Relative energy approach to a diffuse interface model of a compressible two-phase flow

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Outline of the talk

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1. Introduction of the model
2. Dissipative weak solutions
3. Relative energy inequality
4. Applications: weak-strong uniqueness principle
5. Applications: low Mach number limit
6. Applications: some numerics on the model

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Introduction of the model

Aim: describe the dynamics of a binary mixture of compressible, viscous and macroscopically immiscible fluids in a bounded domain

Possible approaches:
1.Classical approach: the interface between the two fluids is sharp and is evolving in time along with the fluid, the movement of the interface at each time is determined by a set of interfacial balance conditions
2.Phase-field approach: the interface between the two fluids is seen as a transition layer of small but non-zero width

Illustration of a diffuse interface versus a sharp interface

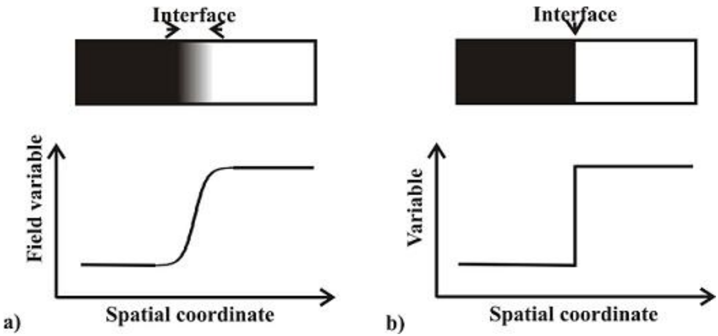

Figure: Illustration of a diffuse interface and of a sharp interface for a mixture of two immiscible fluids

Phase-field approach

- Interface of finite width ε
- c is an order parameter (e.g. the concentration difference) that takes value 1 in the bulk of one fluid and value -1 in the bulk of the other fluid, varying continuously between -1 and 1 at the interface
- c satisfies an equation of Allen-Cahn type
- the fluid density and respectively velocity $(\rho, \boldsymbol{v})$ satisfy the compressible Navier-Stokes equations coupled to the Allen-Cahn equation through a capillarity force $(\mathrm{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2 \mathbb{I}\right))$

Advantages: Captures the evolution of complex interfaces and allows the treatment of topological changes of the interface. Easier to use in order to study numerically the dynamics of the interface, since the simulations can be done on a fixed grid.

The compressible Navier-Stokes-Allen-Cahn Model

The Blesgen model:

$$\partial_t \rho + \operatorname{div}_x(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, c) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - \operatorname{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right), \tag{2}$$

$$\partial_t(\rho c) + \operatorname{div}_x(\rho c \mathbf{u}) = -\mu, \tag{3}$$

$$\rho \mu = -\Delta_x c + \rho \frac{\partial f(\rho, c)}{\partial c}. \tag{4}$$

The tensor $\mathbb{S}(\nabla_x \mathbf{u}) = \nu \left(\nabla_x \mathbf{u}^t + \nabla_x^t \mathbf{u} - \frac{2}{N} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}, \ \nu > 0.$
The free energy is $E_{\text{free}}(\rho, c, \nabla_x c) = \frac{1}{2} |\nabla_x c|^2 + \rho f(\rho, c)$.
The pressure is derived from the free energy $p(\rho, c) = \rho^2 \frac{\partial f(\rho, c)}{\partial \rho}$.

Overview on the existing results

Theoretical study of the model: existence of weak and strong solutions, presence of initial vacuum

Blesgen (1999): introduction of the model

Feireisl, Petzeltová, Rocca, Schimperna (2010): existence of global weak solutions when the initial density is bounded and away from vacuum, $\gamma > 6$ where $\rho f(\rho, c) = \rho^\gamma + h(\rho, c)$

Kotschote (2012): existence and uniqueness of local strong solutions for arbitrary initial data for a more advanced model where the stress tensor is multiplied by the density

Ding, Li, Luo (2013): existence and uniqueness of a global strong solution for the $1D$ case without initial vacuum, $\gamma > 1$

Chen, Guo (2017): existence and uniqueness of a global strong solution for the $1D$ case with initial vacuum, $\gamma > 1$

Chen, Wen, Zhu (2019): global existence of weak solution with finite energy for the case $\gamma > 2$, $\rho_0 \geqslant 0$

Overview on the existing results

Theoretical study of the model: stationary solutions, travelling waves, stability study

Freistühler (2014): existence of travelling waves for the model introduced by Kotschote (2012)

Axmann, Mucha (2016): existence of weak stationary solutions with bounded densities, $\gamma > 3$

Kotschote (2017): spectral analysis for travelling waves

A simplified version

We consider the free energy in a simplified form

$$E_{\text{free}} = \frac{1}{2}|\nabla_x c|^2 + F_c(c) + F_e(\rho), \tag{5}$$

yielding the pressure

$$p(\rho, c) = p_e(\rho) - F_c(c),\ \ p_e(\rho) = \rho F_e'(\rho) - F_e(\rho)\,. \tag{6}$$

The Allen-Cahn equation taken in a simplified form:

$$\partial_t c + \mathbf{u} \cdot \nabla_x c = \Delta_x c - F_c'(c). \tag{7}$$

Boundary conditions: $\mathbf{u} = 0$, $c = c_b$ on $\partial\Omega$.

$$\begin{aligned} & p_e \in C[0,\infty) \cap C^\infty(0,\infty), \\ & p_e'(\rho) > 0 \text{ for } \rho > 0,\ \liminf_{\rho\to\infty} p_e'(\rho) > 0,\ p_e(\rho) \leqslant c\,(1 + F_e(\rho)) \text{ for all } \rho \geqslant 0, \end{aligned} \tag{8}$$

$$F_c \in C^\infty(R),\ \ F_c''(c) > 0 \text{ for all } c \in (-\infty, -\overline{c}] \cup [\overline{c}, \infty),\ \ \overline{c} > 0. \tag{9}$$

Typically $F_e(\rho) = a\rho^\gamma$, $a > 0$, $\gamma > 1$, $F_c = (c^2 - 1)^2$ regular double well potential.

Dissipative weak solutions

Let the initial data,

$$\rho(0,\cdot) = \rho_0,\ \rho\mathbf{u}(0,\cdot) = (\rho\mathbf{u})_0,\ c(0,\cdot) = c_0, \tag{10}$$

be given in the class

$$\begin{gathered}\rho_0 \geqslant 0 \text{ a.e. in } \Omega,\ \int_\Omega \left[\frac{|(\rho\mathbf{u})_0|^2}{\rho_0} + F_e(\rho_0)\right]\ \mathrm{d}x < \infty, \\ c_0 \in W^{1,2}(\Omega)\cap L^\infty(\Omega),\ c_0|_{\partial\Omega} = c_b.\end{gathered} \tag{11}$$

We search for $[\rho, \mathbf{u}, c]$ in the class of functions

$$\rho, F_e(\rho) \in L^\infty(0,T; L^1(\Omega)),\ \rho \geqslant 0 \text{ a.a. in } (0,T)\times\Omega,$$
$$\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^N)),$$
$$c \in L^\infty(0,T; W^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega)) \cap L^\infty((0,T)\times\Omega),\ c|_{\partial\Omega} = c_b;$$

Dissipative weak solutions

- The integral identity for ρ

$$\left[\int_\Omega \rho\varphi \; \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\rho\partial_t\varphi + \rho\mathbf{u}\cdot\nabla_x\varphi] \; \mathrm{d}x\,\mathrm{d}t, \; \forall\tau \geqslant 0, \; \varphi \in C^1([0,T]\times\overline{\Omega});$$

- The integral identity for the momentum equation

$$\left[\int_\Omega \rho\mathbf{u}\cdot\boldsymbol{\varphi} \; \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\rho\mathbf{u}\partial_t\boldsymbol{\varphi} + \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p_e(\rho)\mathrm{div}_x\boldsymbol{\varphi}] \; \mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} \; \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right) : \nabla_x\boldsymbol{\varphi} \; \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega F_c(c)\mathrm{div}_x\boldsymbol{\varphi} \; \mathrm{d}x\,\mathrm{d}t, \; \forall\,\tau \geqslant 0,$$
$$\forall\,\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega}; R^N), \; \boldsymbol{\varphi}|_{\partial\Omega} = 0,$$

Dissipative weak solutions

- The Allen-Cahn equation:

$$\partial_t c + \mathbf{u} \cdot \nabla_x c = \Delta_x c - F_c'(c), \text{ a.a. } (0, T) \times \Omega, \ c(0, \cdot) = c_0, \ c|_{\partial\Omega} = c_b;$$

- The energy inequality:

$$\left[\int_\Omega \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + F_e(\rho) + F_c(c) \right] \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega [\Delta_x c - F_c'(c)]^2 \ \mathrm{d}x \, \mathrm{d}t \leqslant 0, \ \forall \tau \geqslant 0.$$

Weak-strong uniqueness

Theorem

Let $\Omega \subset R^N$, $N = 1, 2, 3$ *be a bounded Lipschitz domain. Let* $[\rho, \mathbf{u}, c]$ *be a dissipative weak solution in* $(0, T) \times \Omega$ *in the sense specified above. Suppose that the same problem admits a classical solution* $[r, \mathbf{U}, C]$, $r > 0$ *defined on the same time interval. Then*

$$\rho = r, \ \mathbf{u} = \mathbf{U}, \ c = C \text{ in } (0, T) \times \Omega.$$

Low Mach number limit

After rescaling the elastic pressure:

$$\begin{aligned} \partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) &= 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p_e(\rho) &= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \\ -\mathrm{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) &+ \nabla_x F_c(c), \\ \partial_t c + \mathbf{u} \cdot \nabla_x c &= \Delta_x c - F_c'(c), \end{aligned} \tag{12}$$

with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ c|_{\partial\Omega} = 0. \tag{13}$$

We study the limit when $\varepsilon \to 0$.

Low Mach number limit

We prove the convergence towards the incompressible limit:

$$\begin{aligned} \mathrm{div}_x \mathbf{U} &= 0, \\ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} + \nabla_x \Pi &= \nu \Delta_x \mathbf{U} - \mathrm{div}_x (\nabla_x C \otimes \nabla_x C), \\ \partial_t C + \mathbf{U} \cdot \nabla_x C &= \Delta_x C - F_c'(C), \end{aligned} \tag{14}$$

with the boundary conditions $\mathbf{U}|_{\partial\Omega} = 0, \ C|_{\partial\Omega} = 0.$

Low Mach number limit

Theorem

Suppose that problem (14) *admits a smooth solution* $[\mathbf{U}, C]$, *with the initial data* $[\mathbf{U}_0, C_0]$, *on a time interval* $[0, T]$. *Suppose*

$$\rho(0,\cdot) = \rho_{0,\varepsilon} = 1 + \varepsilon\rho^{(1)}_{0,\varepsilon}, \ \int_\Omega \rho^{(1)}_{0,\varepsilon} \, \mathrm{d}x = 0,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \ \rho^{(1)}_{0,\varepsilon} \to 0 \text{ in } L^\infty(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^2(\Omega; R^N),$$

$$c(0,\cdot) = c_{0,\varepsilon} \in L^\infty \cap W^{1,2}_0(\Omega), \ \|c_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1, \ c_{0,\varepsilon} \to C_0 \text{ in } W^{1,2}_0(\Omega)$$

as $\varepsilon \to 0$. *Let* $[\rho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]_{\varepsilon>0}$ *be a dissipative weak solution of the compressible NSAC with the initial data* $[\rho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, c_\varepsilon]$, $\|c_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \lesssim 1$. *Then*

$$\rho_\varepsilon(t,\cdot) \to 1 \text{ in } L^1(\Omega), \ \mathbf{u}_\varepsilon(t,\cdot) \to \mathbf{U}(t,\cdot) \text{ in } L^2(\Omega; R^N), \ c_\varepsilon(t,\cdot) \to C(t,\cdot) \text{ in } W^{1,2}(\Omega)$$

uniformly for $t \in [0, T]$.

Relative energy

$\mathcal{E}\left(\rho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right)$ = a "distance" between a weak dissipative solution and some regular functions

Relative entropy technique: long time behavior of certain quasilinear parabolic equations when the regular function is a stationary solution, incompressible and/or inviscid limits, weak-strong uniqueness

Ukai (1986), Grenier (1997), Masmoudi (2001), Carillo, Jüngel, Markowich, Toscani, Unterreiter (2001), Wang, Jiang (2006), Saint-Raymond (2009), Leger, Vasseur (2011), Feireisl, Jin, Novotny (2012), Feireisl, Lu, Malek (2014), Feireisl, Lu, Novotny (2018)

Incompressible NSAC:

Zhao, Guo, Huang (2012): vanishing viscosity limit

Hosek, Macha (2018): weak-strong uniqueness

Relative energy:

$$\mathcal{E}\left(\rho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) = \int_\Omega \left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2 + \frac{1}{2}|\nabla_x c - \nabla_x C|^2 + F_e(\rho) - F_e'(r)(\rho - r) - F_e(r)\right] \,\mathrm{d}x.$$

Relative energy inequality

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2\right] \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant - \int_0^\tau \int_\Omega \left[F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)\right] \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\rho(\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \rho \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x^t \mathbf{U} \cdot (\mathbf{U} - \mathbf{u})\right] \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\rho \partial_t F_e'(r) + \rho \mathbf{u} \cdot \nabla_x F_e'(r)\right] \, \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega p_e(\rho) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I}\right) : \nabla_x \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \left[\mu - \mathbf{u} \cdot \nabla_x c\right] \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \left(\frac{1}{2} |\nabla_x C|^2 + p_e(r)\right) \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
$$

for all $(r, C, \mathbf{U})$ regular enough.

Weak-strong uniqueness

Idea: Take $[r, \mathbf{U}, C]$ a strong solution of the problem, use it as test function in the relative energy inequality and apply a Gronwall type argument.
Convective term in the equation of continuity:

$$\int_0^\tau \int_\Omega \left[\rho(\mathbf{U}-\mathbf{u}) \cdot \partial_t \mathbf{U} + \rho \mathbf{u} \cdot \nabla_x^t \mathbf{U} \cdot (\mathbf{U}-\mathbf{u})\right] \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \left[\rho(\mathbf{U}-\mathbf{u}) \cdot \partial_t \mathbf{U} + \rho(\mathbf{U}-\mathbf{u}) \cdot \nabla_x^t \mathbf{U} \cdot \mathbf{U}\right] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \rho(\mathbf{u}-\mathbf{U}) \cdot \nabla_x^t \mathbf{U} \cdot (\mathbf{U}-\mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_0^\tau \int_\Omega \left[\left(\frac{\rho}{r}-1\right)(\mathbf{U}-\mathbf{u}) \cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\Big)\right] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ c_1 \int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u} \,\Big|\, r, C, \mathbf{U}\right) \, \mathrm{d}t + \int_0^\tau \int_\Omega \left[(\mathbf{U}-\mathbf{u}) \cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C\Big)\right] \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) : \mathbb{S}(\nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t.$$

We can prove that:

$$\int_0^\tau \int_\Omega \left[\left(\frac{\rho}{r} - 1 \right) (\mathbf{U} - \mathbf{u}) \cdot \left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \right) \right] \, \mathrm{d}x \, \mathrm{d}t$$
$$\leqslant c_2 \int_0^\tau \mathcal{E} \left(\rho, c, \mathbf{u} \middle| r, C, \mathbf{U} \right) \, \mathrm{d}t + \frac{1}{2} \int_0^\tau \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \right) \, \mathrm{d}x \, \mathrm{d}t.$$

Technical ingredient: We introduce a cut-off function $\Psi \in C_c^\infty(0, \infty)$,
$0 \leqslant \Psi \leqslant 1$, $\Psi \equiv 1$ in $[\delta, \frac{1}{\delta}]$, where δ s.t. $r(t, x) \in [2\delta, \frac{1}{2\delta}] \ \forall \ (t, x) \in [0, T] \times \overline{\Omega}$.
For $h \in L^1((0, T) \times \Omega)$, we set
$h = h_{\mathrm{ess}} + h_{\mathrm{res}}$, $h_{\mathrm{ess}} = \Psi(\rho) h$, $h_{\mathrm{res}} = (1 - \Psi(\rho)) h$. We can prove:

$$F_e(\rho) - F_e'(r)(\rho - r) - F_e(r) \gtrsim (\rho - r)^2_{\mathrm{ess}} + (1 + \rho)_{\mathrm{res}}$$

and

$$\mathcal{E} \left(\rho, \mathbf{u}, c \middle| r, \mathbf{U}, C \right) \gtrsim \int_\Omega \left(\rho |\mathbf{u} - \mathbf{U}|^2 + [\rho - r]^2_{\mathrm{ess}} + 1_{\mathrm{res}} + \rho_{\mathrm{res}} \right) \, \mathrm{d}x.$$

Thus,

$$\int_0^\tau \int_\Omega \left[\left(\frac{\rho}{r} - 1 \right) (\mathbf{U} - \mathbf{u}) \cdot \left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \right) \right] \, \mathrm{d}x \, \mathrm{d}t$$

$$\lesssim \int_0^\tau \int_\Omega |\rho - r| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_0^\tau \int_\Omega |[\rho - r]_{\mathrm{ess}}| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega |[\rho - r]_{\mathrm{res}}| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant c(\delta) \int_0^\tau \int_\Omega [\rho - r]_{\mathrm{ess}}^2 + 1_{\mathrm{res}} + \rho_{\mathrm{res}} + \rho |\mathbf{u} - \mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t + \delta \int_0^\tau \int_\Omega |\mathbf{u} - \mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t$$

Using the Korn–Poincaré inequality

$$\int_\Omega |\mathbf{u} - \mathbf{U}|^2 + |\nabla_x (\mathbf{u} - \mathbf{U})|^2 \, \mathrm{d}x \leqslant c_{kp} \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x,$$

we get the desired inequality.

Elastic pressure terms:

$$
\begin{aligned}
-\int_0^\tau \int_\Omega [\rho\partial_t F_e'(r) + \rho\mathbf{u}\cdot\nabla_x F_e'(r)] \, \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega p_e(\rho)\mathrm{div}_x\mathbf{U} \, \mathrm{d}x\,\mathrm{d}t \\
- \int_0^\tau \int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla_x p_e(r) \, \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \partial_t p_e(r) \, \mathrm{d}x\,\mathrm{d}t = \\
\int_0^\tau \int_\Omega \left[p_e(r) - p_e'(r)(r-\rho) - p_e(\rho)\right]\mathrm{div}_x\mathbf{U} + (r-\rho)(\mathbf{u}-\mathbf{U})\cdot\nabla_x F_e'(r) \, \mathrm{d}x\,\mathrm{d}t \\
\lesssim \int_0^\tau \int_\Omega F_e(\rho) - F_e'(r)(\rho - r) - F_e(r) \, \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega |r-\rho||\mathbf{u}-\mathbf{U}| \, \mathrm{d}x\,\mathrm{d}t \\
\leqslant \frac{1}{4}\int_0^\tau \int_\Omega \Big[(\mathbb{S}(\nabla_x\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U})) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) \Big] \, \mathrm{d}x\,\mathrm{d}t + c_2\int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right) \, \mathrm{d}t,
\end{aligned}
$$

where we used:

$\partial_t F_e'(r) + \mathbf{U}\cdot\nabla_x F_e'(r) + p_e'(r)\mathrm{div}_x\mathbf{U} = 0$, $\partial_t p_e(r) = r\partial_t F_e'(r)$,
$\nabla_x p_e(r) = r\nabla_x F_e'(r)$.

Terms containing the order parameter:

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \frac{1}{4} \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega |\Delta_x c - \Delta_x C|^2 \ \mathrm{d}x \, \mathrm{d}t \\
&\leqslant \int_0^\tau \int_\Omega \Delta_x C (\nabla_x C - \nabla_x c) \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \nabla_x \mathbf{U} : \left[\nabla_x (C - c) \otimes \nabla_x (C - c) - \frac{1}{2} |\nabla_x (C - c)|^2 \mathbb{I}\right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \mathrm{div}_x \left(\mathbf{U} - \mathbf{u}\right) \left(F_c(c) - F_c(C)\right) \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\Delta_x (C - c)(F_c'(c) - F_c'(C))\right] \ \mathrm{d}x \, \mathrm{d}t + c_4 \int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right) \ \mathrm{d}t.
\end{aligned}
$$

Using the boundedness of c and C, we get:

$$\left|\mathrm{div}_x\,(\mathbf{U}-\mathbf{u})\,(F_c(c)-F_c(C))\right| + \left|\Delta_x(C-c)(F_c'(c)-F_c'(C))\right| \lesssim \big(|\nabla_x(\mathbf{u}-\mathbf{U})| + |\Delta_x c - \Delta_x C|\big)|c-C|.$$

Using the Korn-Poincaré inequality, we obtain:

$$\left[\mathcal{E}\left(\rho, c, \mathbf{u}\,\middle|\, r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \lesssim \int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u}\,\middle|\, r, C, \mathbf{U}\right)\ \mathrm{d}t;$$

whence Gronwall's lemma completes the proof.

Application: incompressible limit

Existence of the limit problem (incompressible NSAC):

Lin, Liu (1995): Existence of global weak solutions, existence of strong solution globally in 2D and locally in 3D

Thanks to the energy inequality and the continuity equation, $[\rho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]$ satisfy:

$$\left[\int_\Omega \left[\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{2}|\nabla_x c_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(F_e(\rho_\varepsilon) - F_e'(1)(\rho_\varepsilon - 1) - F_e(1)\right) + F_c(c_\varepsilon)\right] \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega [\Delta_x c_\varepsilon - F_c'(c_\varepsilon)]^2 \ \mathrm{d}x\,\mathrm{d}t \leqslant 0,$$

as $F_e'(1)(\rho_\varepsilon - 1) + F_e(1)$ is a constant of motion.

Since

$$\frac{1}{\varepsilon^2}\left(F_e(\rho_{0,\varepsilon}) - F_e'(1)(\rho_{0,\varepsilon} - 1) - F_e(1)\right) \lesssim \left(\frac{\rho_{0,\varepsilon} - 1}{\varepsilon}\right)^2,$$

$[\rho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]_{\varepsilon>0}$ admits energy bounds uniformly for $\varepsilon \to 0$, and converge weakly to some triple $[\rho, \mathbf{u}, c]$.

We can write

$$\frac{1}{\varepsilon^2}\left(F_e(\rho_\varepsilon) - F_e'(1)(\rho_\varepsilon - 1) - F_e(1)\right) \gtrsim \left(\frac{\rho_\varepsilon - 1}{\varepsilon}\right)^2_{\mathrm{ess}} + \frac{(1+\rho_\varepsilon)_{\mathrm{res}}}{\varepsilon^2}. \tag{15}$$

Using the boundedness of the energy

$$\rho_\varepsilon(t,\cdot) \to 1 \text{ in } L^1(\Omega) \text{ uniformly in } t \in [0,T]. \tag{16}$$

Passing to the limit in the equation of continuity,

$$\mathrm{div}_x \mathbf{u} = 0. \tag{17}$$

It remains to identify the couple $[\mathbf{u}, c]$ with the solution $[\mathbf{U}, C]$ of the limit, incompressible model. We use the relative energy inequality with $r \equiv 1$, and $[\mathbf{U}, C]$ solution of the incompressible NSAC.

$$\mathcal{E}_\varepsilon\left(\rho, \mathbf{u}, c \,\middle|\, \mathbf{U}, C\right) = \int_\Omega \left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2 + \frac{1}{2}|\nabla_x c - \nabla_x C|^2 + \frac{1}{\varepsilon^2}\left(F_e(\rho) - F_e'(1)(\rho-1) - F_e(1)\right)\right]\,\mathrm{d}x$$

$$\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\middle|\, C, \mathbf{U}\right)\Big]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \left[\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : (\nabla_x\mathbf{u}_\varepsilon - \nabla_x\mathbf{U}) + \mu_\varepsilon^2\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant -\int_0^\tau\int_\Omega \left[F_c(c_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon + \mu_\varepsilon F_c'(c_\varepsilon)\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \left[\rho_\varepsilon(\mathbf{U}-\mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{U} + \rho_\varepsilon\mathbf{u}_\varepsilon\cdot(\mathbf{U}-\mathbf{u}_\varepsilon)\cdot\nabla_x\mathbf{U}\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^\tau\int_\Omega (\nabla_x c_\varepsilon\otimes\nabla_x c_\varepsilon) : \nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \partial_t\frac{1}{2}|\nabla_x C|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \partial_t(\Delta_x C)c_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \Delta_x C\left[\mu_\varepsilon - \mathbf{u}_\varepsilon\cdot\nabla_x c_\varepsilon\right]\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\mu_\varepsilon = \Delta_x c_\varepsilon - F_c'(c_\varepsilon).$$

Using the same treatment for the convective terms, we get:

$$\begin{aligned}
&\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\middle|\, C, \mathbf{U}\right)\Big]_{t=0}^{t=\tau} + \frac{1}{2}\int_0^\tau\int_\Omega \left[(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2\right]\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant -\int_0^\tau\int_\Omega \left[F_c(c_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon + \mu_\varepsilon F_c'(c_\varepsilon)\right]\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon)\cdot(-\nabla_x \Pi - \mathrm{div}_x(\nabla_x C \otimes \nabla_x C))\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_\Omega (\nabla_x c_\varepsilon \otimes \nabla_x c_\varepsilon) : \nabla_x \mathbf{U}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \partial_t(\Delta_x C)c_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \Delta_x C\left[\mu_\varepsilon - \mathbf{u}_\varepsilon\cdot\nabla_x c_\varepsilon\right]\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \partial_t \frac{1}{2}|\nabla_x C|^2\,\mathrm{d}x\,\mathrm{d}t + c_1\int_0^\tau \mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\middle|\, C, \mathbf{U}\right)\mathrm{d}t.
\end{aligned} \tag{18}$$

The only term different to the weak-strong uniqueness case is the one in $\nabla_x \Pi - \nabla_x F_c(C)$:

$$\left[\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \lesssim \int_0^\tau \mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, C, \mathbf{U}\right) dt + \int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \nabla_x(\Pi - F_c(C)) \,\mathrm{d}x\,\mathrm{d}t.$$

From the assumptions on the initial conditions $\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, C, \mathbf{U}\right)(0) \to 0$ as $\varepsilon \to 0$.

Using the continuity equation we also have:

$$\int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \nabla_x \tilde{\Pi} \,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau \int_\Omega (1 - \rho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \tilde{\Pi} \,\mathrm{d}x\,\mathrm{d}t - \left[\int_\Omega \rho_\varepsilon \tilde{\Pi} \,\mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \rho_\varepsilon \partial_t \tilde{\Pi} \,\mathrm{d}x\,\mathrm{d}t,$$

with $\tilde{\Pi} = \Pi - F_c(C)$ and this term converges uniformly to 0 over $\tau \in [0, T]$.

Application: numerical approximation

Work in collaboration with Eduard Feireisl and Bangwei She:
1. Discretise the problem by a stable and consistent scheme
2. Prove that the numerical solutions converge to a dissipative weak solution and thus, by the weak-strong uniqueness principle, prove the unconditional convergence to a strong solution if the later exists

Starting point:
T. Karper: A convergent FEM-DG method for compressible Navier-Stokes system, 2013 ($\gamma > 3$)
E. Feireisl and M. Lukáčová-Medvid'ová: Convergence of a mixed finite element–discontinuous Galerkin scheme for the isentropic Navier-Stokes system via dissipative measure-valued solutions, 2018 ($\gamma > 1$)
Here and hereafter we suppose that

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p(0)=0,\ p'(\rho) > 0 \text{ for } \rho > 0;$$
$$\text{the pressure potential } P \text{ determined by } P'(\rho)\rho - P(\rho) = p(\rho) \text{ satisfies } P(0)=0,$$
$$\text{and } P - \underline{a}p,\ \overline{a}p - P \text{ are convex functions for certain constants } \underline{a} > 0,\ \overline{a} > 0, \tag{19}$$

which implies that there exists $\gamma > 1$ such that

$$P(\rho) \geqslant a\rho^{\gamma} \text{ for some } a > 0 \text{ and all } \rho \geqslant 1. \tag{20}$$

Mesh

Let $\mathcal{T}$ be a regular, periodic and quasi-uniform triangulation of $\Omega \equiv \left([-1,1]|_{\{-1,1\}}\right)^d$.
Notation:

- $\Omega = \cup_{K\in\mathcal{T}} K$. For any element K we denote $|K|$ its volume and h_K its diameter. $h = \max_{K\in\mathcal{T}} h_K$ is the size the mesh.
- $\mathcal{E}$ the set of all faces, $\mathcal{E}(K)$ the set of faces of an element $K \in \mathcal{T}$. By $|\sigma|$ we denote the volume of the face $\sigma \in \mathcal{E}$. Note that each $\sigma \in \mathcal{E}$ is an interior edge due to the periodicity assumption, i.e., there exist two different elements $K \in \mathcal{T}$ and $L \in \mathcal{T}$ such that $\sigma = \mathcal{E}(K) \cap \mathcal{E}(L)$ for all $\sigma \in \mathcal{E}$, which we often note $\sigma = K|L$.
- For each face $\sigma \in \mathcal{E}$, we denote by $\mathbf{n}$ its outer normal vector. If $\sigma \in \mathcal{E}(K)$ (resp. $\mathcal{E}(L)$) we write it as $\mathbf{n}_K$ (resp. $\mathbf{n}_L$).
- Assume $|K| \approx h^d$ and $|\sigma| \approx h^{d-1}$ for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$.

Function spaces

$\mathcal{P}^{\ell}_{d}(K)$ the space of polynomials of degree not greater than ℓ on K for $d-$dimensional vector-valued functions.

$$Q_h = \{v \in L^1(\Omega) | v_K \in \mathcal{P}^0_1(K)\ \forall K \in \mathcal{T}\},$$

$$\mathbf{V_h} = \{\mathbf{v} \in L^2(\Omega) | \mathbf{v}_K \in \mathcal{P}^1_d(K)\ \forall K \in \mathcal{T}; \int_{\sigma} [[\mathbf{v}]]\,\mathrm{d}\sigma = 0\ \forall \sigma \in \mathcal{E}\},$$

$$X_h = \{v \in L^2(\Omega) | v_K \in \mathcal{P}^1_1(K)\ \forall K \in \mathcal{T}\},$$

where

$$v^{\mathrm{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \mathbf{n}), \quad v^{\mathrm{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \mathbf{n}),$$

$$\{\!\{v\}\!\}(x) = \frac{v^{\mathrm{in}}(x) + v^{\mathrm{out}}(x)}{2}, \quad [[v]] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

whenever $x \in \sigma \in \mathcal{E}$.

The associated projection operator for Q_h is:

$$\Pi_h^Q : L^1(\Omega) \to Q_h,$$

where

$$\Pi_h^Q \phi = \sum_{K \in \mathcal{T}} \frac{1_K(x)}{|K|} \int_K \phi \, \mathrm{d}x, \quad 1_K = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

We shall frequently use the notation $\widehat{\phi} = \Pi_h^Q \phi$. For any $K \in \mathcal{T}$ we denote:

$$\nabla_h v|_K = \nabla_x v|_K, \quad \mathrm{div}_h \mathbf{u}|_K = \mathrm{div}_x \mathbf{u}|_K$$

for any $v \in \mathbf{V_h} \cup X_h$, $\mathbf{u} \in \mathbf{V_h}$.

Discrete Laplace operator

For any $v \in X_h$ we define $\Delta_h v \in W_h := \{v \in X_h | \int_\Omega v \, \mathrm{d}x = 0\}$ such that

$$-\int_\Omega \Delta_h v \, w \, \mathrm{d}x = B(v, w) \quad \text{for any } w \in W_h, \tag{21}$$

$$B(v, w) = \int_\Omega \nabla_h v \cdot \nabla_h w \, \mathrm{d}x + \sum_{\sigma \in \mathcal{E}} \int_\sigma \left([[w]]\mathbf{n} \cdot \{\!\{\nabla_h v\}\!\} + [[v]]\mathbf{n} \cdot \{\!\{\nabla_h w\}\!\} + \frac{c_B}{h^{1+\beta}} [[v]][[w]] \right) \mathrm{d}\sigma,$$

where $c_B > 0$ is a sufficient large constant to ensure the coercivity. The following identity holds:

$$B(v, v - w) = \frac{1}{2} B(v, v) - \frac{1}{2} B(w, w) + \frac{1}{2} B(v - w, v - w).$$

Furthermore, we define the following seminorms

$$|||v|||^2 = \|\nabla_h v\|_{L^2}^2 + \sum_{\sigma \in \mathcal{E}} \int_\sigma \left(\frac{1}{h^{1+\beta}} [[v]]^2 + h \{\!\{\nabla_h v\}\!\}^2 \right) \mathrm{d}\sigma, \quad |||v|||_B^2 = B(v, v),$$

and $|||v||| \approx |||v|||_B$.

Diffusive upwind flux

Given $\mathbf{v} \in \mathbf{V_h}$, the upwind flux for any function $r \in Q_h$ is specified at each face $\sigma \in \mathcal{E}$ by

$$\mathrm{Up}[r, \mathbf{v}]|_\sigma = r^{\mathrm{up}} \mathbf{v}_\sigma \cdot \mathbf{n} = r^{\mathrm{in}} [\mathbf{v}_\sigma \cdot \mathbf{n}]^+ + r^{\mathrm{out}} [\mathbf{v}_\sigma \cdot \mathbf{n}]^- = \{\!\{r\}\!\}\ \mathbf{v}_\sigma \cdot \mathbf{n} - \frac{1}{2} |\mathbf{v}_\sigma \cdot \mathbf{n}| [[r]],$$

where

$$\mathbf{v}_\sigma = \frac{1}{|\sigma|} \int_\sigma \mathbf{v} \, \mathrm{d}\sigma, \quad [f]^\pm = \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\mathrm{up}} = \begin{cases} r^{\mathrm{in}} & \text{if } \mathbf{v}_\sigma \cdot \mathbf{n} \geqslant 0, \\ r^{\mathrm{out}} & \text{if } \mathbf{v}_\sigma \cdot \mathbf{n} < 0. \end{cases}$$

Diffusive numerical flux:

$$\mathrm{F}_h^{\mathrm{up}}(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\varepsilon [[r]], \ \varepsilon > 0.$$

Numerical scheme

$$\int_\Omega D_t \rho_h^k \phi_h \, \mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathrm{F}_h^{\mathrm{up}}(\rho_h^k, \mathbf{u}_h^k)[[\phi_h]]\,\mathrm{d}\sigma = 0, \quad \forall \phi_h \in Q_h;$$

$$\int_\Omega D_t(\rho_h^k \widehat{\mathbf{u}_h^k}) \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{F}_h^{\mathrm{up}}(\rho_h^k \widehat{\mathbf{u}_h^k}, \mathbf{u}_h^k) \cdot [[\widehat{\boldsymbol{\varphi}_h}]]\,\mathrm{d}\sigma + \nu \int_\Omega \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\varphi}_h \, \mathrm{d}x$$

$$+ \eta \int_\Omega \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\varphi}_h \, \mathrm{d}x = \int_\Omega p_h^k \mathrm{div}_h \boldsymbol{\varphi}_h \, \mathrm{d}x + \int_\Omega (f_h^k - \Delta_h \mathfrak{c}_h^k) \nabla_h \mathfrak{c}_h^k \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x, \ \forall \boldsymbol{\varphi}_h \in \mathbf{V_h};$$

$$\int_\Omega (D_t \mathfrak{c}_h^k + \mathbf{u}_h^k \cdot \nabla_h \mathfrak{c}_h^k)\psi_h \, \mathrm{d}x = \int_\Omega \left(\Delta_h \mathfrak{c}_h^k - f_h^k\right)\psi_h \, \mathrm{d}x, \quad \forall \psi_h \in X_h,$$

where $D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$ for all $k = 1, \dots, N_T$, $p_h^k = p(\rho_h^k)$, $\eta = \frac{d-2}{d}\nu + \lambda > 0$ and

$$f_h^k = \begin{cases} 2(\mathfrak{c}_h^k + 1) & \text{if } \mathfrak{c}_h^k \in (-\infty, -1), \\ (\mathfrak{c}_h^k)^3 - \mathfrak{c}_h^{k-1} & \text{if } \mathfrak{c}_h^k \in [-1, 1], \\ 2(\mathfrak{c}_h^k - 1) & \text{if } \mathfrak{c}_h^k \in (1, \infty). \end{cases}$$

Some estimates

Lemma (Sobolev inequality)

Let $r \geqslant 0$ be a function defined on $\Omega \subset \mathbb{R}^d$ such that

$$0 < c_M \leqslant \int_\Omega r \; \mathrm{d}x, \; \textit{and} \int_\Omega r^\gamma \; \mathrm{d}x \leqslant c_E \; \textit{for} \; \gamma > 1,$$

where c_M and c_E are some positive constants. Then

$$\|v_h\|^2_{L^q(\Omega)} \lesssim c(\|\nabla_h v_h\|^2_{L^2(\Omega)} + \int_\Omega r|\hat{v}_h|^2 \; \mathrm{d}x)$$

for any $v_h \in \mathbf{V_h}$, and $1 \leqslant q \leqslant 6$ for $d = 3$, $1 \leqslant q < \infty$ for $d = 2$, where $c = c(c_M, c_E)$.

Stability

Conservation of mass:

$$\int_\Omega \rho_h^k \,\mathrm{d}x = \int_\Omega \rho_h^{k-1} \,\mathrm{d}x = \cdots = \int_\Omega \rho_h^0 \,\mathrm{d}x = \int_\Omega \Pi_h^Q \rho_0 \,\mathrm{d}x = \int_\Omega \rho_0 \,\mathrm{d}x, \forall k = 1, \ldots, N_T.$$

Discrete energy balance: Let $(\rho_h^k, \mathbf{u}_h^k, \mathfrak{c}_h^k)$ satisfy the numerical scheme, then:

$$D_t \int_\Omega \frac{1}{2}\rho_h^k \left|\widehat{\mathbf{u}_h^k}\right|^2 + P(\rho_h^k) \,\mathrm{d}x + D_t \left(\int_\Omega F(\mathfrak{c}_h^k) \,\mathrm{d}x + \frac{1}{2} |||\mathfrak{c}_h^k|||_B^2 \right)$$
$$+ \nu \|\nabla_h \mathbf{u}_h^k\|_{L^2}^2 + \eta \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2}^2 + \|D_t \mathfrak{c}_h^k + \mathbf{u}_h^k \cdot \nabla_h \mathfrak{c}_h^k\|_{L^2}^2 = -D_{\mathrm{num}},$$

where $D_{\mathrm{num}} \geqslant 0$ is the numerical dissipation

$$D_{\mathrm{num}} = \frac{\Delta t}{2} \int_\Omega \rho_h^{k-1} |D_t \widehat{\mathbf{u}_h^k}|^2 \,\mathrm{d}x + \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\rho_h^k)^{\mathrm{up}} \left|\mathbf{u}_\sigma^k \cdot \mathbf{n}\right| [[\widehat{\mathbf{u}_h^k}]]^2 \,\mathrm{d}\sigma + h^\varepsilon \sum_{\sigma \in \mathcal{E}} \int_\sigma \left\{\left\{ \rho_h^k \right\}\right\} [[\widehat{\mathbf{u}_h^k}]]^2 \,\mathrm{d}\sigma$$
$$+ \frac{\Delta t}{2} \int_\Omega P''(\xi) |D_t \rho_h^k|^2 \,\mathrm{d}x + \sum_{\sigma \in \mathcal{E}} \int_\sigma P''(\zeta) [[\rho_h^k]]^2 \left(h^\varepsilon + \frac{1}{2} |\mathbf{u}_\sigma^k \cdot \mathbf{n}| \right) \mathrm{d}\sigma$$
$$+ \frac{\Delta t}{2} |||D_t \mathfrak{c}_h^k|||_B^2 + \int_\Omega [1 + \frac{1}{2}(1_{|\mathfrak{c}_h^k|>1} + 3(\mathfrak{c}_h^{k,*})^2 1_{|\mathfrak{c}_h^k| \leqslant 1})] \Delta t \left| D_t \mathfrak{c}_h^k \right|^2 \,\mathrm{d}x,$$

where $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$, $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ and $\mathfrak{c}_h^{k,*} \in \mathrm{co}\{\mathfrak{c}_h^{k-1}, \mathfrak{c}_h^k\}$.

Idea of the proof: Take $\varphi_h = \mathbf{u}_h^k \in \mathbf{V_h}$, $\phi_h = \frac{1}{2}\left|\widehat{\mathbf{u}_h^k}\right|^2 \in Q_h$ and substract:

$$\int_\Omega D_t(\rho_h^k \widehat{\mathbf{u}_h^k}) \cdot \mathbf{u}_h^k \, \mathrm{d}x - \int_\Omega D_t \rho_h^k \frac{1}{2}\left|\widehat{\mathbf{u}_h^k}\right|^2 \, \mathrm{d}x + \nu \|\nabla_h \mathbf{u}_h^k\|_{L^2}^2 + \eta \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2}^2$$

$$= \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{F}_h^{\mathrm{up}}(\rho_h^k \widehat{\mathbf{u}_h^k}, \mathbf{u}_h^k) \cdot [[\widehat{\mathbf{u}_h^k}]] \, \mathrm{d}\sigma - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathrm{F}_h^{\mathrm{up}}[\rho_h^k, \mathbf{u}_h^k][[\frac{1}{2}\left|\widehat{\mathbf{u}_h^k}\right|^2]] \, \mathrm{d}\sigma$$

$$+ \int_\Omega p_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x + \int_\Omega (f_h^k - \Delta_h \mathfrak{c}_h^k) \nabla_h \mathfrak{c}_h^k \cdot \mathbf{u}_h^k \, \mathrm{d}x.$$

$$\int_\Omega D_t(\rho_h \widehat{\mathbf{u}}_h)^k \cdot \mathbf{u}_h^k - D_t \rho_h^k \frac{\left|\widehat{\mathbf{u}_h^k}\right|^2}{2} \, \mathrm{d}x = \int_\Omega D_t\left(\frac{1}{2}\rho_h^k \left|\widehat{\mathbf{u}_h^k}\right|^2\right) + \frac{\Delta t}{2}\rho_h^{k-1}\left|D_t \widehat{\mathbf{u}_h^k}\right|^2 \, \mathrm{d}x.$$

$$\sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{F}_h^{\mathrm{up}}(\rho_h^k \widehat{\mathbf{u}_h^k}, \mathbf{u}_h^k) \cdot [[\widehat{\mathbf{u}_h^k}]] - \mathbf{F}_h^{\mathrm{up}}(\rho_h^k, \mathbf{u}_h^k)[[\frac{1}{2}|\widehat{\mathbf{u}_h^k}|^2]] \, \mathrm{d}\sigma$$

$$= -\frac{1}{2}\sum_{\sigma\in\mathcal{E}} \int_\sigma \rho_h^{\mathrm{up},k}[[\widehat{\mathbf{u}_h^k}]]^2 \left|\mathbf{u}_\sigma^k \cdot \mathbf{n}\right| \, \mathrm{d}\sigma - h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \{\{\rho_h^k\}\} [[\widehat{\mathbf{u}_h^k}]]^2 \, \mathrm{d}\sigma,$$

There exist $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ and $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that

$$\int_\Omega D_t P(\rho_h^k)\ \mathrm{d}x + \int_\Omega p(\rho_h^k)\mathrm{div}_h \mathbf{u}_h^k\ \mathrm{d}x$$
$$= -\frac{\Delta t}{2}\int_\Omega P''(\xi)|D_t\rho_h^k|^2\ \mathrm{d}x - \sum_{\sigma\in\mathcal{E}}\int_\sigma P''(\zeta)[[\rho_h^k]]^2\left(h^\varepsilon + \frac{1}{2}|\mathbf{u}_\sigma^k\cdot\mathbf{n}|\right)\mathrm{d}\sigma \leqslant 0.$$

Take $\psi_h = D_t\mathfrak{c}_h^k + \mathbf{u}_h^k\cdot\nabla_h\mathfrak{c}_h^k \in X_h$:

$$\int_\Omega \left|D_t\mathfrak{c}_h^k + \mathbf{u}_h^k\cdot\nabla_h\mathfrak{c}_h^k\right|^2\ \mathrm{d}x = \int_\Omega (\Delta_h\mathfrak{c}_h^k - f_h^k)(D_t\mathfrak{c}_h^k + \mathbf{u}_h^k\cdot\nabla_h\mathfrak{c}_h^k)\ \mathrm{d}x$$
$$= \int_\Omega (\Delta_h\mathfrak{c}_h^k - f_h^k)\mathbf{u}_h^k\cdot\nabla_h\mathfrak{c}_h^k\ \mathrm{d}x - B(\mathfrak{c}_h^k, D_t\mathfrak{c}_h^k) - \int_\Omega f_h^k D_t\mathfrak{c}_h^k\ \mathrm{d}x$$
$$= \int_\Omega (\Delta_h\mathfrak{c}_h^k - f_h^k)\mathbf{u}_h^k\cdot\nabla_h\mathfrak{c}_h^k\ \mathrm{d}x - \frac{1}{2}D_t|||\mathfrak{c}_h^k|||_B^2 - \frac{\Delta t}{2}|||D_t\mathfrak{c}_h^k|||_B^2$$
$$-\int_\Omega D_t F(\mathfrak{c}_h^k) + [1 + \frac{1}{2}(1_{|\mathfrak{c}_h^k|>1} + 3(\mathfrak{c}_h^{k,*})^2 1_{|\mathfrak{c}_h^k|\leqslant 1})]\Delta t\left|D_t\mathfrak{c}_h^k\right|^2\ \mathrm{d}x.$$

Sum all the equations.

Uniform bounds

Lemma

Let $(\rho_h, \mathbf{u}_h, \mathfrak{c}_h)$ *be a solution to the numerical scheme for* $\gamma > 1$. *Then:*

$$\|\rho_h|\widehat{\mathbf{u}}_h|^2\|_{L^\infty L^1} \lesssim 1, \quad \|\rho_h\|_{L^\infty L^\gamma} \lesssim 1, \quad \|\rho_h\widehat{\mathbf{u}}_h\|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \lesssim 1,$$

$$\|\nabla_h \mathbf{u}_h\|_{L^2L^2} \lesssim 1, \quad \|\mathrm{div}_h \mathbf{u}_h\|_{L^2L^2} \lesssim 1, \quad \|\mathbf{u}_h\|_{L^2L^p} \lesssim 1, \quad \|\rho_h\mathbf{u}_h\|_{L^2L^q} \lesssim 1,$$

$$\sup_{t\in(0,T)} |||\mathfrak{c}_h||| \approx \sup_{t\in(0,T)} |||\mathfrak{c}_h(t)|||_B \lesssim 1, \quad \|f_h\|_{L^\infty L^2} \approx \|\mathfrak{c}_h\|_{L^\infty L^2} \lesssim \|F(\mathfrak{c}_h)\|_{L^\infty L^1} \lesssim 1,$$

$$\|\mathfrak{c}_h\|_{L^\infty L^p} \lesssim \sup_{t\in(0,T)} |||\mathfrak{c}_h|||_B + \|\mathfrak{c}_h\|_{L^\infty L^2} \lesssim 1, \quad \|D_t\mathfrak{c}_h + \mathbf{u}_h \cdot \nabla_h \mathfrak{c}_h\|_{L^2L^2} \lesssim 1,$$

$$\|\Delta_h \mathfrak{c}_h\|_{L^2L^2} \lesssim 1, \quad \|D_t\mathfrak{c}_h\|_{L^2L^{3/2}} \lesssim 1.$$

where $p \in [1, \infty)$, $q \in [1, \gamma)$ *if* $d = 2$ *or* $p = 6$, $q = \frac{6\gamma}{\gamma+6}$ *if* $d = 3$.

Existence of a solution and positivity of the density

Given $\rho_0 > 0$. For every $k = 1, \dots, N_T$ there exists a solution $(\rho_h^k, \mathbf{u}_h^k, \mathfrak{c}_h^k) \in Q_h \times \mathbf{V_h} \times X_h$ to the numerical scheme. Moreover, any solution to the numerical scheme preserves the positivity of the density, i.e. $\rho_h^k > 0$ for any $k = 1, \dots, N_T$.

Theorem (Topological degree theory, see Gallouët et al.)

Let $M, N \in \mathbb{N}$. Let $C_1 > \varepsilon > 0$ and $C_2 > 0$ be real numbers. Let

$$V = \{(r, U) \in R^M \times R^N;\ r_i > 0\ \forall\ i = 1, \dots, M\},$$
$$W = \{(r, U) \in R^M \times R^N; |U| \leqslant C_2 \text{ and } \varepsilon < r_i < C_1\ \forall\ i = 1, \dots, M\}.$$

Let $\mathcal{F} : V \times [0, 1] \to R^M \times R^N$ be a continuous mapping and:

1. *$f \in W$ if $f \in V$ satisfies $\mathcal{F}(f, \zeta) = \mathbf{0}$ for all $\zeta \in [0, 1]$;*
2. *Equation $\mathcal{F}(f, 0) = \mathbf{0}$ is a linear system w.r.t. f and admits a solution in W.*

Then there exists $f \in W$ such that $\mathcal{F}(f, 1) = \mathbf{0}$.

The idea of the proof is to construct a mapping $\mathcal{F}$ that satisfies the topological degree theory. Define:

$$V = \left\{(\rho_h^k, \mathbf{U}_h^k) \in Q_h \times \mathcal{M}_h,\ \rho_h^k > 0\right\},$$
$$W = \left\{(\rho_h^k, \mathbf{U}_h^k) \in Q_h \times \mathcal{M}_h,\ \|\mathbf{U}_h^k\| \leqslant C_2, \epsilon < \rho_h^k < C_1\right\},$$

where $\mathbf{U}_h^k := (\mathbf{u}_h^k, \mathfrak{c}_h^k) \in \mathbf{V_h} \times X_h =: \mathcal{M}_h$ and the norm $\|U_h^k\|$ is given by $\|U_h^k\| \equiv \|\mathbf{u}_h^k\|_{L^6} + \|\mathfrak{c}_h^k\|_{L^2}$.

Next, for $\zeta \in [0,1]$ and $U^\star = (\mathbf{u}^\star, \mathfrak{c}^\star)$ we define

$$\mathcal{F} : \; V \times [0,1] \to Q_h \times \mathcal{M}_h, \quad (\rho_h^k, \mathbf{U}_h^k, \zeta) \longmapsto (\rho^\star, U^\star) = \mathcal{F}(\rho_h^k, \mathbf{U}_h^k, \zeta),$$

where $(\rho^\star, U^\star)$ is defined by:

$$\int_\Omega \rho^\star \phi_h \, \mathrm{d}x = \int_\Omega \frac{\rho_h^k - \rho_h^{k-1}}{\Delta t} \phi_h \, \mathrm{d}x - \zeta \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathrm{F}_h^{\mathrm{up}}(\rho_h^k, \mathbf{u}_h^k)[[\phi_h]] \, \mathrm{d}\sigma,$$

$$\begin{aligned}\int_\Omega \mathbf{u}^\star \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x &= \int_\Omega \frac{\rho_h^k \widehat{\mathbf{u}_h^k} - \rho_h^{k-1} \widehat{\mathbf{u}_h^{k-1}}}{\Delta t} \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x + \nu \int_\Omega \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\varphi}_h \, \mathrm{d}x \\ &+ \zeta\eta \int_\Omega \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\varphi}_h \, \mathrm{d}x - \zeta \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h^{\mathrm{up}}(\rho_h^k \widehat{\mathbf{u}_h^k}, \mathbf{u}_h^k) \cdot [[\widehat{\boldsymbol{\varphi}_h}]] \, \mathrm{d}\sigma \\ &- \zeta \int_\Omega p(\rho_h^k) \mathrm{div}_h \boldsymbol{\varphi}_h \, \mathrm{d}x - \zeta \int_\Omega (f_h^k - \Delta_h \mathfrak{c}_h^k) \nabla_h \mathfrak{c}_h^k \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x,\end{aligned}$$

$$\int_\Omega \mathfrak{c}^\star \psi_h \, \mathrm{d}x = \int_\Omega \frac{\mathfrak{c}_h^k - \mathfrak{c}_h^{k-1}}{\Delta t} \psi_h \, \mathrm{d}x + \zeta \int_\Omega \mathbf{u}_h^k \cdot \nabla_h \mathfrak{c}_h^k \psi_h \, \mathrm{d}x - \int_\Omega \left(\Delta_h \mathfrak{c}_h^k - \zeta f_h^k\right) \psi_h \, \mathrm{d}x,$$

for any $\phi_h \in Q_h$ and $\boldsymbol{\varphi}_h \times \psi_h \in \mathcal{M}_h$, where $\boldsymbol{\varphi}_h = (\phi_{1,h}, \ldots, \phi_{d,h})$.

Consistency

Theorem

Let $(\rho_h, \mathbf{u}_h, \mathfrak{c}_h)$ *be a solution of the numerical scheme on* $[0, T]$ *with* $\Delta t \approx h$, $\varepsilon > 0$ *for* $\gamma \geqslant 2$ *and* $0 < \varepsilon < \min\{1, 2(\gamma - 1 - d/3)\}$ *for* $\gamma \in (4d/(1+3d), 2)$. *Then*

$$-\int_\Omega \rho_h^0 \phi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_\Omega [\rho_h \partial_t \phi + \rho_h \mathbf{u}_h \cdot \nabla_x \phi]\ \mathrm{d}x\,\mathrm{d}t + \int_0^T e_{1,h}(t,\phi)\,\mathrm{d}t,$$

$$\begin{aligned}
-\int_\Omega \rho_h^0 \widehat{\mathbf{u}_h^0} \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x &= \int_0^T \int_\Omega \left[\rho_h \widehat{\mathbf{u}_h} \cdot \partial_t \boldsymbol{\varphi} + \rho_h \widehat{\mathbf{u}_h} \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\varphi} + p_h \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\,\mathrm{d}t,\\
&- \nu \int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t - \eta \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^T \int_\Omega (f_h - \Delta_h \mathfrak{c}_h) \nabla_h \mathfrak{c}_h \cdot \boldsymbol{\varphi}\ \mathrm{d}x + \int_0^T e_{2,h}(t,\boldsymbol{\varphi})\,\mathrm{d}t
\end{aligned}$$

$$-\int_\Omega \mathfrak{c}_h^0 \psi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_\Omega \mathfrak{c}_h \partial_t \psi - \mathbf{u}_h \cdot \nabla_x \mathfrak{c}_h \psi + (\Delta_h \mathfrak{c}_h - f_h)\psi\ \mathrm{d}x\,\mathrm{d}t + \int_0^T e_{3,h}(t,\psi)\,\mathrm{d}t,$$

$\forall\, \phi, \boldsymbol{\varphi} \in C_c^2([0,T) \times \Omega)$, $\psi \in C_c^1([0,T) \times \Omega)$ *with* $\|e_{i,h}(\cdot)\|_{L^1(0,T)} \lesssim h^\beta$ *for some* $\beta > 0$.