that makes it impossible to proceed as before. The mapping $(\eta, c) \mapsto u(\eta, c)$ is not differentiable. At least not in the natural function spaces.
This comes from the fact that

$$
\begin{aligned}
u \cdot \nabla \tau &= 1 \\
\tau|_{x_1} &= 0
\end{aligned}
$$

gives a $\tau[u] \in C^s$ for every $u \in C^s$ that is not differentiable with respect to u as a mapping $C^s \to C^s$.

If we view $u \mapsto \tau[u]$ as a a mapping $C^s \to C^{s-2}$ it is differentiable and as a mapping $C^s \to C^{s-3}$ it is twice differentiable.

Without going into the technical details this eventually leads to the mapping $(\eta, c) \mapsto u(\eta, c)$, which actually maps $C^{s+1} \times \mathbb{R} \to C^s$, is only twice differentiable as a mapping $C^{s+1} \times \mathbb{R} \to C^{s-4}$.

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Plugging this $u(\eta, c)$ into the dynamic boundary condition gives us the surface equation

$$F(\eta, c) + R(\eta, c) = 0$$

where $F(\eta, c)$ is the same as in the irrotational case and $R(\eta, c)$ consists the 'problematic' terms. With some weak conditions imposed on R we can find a solution by performing a Lyapunov-Schmidt reduction as long as F satisfies the same conditions as in the irrotational case.

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The first step is to find the $\tilde{\eta}(s, c)$ solving

$$PF(s\eta_1 + \tilde{\eta}, c) + PR(s\eta_1 + \tilde{\eta}, c) = 0$$

This is done by solving the equivalent fixed point equation

$$G(\tilde{\eta}) := \tilde{\eta} - L^{-1}PF(s\eta_1 + \tilde{\eta}, c) - L^{-1}PR(s\eta_1 + \tilde{\eta}, c) = \tilde{\eta}.$$

For sufficiently small $|\beta|$ the operator G is a contraction because $R \to 0$ as $|\beta| \to 0$.

The second step is to find $c(s)$ solving

$$H(s,c) := (I-P)F(s\eta_1+\tilde{\eta}(s,c),c)+(I-P)R(s\eta_1+\tilde{\eta}(s,c),c) = 0.$$

This can be done because even though both R and $\tilde{\eta}$ are only differentiable with some loss of regularity H is differentiable because it is a mapping between finite dimensional spaces. Moreover the fact that $D_\eta D_c R[0,c] = 0$ (in the sense that it is differentiable) allows us to use the implicit function theorem in the same manner as if $R \equiv 0$.

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The final obstacle is to prove that we indeed get a solution that is not irrotational. This can be done both implicitly by a contradiction argument or explicitly by expanding the solutions. The latter is preferable because it gives more information about the properties of the solutions.

Theorem (Varholm, Wahlén, S.)

There exists a curve of solutions to the three dimensional water wave problem under the traveling wave assumption, where the surface is doubly periodic with respect to some lattice and the water have nonzero vorticity. Moreover, the velocity field of the water is not a Beltrami field.

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Thank you for listening!