# Non-local Models in Replicator Dynamics


Nikos Kavallaris
(Joint work with J. Lankeit, M. Winkler)


KAAS, Karlstads University
30-11-2022

"**Evolutionary game theory (EGT)** is the application of game theory to evolving populations in biology. It defines a framework of contests, strategies, and analytics into which Darwinian competition can be modelled. It originated in 1973 with John Maynard Smith and George R. Price's formalisation of contests, analysed as strategies, and **the mathematical criteria** that can be used to predict the results of **competing strategies (Smith & Price (1973)).**
Evolutionary game theory **differs** from classical game theory in focusing more on the **dynamics of strategy change**. This is influenced by the frequency of the **competing strategies** in the population.
Evolutionary game theory has helped to explain the basis of altruistic behaviours in Darwinian evolution. It has in turn become of interest to **economists, sociologists, anthropologists, and philosophers**."
**(Wikipedia)**

**Some basic principles of evolutionary game theory**

1. The main subject of evolutionary game dynamics is to explain how a population of players **update their strategies in the course of a game according to their success**.
2. Strategies with **high pay-off** will spread within the population through **learning, imitation or inheriting processes or even by infection**.
3. There is a variety of different dynamics in evolutionary game theory: **replicator dynamics, imitation dynamics, best response dynamics, Brown-von Neumann-Nash dynamics e.t.c..**

# Replicator Dynamics(Key Ideas)

**Finite Strategy Space**

1. The dynamics most widely used and studied in the literature on evolutionary game theory is the **replicator dynamics**.
2. Such kind of dynamics illustrates the idea that in a dynamic process of evolution a strategy should increase in frequency if it is a **successful strategy**. In other words, **the more successful a strategy is then the more individuals playing (following) this strategy to obtain a higher than average payoff.**
3. Consider a finite strategy space $S = \{1, 2, ..., m\}$, with corresponding frequency (probability) vector $p(t) = (p_1(t), p_2(t), ..., p_m(t))^T$ for any $t \geq 0$. Then $p(t)$ belongs to the invariant simplex

$$S(m) = \left\{ y = (y_1, y_2, ..., y_m)^T \in \mathbb{R}^m : y_i \geq 0, i = 1, 2, ..., m, \sum_{i=1}^{m} y_i = 1 \right\}.$$

# Replicator Dynamics (Pay-off )

1. The game is actually determined by **the pay-off matrix** $A = (a_{ij})$, which is a real $m \times m$ matrix. **Pay-off means expected gain**, and if an individual plays strategy $i$ against another individual following strategy $j$, then the pay-off to player $i$ is defined to be $a_{ij}$ while the pay-off to player $j$ is $a_{ji}$. For symmetric games matrix $A$ is considered to be symmetric.
2. Note that in the case of a biological population **pay-off represents fitness, or reproductive success.**
3. Then the **expected pay-off** for an individual playing strategy $i$ can be expressed as

$$(A \cdot p(t))_i = \sum_{j=1}^{m} a_{ij} p_j(t),$$

whereas the **average pay-off** over the whole population is given by

$$(p(t)^T \cdot A \cdot p(t)) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} p_i(t) p_j(t),$$

(**Bomze (1990)**).

# Replicator dynamics (relevant equations)

1. Consider symmetric games with **infinitely many players** then we can assume that $p_i(t)$ evolve as **differentiable functions**.
2. For biological populations the above assumption means that **infinitely big populations** are considered and their generations **blend continuously** to each other.
3. A reasonable assumption, which is also in agreement with the basic tenet of Darwinism, is that **the per capita rate of growth (i.e. the logarithmic derivative)** $\dot{p}_i/p_i$ is given by the difference between the pay-off for strategy $i$ and the average pay-off. This yields **the replicator dynamics system** ,

$$\dot{p}_i := \frac{dp_i}{dt} = \left( \sum_{j=1}^{m} a_{ij} p_j(t) - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} p_i(t) p_j(t) \right) p_i(t), \quad i = 1, 2, ..., m, \quad t > 0, \tag{1}$$

   (**Bomze (1990)**).
4. The dynamical system (1) actually describes the mechanism that individuals tend to switch to strategies that are doing well.
5. In the context of biology (1) yields that individuals bear offspring tend to use the same strategies as their parents, and thus the fitter the individual, the more numerous his offspring.

# Continuous Strategy Space

1. Consider now games with pure strategies belonging to a **continuum**. For instance, this could be **the aspiration level** of a player or **the size of an investment** in economics (**Hofbauer & Sigmund (2003)** ) or it might arise in situations where the pure strategies correspond to **geographical points** as in economic geography (**Krugman (1996)** ).
2. On the other hand, in biology a continuous strategy space might correspond to some **continuously varying trait** such as the **sex ratio in a birth or the virulence of an infection** (**Haccou & Iwasa (1998)**).
3. Consider the case the strategy set $\Omega$ is an arbitrary, not necessarily bounded, Borel set of $\mathbb{R}^N, N \geq 2$, hence strategies can be identified by $x \in \Omega$.
4. For the case of **symmetric two-player games**, the pay-off can be given by a Borel measurable function $f : \Omega \times \Omega \to \mathbb{R}$, **where** $f(x, y)$ **is the pay-off for player** $1$ **when she follows strategy** $x$ **and player** $2$ **plays strategy** $y$.
5. A population is now characterized by its state, **a probability measure** $\mathcal{P}$ in the measure space $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is the Borel algebra of subsets of $\Omega$. **The average (mean) pay-off of a sub-population in state** $\mathcal{P}$ **against the overall population in state** $\mathcal{Q}$ is given by the form

$$E(\mathcal{P}, \mathcal{Q}) := \int_\Omega \int_\Omega f(x, y)\mathcal{Q}(dy)\mathcal{P}(dx).$$

# Continuous Strategy Space ( cont.)

1. Then, **the success (or lack of success)** of a strategy $x$ followed by population $\mathcal{Q}$ is provided by the difference

$$\sigma(x, \mathcal{Q}) := \int_\Omega f(x,y)\mathcal{Q}(dy) - \int_\Omega \int_\Omega f(x,y)\mathcal{Q}(dy)\mathcal{Q}(dx) = E(\delta_x, \mathcal{Q}) - E(\mathcal{Q}, \mathcal{Q}),$$

   where $\delta_x$ is the unit mass concentrated on the strategy $x$ (Dirac measure).

2. **The evolution in time** of the population state $\mathcal{Q}(t)$ is given by **the replicator dynamics equation**

$$\frac{d\mathcal{Q}}{dt}(A) = \int_A \sigma(x, \mathcal{Q}(t))\mathcal{Q}(t)(dx), \; t > 0, \quad \mathcal{Q}(0) = \mathcal{P}, \tag{2}$$

   for any $A \in \mathcal{A}$, where the time derivative should be understood with respect to the variational norm of a subspace of the linear span $\mathcal{M}$ of $\mathcal{A}$.

3. **The abstract form of equation** (2) does not actually allow us to obtain insight on the form of its solutions and thus a better understanding of the evolutionary dynamics of the corresponding game.

4. We now restrict our attention to measures $\mathcal{Q}(t)$ which, for each $t > 0$, are **absolutely continuous with respect to the Lebesgue measure**, with **probability density** $u(x,t)$. Then the replicator dynamics equation (2) can be reduced to the following integro-differential equation

$$\frac{\partial u}{\partial t} = \left(\int_\Omega f(x,y)u(y,t)\,dy - \int_\Omega \int_\Omega f(z,y)u(y,t)u(z,t)dy\,dz\right)u(x,t), \; t > 0, \;\; x \in \Omega. \tag{3}$$

# An example from economic geography (Krugman 1996)

Let $\Omega$ denote a geographical space and $x \in \Omega$ a location (site) on this space. For example $\Omega$ could be the circle $S^1$ and so $x \in [0, 2\pi)$ means a location on this circle. A fixed large number of firms choose locations on the circle. Let $u(x,t)$ be **the probability density (or the proportion) of firms located at** $x \in S^1$ **in time** $t$. Adoption of a strategy means decision to locate at $x \in [0, 2\pi)$. **The payoff** for a firm located at $x$ depends on the **desirability** of the location. Desirability is reflected by

$$P(x,t) = \int_{S^1} h(D_{xz}) u(z,t)\, dz$$

where $D_{xz}$ denotes distance between $x$ and $z$. The function $h(D_{xz})$ is the so called **market potential** that incorporates both **centripetal agglomerative** and **dispersing centrifugal forces**. Krugman suggests

$$h(D_{xz}) = A_1 \exp(-r_1|x-z|) - A_2 \exp(-r_2|x-z|), \quad A_1, A_2, r_1, r_2 > 0.$$

The **average market potential** is defined as

$$\bar{P}(t) = \int_{S^1} P(x,t) u(x,t)\, dx.$$

It is assumed that firms immigrate towards locations with market potential above the average. This can be modelled by the replicator dynamics equation

$$\frac{\partial u}{\partial t} = (P - \bar{P})\, u.$$

# Gaussian-type pay-off function

1. Inspired by the previous example we can consider pay-off functions of the form

$$f(x,y) = K(|x-y|)$$

2. There are applications both in biology (**Haccou & Iwasa (1998)**) as well as in computer science ( **Krause & Ong (2011), Marecki (2011)**) where the pay-off kernel has the form $f(x,y) = G(|x-y|)$ with $G$ being **a steep function of Gaussian type.**
3. This case refers to games where **the pay-off is measured as the distance from some reference strategy** and finally under some proper scaling leads to

$$\int_\Omega f(x,y)u(y,t)\,dy \approx c_0 u(x,t) + \overrightarrow{c_1}\cdot\nabla u + c_2\Delta u(x,t),$$

where the coefficients $c_0$, $\overrightarrow{c_1}$ and $c_2$ are related to the moments of the kernel $G$.
Ignoring the lower order terms and after some scaling we end up with

$$\int_\Omega f(x,y)u(y,t)\,dy \approx \Delta u(x,t).$$

# The non-local model

1. For a bounded and smooth domain $\Omega \subset \mathbb{R}^N$ then via integration by parts the non-local integro-differential dynamics equation (3) is approximated by the degenerate non-local parabolic equation

$$\frac{\partial u}{\partial t} = u\Big(\Delta u + \int_\Omega |\nabla u|^2\, dx\Big), \qquad x \in \Omega,\ t > 0,\ \int_\Omega u(x,t) = 1, \tag{4}$$

   **(Kravvaritis, Papanicolaou, Xepapadeas & Yannacopoulos (2010)).**

2. The non-local equation (4) is associated with initial condition

$$u(x,0) = u_0(x),\ \ x \in \Omega, \tag{5}$$

   and homogeneous **Dirichlet boundary conditions**

$$u(x,t) = 0,\ \ x \in \partial\Omega,\ \ t > 0, \tag{6}$$

   when the agents avoid to play the strategies locating on the boundary of the strategy space since **they are supposed to be too risky** (or the individuals of the biological population do not interact when they are close to the spatial boundary **where probably the "food" is less**).

3. When on the boundary of the strategy space **individuals do not really distinguish between nearby strategies** and hence populate them equally, then the non-local equation (4) should rather be complemented by homogeneous **Neumann boundary conditions.**

# Key features of the model

$$u_t = u\left(\Delta u + \int_\Omega |\nabla u|^2\,dx\right), \qquad x \in \Omega,\ t > 0, \tag{7}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{8}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{9}$$

1. From a mathematical perspective, evolution equation (7) is governed by two characteristic mechanisms, each of which already gives rise to considerable challenges on its own.
2. Firstly, diffusion in (7) is strongly degenerate at small densities in the sense that near points where $u = 0$ typical diffusive effects are substantially inhibited.
3. Equation (7) is also non-local due to the presence of the term $\int_\Omega |\nabla u|^2\,dx$.

# Notion of solution for problem (7)-(9)

## Definition

*Let* $T \in (0, \infty]$. *By a* weak solution *of* (7)-(9) *in* $\Omega \times (0, T)$ *we mean a nonnegative function*

$$u \in L^{\infty}_{loc}(\bar{\Omega} \times [0, T)) \cap L^2_{loc}([0, T); W^{1,2}_0(\Omega)) \qquad \text{with} \qquad u_t \in L^2_{loc}(\bar{\Omega} \times [0, T)),$$

*which satisfies*

$$-\int_0^T\int_\Omega u\varphi_t \, dxdt + \int_0^T\int_\Omega \nabla u \cdot \nabla(u\varphi) \, dxdt = \int_\Omega u_0\varphi_0 \, dx + \int_0^T \Big( \int_\Omega u\varphi \, dx \Big) \cdot \Big( \int_\Omega |\nabla u|^2 \, dx \Big) dt$$

*for all* $\varphi \in C^{\infty}_0(\Omega \times [0, T))$ *with* $\varphi(x, 0) = \varphi_0$.
*A weak solution* $u$ *of* (7)-(9) *in* $\Omega \times (0, T)$ *will be called* locally positive *if* $\frac{1}{u} \in L^{\infty}_{loc}(\Omega \times [0, T])$.

## Remark

*Equation* (7) *is a non-local perturbation of* $u_t = u\Delta u$ *for which there is no a unique solution for the associated Dirichlet problem* **(Luckhaus & Dal Passo (1987))** . *So we do not anticipate uniqueness for problem* (7)-(9).

# Initial Conditions

In order to construct such locally positive weak solutions we consider initial data of the form

(H1) $u_0 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$

(H2) $u_0 \geq 0$ and $\frac{1}{u_0} \in L^\infty_{loc}(\Omega)$ and

(H3) there exists $0 < L < \infty$ such that $\|u_0\|_{\Phi,\infty} \leq L$.

For a measurable function $v\colon \Omega \to \mathbb{R}$ we set

$$\|v\|_{\Phi,\infty} := \operatorname*{ess\,sup}_{x\in\Omega} \left|\frac{v}{\Phi}\right|,$$

where $\Phi \in C^2(\overline{\Omega})$ denotes the solution to

$$-\Delta\Phi = 1 \quad \text{in } \Omega, \qquad \Phi|_{\partial\Omega} = 0.$$

# An approximating regularized problem

Following an approach well-established in the context of degenerate parabolic equations, we construct a solution $u$ to (7)-(9) as the limit $\varepsilon \to 0$ of solutions to the following regularized problem

$$\begin{cases} u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + u_\varepsilon \cdot \rho_\varepsilon\big( \int_\Omega |\nabla u_\varepsilon|^2 \big), & x \in \Omega,\ t > 0, \\ u_\varepsilon(x,t) = \varepsilon, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{10}$$

where

$$\rho_\varepsilon(z) := \min\left\{ z,\ \frac{1}{\varepsilon} \right\} \qquad \text{for } z \geq 0,$$

and $(u_{0\varepsilon})_{\varepsilon=\varepsilon_j} \subset C^3(\bar{\Omega})$ with the properties

$$u_{0\varepsilon} \geq \varepsilon \text{ in } \Omega, \qquad u_{0\varepsilon} = \varepsilon \text{ on } \partial\Omega, \qquad \Delta u_{0\varepsilon} = -\int_\Omega |\nabla u_{0\varepsilon}|^2 \text{ on } \partial\Omega \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$$

and

$$\limsup_{\varepsilon=\varepsilon_j \searrow 0} \|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty} \leq L,$$

with $L > \max\left\{ \int_\Omega |\nabla u_0|^2, \|u_0\|_{\Phi,\infty} \right\}$, cf. (H3), and

$$u_{0\varepsilon} \to u_0 \quad \text{in } W^{1,2}(\Omega) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

with

$$\int_\Omega u_{0\varepsilon} = \int_\Omega u_0 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}.$$

# Main estimates for the regularized problem (10)

## Proposition 1 (K., Lankeit & Winkler (2017) )

For all sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, problem (10) has a unique classical global-in-time solution $u_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0,\infty))$.

The following lemma essentially derives a uniform pointwise bound for $u_\varepsilon$ from a space-time integral estimate for $|\nabla u_\varepsilon|^2$.

## Lemma 1 (K., Lankeit & Winkler (2017) )

For all $M > 0$ and $B > 0$ there exists $C(M,B) > 0$ with the following property: If

$$u_{0\varepsilon} \le M \quad \text{in } \Omega \qquad \text{and} \qquad \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le B$$

holds for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and $T \in (0,\infty]$ then we have

$$u_\varepsilon \le C(M,B) \qquad \text{in } \Omega \times [0,T).$$

# Main estimates for the regularized problem (10) (cont. 1)

Next, the fact that solutions of (10) cannot blow up immediately can be turned into a quantitative local-in-time boundedness estimate in terms of the norm of the initial data in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$.

## Lemma 2 (K., Lankeit & Winkler (2017) )

i) For all $M > 0$ there exist $T_1(M) > 0$ and $C_1(M) > 0$ such that if

$$u_{0\varepsilon} \le M \quad \text{in } \Omega \qquad \text{and} \qquad \int_\Omega |\nabla u_{0\varepsilon}|^2 \le M$$

hold for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, then

$$u_\varepsilon \le C_1(M) \qquad \text{in } \Omega \times [0, T_1(M)).$$

ii) For each $M > 0$ and $T > 0$ there exist $T_2(M) \in (0, T]$ and $C_2(M) > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that

$$u_\varepsilon \le M \qquad \text{in } \Omega \times (0, T) \qquad \text{and} \qquad \int_\Omega |\nabla u_{0\varepsilon}|^2 \le M$$

are satisfied, then

$$\int_0^{T_2(M)} \int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \sup_{t\in(0,T_2(M))} \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 \le C_2(M).$$

## Main estimates for the regularized problem (10) (cont. 2)

When constructing the solution $u$ of (7)-(9) as the limit of solutions $u_\varepsilon$ of (10), it is comparatively easy to obtain the approximation property $\nabla u_\varepsilon \to \nabla u$ in the sense of $L^2_{loc}(\Omega \times [0,T))$-convergence. For handling the non-local term in the equation, however, it seems appropriate to make sure that also
$\int_\Omega |\nabla u_\varepsilon|^2 \to \int_\Omega |\nabla u|^2$ in $L^1_{loc}([0,T))$.
In order to achieve the latter we exclude certain boundary concentration phenomena of $\nabla u_\varepsilon$ in the following sense.

**Lemma 3 (K., Lankeit & Winkler (2017) )**

For any $T > 0$, $C > 0$, $M > 0$ and $\delta > 0$, there is $K = K(M,C,T,\delta) \subset\subset \Omega$ and $\eta > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that $\varepsilon < \eta$ and

$$\sup_{t\in[0,T]} \int_\Omega |\nabla u_\varepsilon(t)|^2 \le C \quad \text{and} \quad u_\varepsilon \le M, \tag{11}$$

we have

$$\int_0^T \int_{\Omega\setminus K} |\nabla u_\varepsilon|^2 < \delta. \quad \text{(no concentrantion on the boundary)}$$

We are now ready to prove that the $u_\varepsilon$ in fact approaches a weak solution of (7)-(9) that is locally positive. Before we do so, however, we prepare the following estimate for $u_\varepsilon$ that will be useful in proving assertions about the blow-up behaviour of $u$.

**Lemma 4 (K., Lankeit & Winkler (2017) )**

Let $\Omega' \subset\subset \Omega$ be a domain with smooth boundary. Assume also that $\phi$ denotes the solution to $-\Delta\phi = 1$ in $\Omega'$, $\phi|_{\partial\Omega'} = 0$. Then there exists $C_{\Omega'} > 0$ such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and any $t > 0$ the solution $u_\varepsilon$ of (10) satisfies

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \leq$$

$$\int_\Omega |\nabla u_{0\varepsilon}|^2 \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)}\int_\Omega u_\varepsilon(\tau)\right)\left(\int_{\Omega'}\phi\ln u_\varepsilon(\cdot,t) - \int_{\Omega'}\phi\ln u_{0\varepsilon} + \int_0^t\int_{\Omega'} u_\varepsilon\right)\right]. \quad (12)$$

## Theorem 1 (K., Lankeit & Winkler (2017) )

Let $u_0$ satisfy (H1)-(H3). Then there exist $T_{max} \in (0, \infty]$ and a locally positive weak solution $u$ to (7)-(9) in $\Omega \times (0, T_{max})$ which satisfies

$$\text{either } T_{max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

and which is such that for each smoothly bounded subdomain $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ with

$$\int_\Omega |\nabla u(\cdot, t)|^2 \leq \int_\Omega |\nabla u_0|^2 \cdot \exp \left[ \frac{1}{2C_{\Omega'}} \left( \sup_{\tau \in (0,t)} \int_\Omega u(\cdot, \tau) \right) \left( \int_{\Omega'} \phi \ln u(\cdot, t) - \int_{\Omega'} \phi \ln u_0 + \int_0^t \int_{\Omega'} u \right) \right] \tag{13}$$

as well as

$$\|u(\cdot, t)\|_{\Phi, \infty} \leq \max \left\{ \|u_0\|_{\Phi, \infty}, \sup_{\tau \in (0,t)} \int_\Omega |\nabla u(x, \tau)|^2 \, dx \right\}, \tag{14}$$

for a.e. $t \in (0, T_{max})$.

Actually relation (13) guarantees that finite-time gradient blow-up cannot take place!

**Corollary 1 (K., Lankeit & Winkler (2017) )**

By (14) we deduce that

$$\limsup_{t \to T_{max}} ||u(\cdot,t)||_{L^\infty(\Omega)} = \infty \Rightarrow \limsup_{t \to T_{max}} \int_\Omega |\nabla u(x,t)|^2 \, dx = \infty.$$

**Corollary 2 (K., Lankeit & Winkler (2017) )**

Furthermore by virtue of (13) and Corollary 1 we deduce that

$$\limsup_{t \to T_{max}} ||u(\cdot,t)||_{L^\infty(\Omega)} = \infty \Rightarrow \limsup_{t \to T_{max}} \int_\Omega u(x,t) \, dx = \infty. \quad \text{(Global Blow-up?)}$$

# Classification of initial data

## Theorem 2 (K., Lankeit & Winkler (2017) )

Let $u_0$ satisfy (H1)-(H3), and let $u$ and $T_{max}$ denote the corresponding locally positive weak solution of (7)-(9), as well as its maximal time of existence.

(i) If $\int_\Omega u_0 < 1$, then $T_{max} = \infty$ and

$$\int_\Omega u(x,t)\,dx \to 0 \qquad \text{as } t \to \infty.$$

(ii) Suppose that $\int_\Omega u_0 = 1$. Then $T_{max} = \infty$ and

$$\int_\Omega u(x,t)\,dx = 1 \qquad \text{for all } t > 0.$$

(iii) In the case $\int_\Omega u_0\,dx > 1$, we have $T_{max} < \infty$ and

$$\limsup_{t \nearrow T_{max}} \int_\Omega u(x,t)\,dx = \infty.$$

## Remark

*Statement $(ii)$ of the above Theorem says that if the initial data $u_0$ is a probability measure then we have conservation of probability in time. This is actually a desired feature of the replicator dynamics model described by* (7)-(9)*, since $u(\cdot,t)$ stands for a probability distribution of the state of some population of players.*

**Theorem 3 (K., Lankeit & Winkler (2017) )**

Suppose that $\int_\Omega u_0 \, dx > 1$, and let $u$ denote the locally positive weak solution of (7)-(9). Then $u$ blows up globally in the sense that $\mathcal{B} = \overline{\Omega}$, where

$$\mathcal{B} = \Big\{ x \in \overline{\Omega} \quad \Big| \quad \text{there exists a sequence } (x_k, t_k)_{k \in \mathbb{N}} \subset \Omega \times (0, T_{max}) \text{ such that} \\ x_k \to x, t_k \to T_{max} \text{ and } u(x_k, t_k) \to \infty \text{ as } k \to \infty \Big\}$$

is the blow-up set of $u$.

# Convergence Towards Steady States (Nash Equilibria)

## Theorem 4 (Lankeit 2017)

Let $u_0 \in W_0^{1,2}(\Omega)$ satisfy (H1)-(H3), and additionally $\int_\Omega u_0\,dx = 1$. Let also $\Phi$ denote the solution of $-\Delta\Phi = 1$ in $\Omega$, $\Phi|_{\partial\Omega} = 0$. Then

$$\lim_{t\to\infty} \left\| u(\cdot,t) - \frac{\Phi}{\int_\Omega \Phi\,dx} \right\|_{W_0^{1,2}(\Omega)} = 0.$$

The proof is based on a monotonicity property of $J(t) = \int_\Omega |\nabla u|^2\,dx$ along the solution trajectories and the analysis of the associated constrained minimization problem:

$$\min_{v\in\mathcal{M}} \int_\Omega |\nabla v|^2\,dx, \quad \mathcal{M} = \left\{ v \in W_0^{1,2}(\Omega) : \int_\Omega v\,dx = 1 \right\}.$$

## Remark

*The result of the above Theorem is translated in the language of game theory as follows: players' strategies actually converge to Nash equilibria, i.e. in states where almost all the players are happy with the outcome.*

# Key References


• I. Bomze, *Dynamical aspects of evolutionary stability*, Mon. Math. **110**, 189–206 (1990).
• K. Bogdan, K. Burdzy & Z. Chen,c*Censored stable processes,* Probab. Theory Relat. Fields **127**, 89–152 (2003).
• P. Haccou & Y. Iwasa, *Optical mixed strategies in stochastic environments*, Theor. Popul. Biol. **47**, 212–243(1995)
• P. Haccou & Y. Iwasa, *Robustness of optimal mixed strategies*, J. Math. Biol. **36**, 485–496 (1998)
• J. Hofbauer & K. Sigmund, *Evolutionary game dynamics.* Bull. Am. Math. Society **40** 479–519 (2003)
• N. I. Kavallaris, J. Lankeit & M. Winkler, *On a degenerate non-local parabolic problem describing infinite dimensional replicator dynamics*, SIAM J. Math. Anal., **49(2)**, 954–983 (2017).
• N. I. Kavallaris & A.N. Yannacopoulos, *On a non-local diffusion model arising in replicator dynamics*, in progress
• D. Kravvaritis, V.G. Papanicolaou, A. Xepapadeas & A.N. Yannacopoulos, *On a class of operator equations arising in infinite dimensional replicator dynamics.* Nonlin. Anal. RWA **11**, 2537–2556 (2010)
• A. Krause & C.S. Ong, *Contextual gaussian process bandit optimization.* Department of Computer Science, ETH Zurich, Report (2011)
• P. Krugman *The self-organizing economy.* Mitsui Lectures in Economics, Wiley-Blackwell, Cambridge, Mass. , USA, 1996
• J. Lankeit, *Equilibration of unit mass solutions to a degenerate parabolic equation with a nonlocal gradient nonlinearity*, Nonlinear Anal. **135**, 236–248 (2016).
• S. Luckhaus & R. Dal Passo, *A degenerate diffusion equation not in divergence form*, J. Diff. Equations **69**, (1987), 1–14.
• J. Mareck, *Playing in the dark: game theory for security applications,* IBM report (2011)
• J. M. Smith &. G. R. Price, *The Logic of Animal Conflict*, Nature **246**, (1973), 15–18.

Thanks for your attention!