STRONG ADVECTION PROBLEMS IN TURBULENT DIFFUSION

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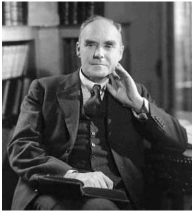
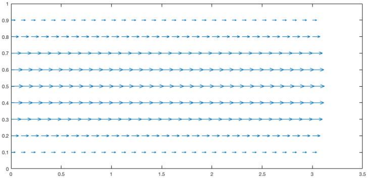
SOME HISTORY



- G.I. TAYLOR posed an **interesting question**.
- Spreading of dissolved solutes inside a tube filled with fluid.
- Interplay between molecular diffusion and advection

$$\partial_t c + \mathbf{b} \cdot \nabla c - d\Delta c = 0.$$



- Scenarios $d \equiv 0$ or $\mathbf{b} \equiv$ constant **do not mix well**.
- Shearing advective field: **Poiseuille** flow in a tube.
- Empirical formula for **effective diffusion**.

[Ref.] G.I.TAYLOR, **Proc. Roy. Soc. Lond. A Math.**, *Vol 219 (1953).*

[Ref.] G.K.BATCHELOR, **The life and legacy of G.I.Taylor**, *(1996).*

MOTIVATION

- Quantity of interest: certain CONCENTRATION FIELD.
- Evolving under the influence of
  - advection by an incompressible field.
  - molecular diffusion.
- Physical quantity immersed in a fluid flow
  - temperature (heat).
  - concentration of some solute.
- Chlorophyll moved around in ocean.
- Heat evolving on the surface of the ocean.

[Ref.] EARTH OBSERVATORY WEBPAGE OF NASA FOR CERTAIN GLOBAL MAPS

https://earthobservatory.nasa.gov

[Media.] TEMPERATURE AND CHLOROPHYLL MAPS (2002-2022)

INTRICATE CONNECTIONS

- Other scientific disciplines such as
  - Geophysics: oceanography.
  - Engineering: chemical engineering.
  - Biology: motor proteins.
- Particular applications namely
  - weak heat fluctuations in fluids.
  - dyes used to visualising flow patterns.
  - pollutants dispersing in the environment.
  - gas exchange in the lungs.
  - blood circulation.

[Ref.] A.MAJDA, P.KRAMER, *Phy. Rep. (1999).*

∗ Initial-boundary value problem for the **scalar** unknown $u^\varepsilon(t,x)$

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0 \qquad \text{in } (0,\ell)\times\Omega,$$

$$u^\varepsilon(0,x) = u^{\text{in}}(x) \qquad \text{in } \Omega,$$

$$\nabla u^\varepsilon \cdot \mathbf{n}(x) = 0 \qquad \text{on } (0,\ell)\times\partial\Omega.$$

- $\varepsilon > 0$ is a parameter (to regulate strength of advective field).
- Molecular diffusion **weak** compared to the strength of advection.
- Advective field $\mathbf{b}(x)$ is **prescribed**, **smooth incompressible** field with **zero normal flux** at the boundary:

$$\nabla\cdot\mathbf{b}(x) = 0 \qquad \text{for a.e. } x\in\Omega,$$

$$\mathbf{b}(x)\cdot\mathbf{n}(x) = 0 \qquad \text{for a.e. } x\in\partial\Omega.$$

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0$$

- Relaxation to equilibrium
  - In the evolution for $u^\varepsilon(t,x)$ **how long** does it take to equilibrate?
  - If we wait long enough, will we reach an **uniform** temperature?
  - Is the **rate** of convergence **uniform** in ε?
  - Are there **special** advective fields which result in **quicker equilibration**?
- Strong advection limit
  - Does there exist a **limit point** for the sequence $\{u^\varepsilon(t,x)\}$?
  - If $\lim_{\varepsilon\to 0} u^\varepsilon = \overline{u}$, how can we **characterise** the limit $\overline{u}$?
  - Is there a **rate** of convergence in terms of ε?
  - What is the **interplay** with the **rate** of convergence and the chosen **advective field**?

Proposition (long time behaviour)

There exists a uniform constant $\gamma > 0$ such that

$$\left\| u^\varepsilon(t,\cdot) - \langle u^{\text{in}} \rangle \right\|_{\mathrm{L}^2(\Omega)} \lesssim e^{-\gamma\, t}$$

where $\langle u^{\text{in}} \rangle$ denotes the average

$$\langle u^{\text{in}} \rangle := \frac{1}{|\Omega|} \int_\Omega u^{\text{in}}(x)\, \mathrm{d}x$$

Multiply the evolution by $u^\varepsilon(t,x)$ and integrate over the spatial domain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left| u^\varepsilon(t,x) \right|^2 \mathrm{d}x + \frac{1}{2\varepsilon} \int_\Omega \mathbf{b}(x) \cdot \nabla \left| u^\varepsilon(t,x) \right|^2 \mathrm{d}x + \int_\Omega \left| \nabla u^\varepsilon(t,x) \right|^2 \mathrm{d}x = 0$$

$$\text{i.e.,} \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left| u^\varepsilon(t,x) \right|^2 \mathrm{d}x = - \int_\Omega \left| \nabla u^\varepsilon(t,x) \right|^2 \mathrm{d}x.$$

Result follows by **Poincaré** inequality and **Grönwall's** inequality.

Definition (Relaxation enhancing fields)

An incompressible field $\mathbf{b}(x)$ *is called relaxation enhancing if for any* $\delta > 0$*, there exists* $\overline{\varepsilon}(\delta) > 0$ *such that* $\forall \varepsilon$ *with* $\varepsilon < \overline{\varepsilon}(\delta)$ *we have*

$$\left\| u^\varepsilon(1,\cdot) - \langle u^{\mathrm{in}} \rangle \right\|_{\mathrm{L}^2(\Omega)} < \delta.$$

[Ref.] P.CONSTANTIN, A.KISELEV, L.RYZHIK, A.ZLATOS, *Ann. Math. (2008).*

[Ref.] H.BERESTYCKI, F.HAMEL, N.NADIRASHSHVILI, *Comm. Pure Appl. Math. (2005).*

[Ref.] B.FAYAD, *Ergodic Theory Dynam. Systems (2002).*

[Ref.] B.FRANKE, C.-R.HWANG, H.-M.PAI, S.-J.SHEU, *Trans. Amer. Math. Soc. (2010).*

Theorem (Constantin et al. (2008))

An incompressible field $\mathbf{b}(x)$ *is relaxation enhancing* **if and only if**

$$\mathcal{N}_{\mathbf{b}} := \left\{ v \in \mathrm{H}^1(\Omega) \text{ such that } \mathbf{b}\cdot\nabla v = 0 \right\}$$

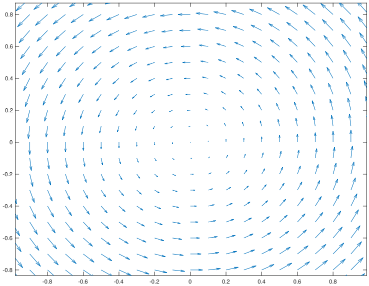
has **no non-trivial elements**.

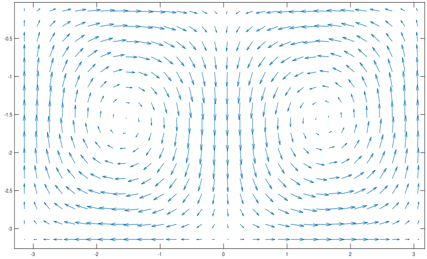
SOME WELL-KNOWN FIELDS

- A rotation in two dimensions

$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

- $x_1^2 + x_2^2 \in \mathcal{N}_{\mathbf{b}}$





- A two dimensional cellular flow

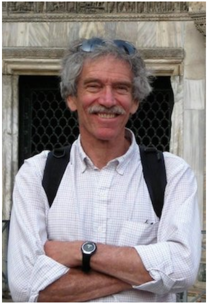
$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -\sin(x_1)\cos(x_2) \\ \cos(x_1)\sin(x_2) \end{pmatrix}$$

- $\sin(x_1)\sin(x_2) \in \mathcal{N}_{\mathbf{b}}$

TALK IS DERIVED FROM

[Ref.] T.HOLDING, H.H, J.RAUCH, **SIAM J. Math. Anal.**, *Vol 49, No 1, pp. 222–271 (2017).*
[Ref.] T.HOLDING, H.H, J.RAUCH, **in preparation**.

SCIENTIFIC COLLABORATORS

Joint work with T. Holding & J. Rauch

- Consider the **null space** and the **range space** of $\mathbf{b}\cdot\nabla$

  $\mathcal{N}_{\mathbf{b}} := \left\{ v \in \mathrm{L}^2(\Omega) \text{ s.t. } \operatorname{div}(\mathbf{b}v) = 0 \text{ in the sense of distributions} \right\}.$

  $\mathcal{W}_{\mathbf{b}} := \left\{ \mathbf{b}\cdot\nabla v \text{ for } v \in \mathrm{H}^1(\Omega) \right\} \subset \mathrm{L}^2(\Omega).$

- Hilbert's theorem yields **orthogonal decomposition**

$$\mathrm{L}^2(\Omega) = \mathcal{N}_{\mathbf{b}} \oplus \overline{\mathcal{W}_{\mathbf{b}}}$$

  i.e., for any $v \in \mathrm{L}^2(\Omega)$, there exists a **unique decomposition**

$$v = v_n + v_r$$

  such that $v_n \in \mathcal{N}_{\mathbf{b}}$ and $v_r \in \mathcal{N}_{\mathbf{b}}^{\perp} = \overline{\mathcal{W}_{\mathbf{b}}}$

- **Projection** on to $\mathcal{N}_{\mathbf{b}}$ denoted $\mathcal{P} : \mathrm{L}^2(\Omega) \mapsto \mathcal{N}_{\mathbf{b}}$

PROJECTION MAP $\mathcal{P} : \mathrm{L}^2(\Omega) \mapsto \mathcal{N}_{\mathbf{b}}$

- For any $v \in \mathrm{L}^2(\Omega)$, the projection $\mathcal{P}v$ can be **computed** as

$$\|v - \mathcal{P}v\|_{\mathrm{L}^2(\Omega)} = \min_{g \in \mathcal{N}_{\mathbf{b}}} \|v - g\|_{\mathrm{L}^2(\Omega)}$$

- More useful way to interpret the projection map $\mathcal{P}$ is due to

von Neumann's ergodic theorem:

For any $v \in \mathrm{L}^2(\Omega)$,

$$\mathcal{P}v(x) := \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} v\left(\Phi_\tau(x)\right) \mathrm{d}\tau$$

with the **flow** $\Phi_\tau(x) : \mathbb{R} \times \Omega \to \Omega$ defined as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x) & = \mathbf{b}\left(\Phi_\tau(x)\right) \\ \Phi_0(x) & = x \end{cases}$$

Theorem (HOLDING, H, RAUCH (2017))

Let $u^\varepsilon(t,x)$ be the solution to the initial-boundary value problem. Then

$$u^\varepsilon \rightharpoonup \overline{u} \qquad \textit{weakly in } \mathrm{L}^2((0,\ell)\times\Omega)$$

with $\overline{u}(t,x)$ being the unique solution to

$$\begin{aligned} \partial_t \overline{u} - \Delta \overline{u} = g \in \mathcal{N}_{\mathbf{b}}^{\perp} \\ \overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}} \\ \overline{u}(0,\cdot) = \mathcal{P}u^{\mathrm{in}}(\cdot) \\ \nabla \overline{u}(t,x)\cdot \mathbf{n}(x) = 0 \qquad \textit{on } (0,\ell)\times\partial\Omega. \end{aligned}$$

- The condition $\overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}}$ is treated as a **constraint**.
- The source $g(t,\cdot) \in \mathcal{N}_{\mathbf{b}}^{\perp}$ is the associated **Lagrange multiplier**.
- The **initial datum** has got **projected** on to the null space.

ESSENTIAL IDEAS

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0$$

- By L^2-weak convergence, we mean that for any $\psi(t,x) \in \mathrm{L}^2$,

$$\lim_{\varepsilon\to 0} \iint\limits_{(0,\ell)\times\Omega} u^\varepsilon(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{(0,\ell)\times\Omega} \overline{u}(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

- For a weak convergence, we cannot give a rate of convergence
- Limit evolution with the constraint should be interpreted as

solving **heat equation** on the subspace $\mathcal{N}_{\mathbf{b}}$.

- Can we improve it to strong convergence (then we explore the rate)

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - \overline{u}\|_{\mathrm{L}^2((0,\ell)\times\Omega)} = 0.$$

- Are there advective fields $\mathbf{b}(x)$ which result in strong convergence?
- One possible obstacle for such a result is

$$u^\varepsilon(0,x) = u^{\mathrm{in}}(x); \qquad \overline{u}(0,x) = \mathcal{P}u^{\mathrm{in}}(x).$$

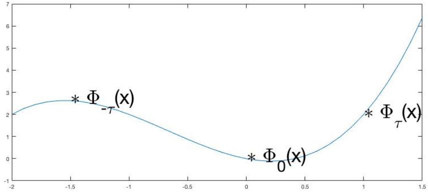
STRATEGY

- WORK IN A MOVING FRAME OF REFERENCE
  - Rather than studying $u^\varepsilon(t,x)$ in a **fixed frame**,
  - we study u^ε taken along a **moving frame**.
  - **Dynamics** of the moving frame dictated by the field $\mathbf{b}(x)$.

MEASURE PRESERVING FLOW

- Consider the **flow** $\Phi_\tau(x) : \mathbb{R}\times\Omega \to \Omega$ defined as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x) & = \mathbf{b}\left(\Phi_\tau(x)\right) \\ \Phi_0(x) & = x \end{cases}$$



IN MOVING FRAME OF REFERENCE

- Rather than studying $u^\varepsilon(t,x)$ we study the family

$$u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)$$

- Flow is evaluated at $\frac{t}{\varepsilon}$
- We introduce a fast time variable $\tau := \frac{t}{\varepsilon}$

$$t = \mathcal{O}(\varepsilon) \implies \tau = \mathcal{O}(1).$$

- For any $\tau \in \mathbb{R}$, the map $x \mapsto \Phi_\tau(x)$ defines a **change-of-variable**
- Associated **Jacobian matrix**

$$J(\tau,x) = \begin{bmatrix} \frac{\partial \Phi^1_\tau}{\partial x_1} & \cdots & \frac{\partial \Phi^1_\tau}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial \Phi^d_\tau}{\partial x_1} & \cdots & \frac{\partial \Phi^d_\tau}{\partial x_d} \end{bmatrix} = \left(\frac{\partial \Phi^i_\tau}{\partial x_j} \right)^d_{i,j=1}$$

- $\mathbf{b}(x)$ incompressible $\implies$ flow $\Phi_\tau(x)$ is **volume preserving**,

$$\text{i.e., } \det\left(J(\tau,x)\right) = 1 \qquad \text{for all } \tau \in \mathbb{R}.$$

Theorem (HOLDING, H, RAUCH (2017))

Let $u^\varepsilon(t,x)$ be the solution to the initial-boundary value problem. Suppose Jacobian matrix $J(\cdot,x) \in \mathcal{A}$, an **algebra with mean value**. *Then for each $t \in (0,\ell)$,*

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(t,\cdot) - u_0\left(t, \Phi_{-t/\varepsilon}(\cdot)\right)\right\|_{\mathrm{L}^2(\Omega)} = 0$$

where $u_0(t,X)$ solves a **diffusion equation**

$$\partial_t u_0 = \nabla_X \cdot \Big(\mathfrak{D}(X) \nabla_X u_0 \Big); \qquad u_0(0,X) \ = u^{\mathrm{in}}(X)$$

with the **diffusion matrix** *given by*

$$\mathfrak{D}(X) = \lim_{\ell\to\infty} \frac{1}{2\ell} \int_{-\ell}^{+\ell} J(\tau,X)^\top J(\tau,X)\,\mathrm{d}\tau.$$

- Computing the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)\Big] = \partial_t u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right) + \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t/\varepsilon}(x)\cdot\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$

$$= \partial_t u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right) + \frac{1}{\varepsilon}\mathbf{b}\left(\Phi_{t/\varepsilon}(x)\right)\cdot\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$

- RHS is the advection term taken along the flow $\Phi_{t/\varepsilon}(x)$.
- X denotes the Lagrangian coordinate.
- Computing the spatial derivative

$$\nabla\Big[u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)\Big] = {}^\top J\left(\frac{t}{\varepsilon},x\right)\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$

where ${}^\top\cdot$ denotes transpose.

- Note the **dependance** of Jacobian on the **fast time variable**.

- Need to compute the Laplacian term along the flow $\Phi_{t/\varepsilon}(x)$.
- Consider the associated energy

$$\int_\Omega \langle \nabla u^\varepsilon(t,x), \nabla u^\varepsilon(t,x)\rangle \, \mathrm{d}x$$

- Perform the change of variables $x \mapsto \Phi_{t/\varepsilon}(x)$ inside the integral

$$\int_\Omega \left\langle {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right), {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right) \right\rangle \underbrace{\frac{\mathrm{d}X}{|\det(J)|}}_{=1}$$

- Hence the Laplacian along the flow $\Phi_{t/\varepsilon}(x)$ becomes

$$\nabla_X \cdot \left(J\left(\frac{t}{\varepsilon}, x\right) {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)\right)$$

Lemma

Suppose $f \in \mathrm{L}^\infty(\mathbb{R})$. *Define the dilated sequence*

$$f^\varepsilon(t) := f\left(\frac{t}{\varepsilon}\right).$$

$$\textit{If} \qquad f^\varepsilon \rightharpoonup M(f) \qquad \textit{weakly} \ * \ \textit{in } \mathrm{L}^\infty(\mathbb{R}) \qquad \textit{as } \varepsilon \to 0$$

where $M(f)$ *is a constant. Then, the limit is characterised as*

$$M(f) = \lim_{\ell\to\infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\tau)\, \mathrm{d}\tau.$$

- By $h^\varepsilon \rightharpoonup h_0$ weakly $*$ in $\mathrm{L}^\infty(\mathbb{R})$, we mean

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} h^\varepsilon(t)\psi(t)\, \mathrm{d}t = \int_{\mathbb{R}} h_0(t)\psi(t)\, \mathrm{d}t \quad \forall \psi \in \mathrm{L}^1.$$

Notation: $\mathcal{B}(\mathbb{R})$ - space of bounded continuous functions.

Definition (Algebra with mean value)

$\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}(\mathbb{R})$ with following properties:

- *$\mathcal{A}$ contains the* **constants**.
- *$\mathcal{A}$ is* **translation invariant**, *i.e. $f(\cdot - a) \in \mathcal{A}$ whenever $f \in \mathcal{A}$.*
- *Any $f \in \mathcal{A}$ possesses a* **mean value** *in the following sense*

$$f\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup M(f) \quad \textit{in } \mathrm{L}^\infty(\mathbb{R})\textit{-weak* as } \varepsilon \to 0.$$

We have already seen that

$$M(f) = \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\tau)\, \mathrm{d}\tau.$$

[Ref.] V.V.JIKOV, E.V.KRIVENKO, **Matem. Zametki** *(1983).*

[Ref.] V.V.JIKOV, S.M.KOZLOV, O.A.OLEINIK, **Springer-Verlag** *(1994).*

SOME EXAMPLES OF ALGEBRA W.M.V.

Example (Periodic functions)

$\mathcal{A} = \mathcal{C}_{\text{per}}$ *be space of continuous functions* **periodic** *with period* 1.

$$M(u) = \int_0^1 u(\tau)\,\mathrm{d}\tau.$$

Example (Functions that converge at infinity)

$\mathcal{A}$ *be space of continuous functions that converge to a limit at infinity*

$$M(u) = \lim_{|\tau|\to\infty} u(\tau).$$

Example (Almost-periodic functions)

$*$ $\mathsf{T}(\mathbb{R})$ *be the set of all trigonometric polynomials, i.e. all* $u(t)$ *that are finite linear combinations of functions in the set*

$$\Big\{\cos(kt), \sin(kt) : k \in \mathbb{R}\Big\}.$$

The space of almost-periodic functions in the sense of **Bohr** *is the closure of* $\mathsf{T}(\mathbb{R})$ *in the supremum norm,*

i.e., given a $\delta > 0$ *and an almost-periodic function* $u(t)$*, there exists a* $g(t) \in \mathsf{T}(R)$ *s.t.*

$$\|u(\cdot) - g(\cdot)\|_{\mathrm{L}^\infty} < \delta.$$

ASYMPTOTIC ANALYSIS STRATEGY

- Fix an arbitrary algebra w.m.v. $\mathcal{A}$.
- Take $\mathbf{b}(x)$ such that Jacobian matrix $J(\cdot, x) \in \mathcal{A}$, i.e., in particular

$$\sup_{\tau \in \mathbb{R}} |J(\tau, x)| < \infty$$

A NEW NOTION OF WEAK CONVERGENCE

Definition (Σ-convergence along flow)

A family $\{u^\varepsilon\} \subset \mathrm{L}^2((0,\ell) \times \Omega)$ is said to **Σ-converge along the flow** *Φ_τ to a limit $u_0(t, X, s) \in \mathrm{L}^2((0,\ell) \times \Omega \times \Delta(\mathcal{A}))$ if, for any smooth test function $\psi(t, x, \cdot) \in \mathcal{A}$, we have*

$$\lim_{\varepsilon \to 0} \iint_{(0,\ell)\times\Omega} u^\varepsilon(t,x)\psi\Big(t, \Phi_{-t/\varepsilon}(x), \frac{t}{\varepsilon}\Big)\, \mathrm{d}x\, \mathrm{d}t$$

$$= \iiint_{(0,\ell)\times\Omega\times\Delta(\mathcal{A})} u_0(t, X, s)\widehat{\psi}(t, X, s)\, \mathrm{d}\beta(s)\, \mathrm{d}X\, \mathrm{d}t.$$

Example (Constant drift)

$$\mathbf{b}(x) = \overline{\mathbf{b}} \in \mathbb{R}^d.$$

Jacobian $J(\cdot)$ **identity** *for all times.*

Example (Asymptotically constant drift)

$$\mathbf{b}(x) = \begin{cases} \mathbf{b}^* & \text{when} \quad x_1 < -a, \\ \mathbf{c}(x) & \text{when} \quad x_1 \in [-a, a], \\ \mathbf{b}^{**} & \text{when} \quad x_1 > a, \end{cases}$$

- $a > 0$, $\mathbf{e}_1 \cdot \mathbf{b}^*, \mathbf{e}_1 \cdot \mathbf{b}^{**} > 0$
- $\mathbf{c}(x)$ *chosen to make* $\mathbf{b}$ *continuously differentiable.*
- *Any integral curve spends only* **finite time** T *in* $\{x_1 \in [-a, a]\}$.

Example (Euclidean motions)

$$\mathbf{b}(x) = \mathbf{A}x + \bar{\mathbf{b}} \qquad \text{with } \mathbf{A} = -^{\top}\mathbf{A} \quad \text{and} \quad \bar{\mathbf{b}} \in \mathbb{R}^d.$$

∗ *Associated flow*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x) = \mathbf{A}\Phi_\tau(x) + \bar{\mathbf{b}}; \qquad \Phi_0(x) = x.$$

∗ *Jacobian $J(\cdot, x)$ is an orthogonal matrix.*

∗ *Jacobian matrix has no growth in τ.*

ROTATION INSIDE A BALL

Example

Let $\Omega \subset \mathbb{R}^2$ *and* $\Omega := B(0;1)$. *Advective field is a rigid rotation*

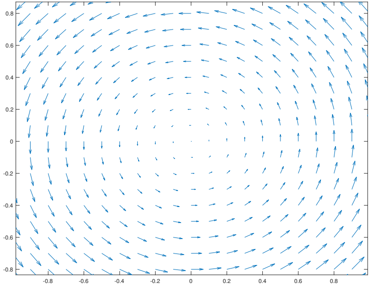
$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

- Associated flow

$$\Phi^1_\tau(x_1, x_2) = -x_2 \sin\tau + x_1 \cos\tau$$
$$\Phi^2_\tau(x_1, x_2) = x_1 \sin\tau + x_2 \cos\tau$$

- Jacobian matrix

$$J(\tau, x_1, x_2) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}$$



- algebra w.m.v. $\mathcal{A} = \mathcal{C}_{\text{per}}$.
- Note that $J^\top J = \text{Id}$.
- Hence diffusion $\mathfrak{D} = \text{Id}$.

STORY SO FAR

- For any incompressible field $\mathbf{b}(x)$, family $u^\varepsilon(t,x)$ converges weakly

$$\lim_{\varepsilon\to 0} \iint_{(0,\ell)\times\Omega} u^\varepsilon(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \iint_{(0,\ell)\times\Omega} \overline{u}(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \quad \forall\psi\in \mathrm{L}^2$$

  with $\overline{u}$ solves an evolution equation with **constraint** $\overline{u}(t,\cdot)\in\mathcal{N}_{\mathbf{b}}$

- For any field $\mathbf{b}(x)$ such that $J(\cdot,x)\in\mathcal{A}$, a certain **algebra w.m.v.**

  Then, for any $t\in(0,\ell)$ we have

$$\lim_{\varepsilon\to 0} \|w^\varepsilon(t,x) - u_0(t,x)\|_{\mathrm{L}^2(\Omega)} = 0$$

  where $w^\varepsilon(t,x) := u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)$ and

  u_0 solves a diffusion equation with diffusivity $\mathfrak{D}$.

- **There is no contradiction.**

  The operative phrase being **moving frame**.

ADVECTIVE FIELDS WITH UNBOUNDED JACOBIANS

- Two dimensional shear flow

$$\mathbf{b}(x) = \begin{pmatrix} a(x_2) \\ 0 \end{pmatrix}$$

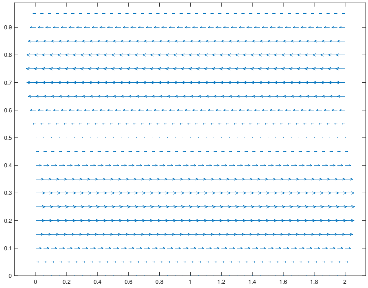
- Measure preserving flow

$$\Phi_\tau(x_1, x_2) = \begin{pmatrix} x_1 + a(x_2)\tau \\ x_2 \end{pmatrix}$$

- Jacobian matrix

$$J(\tau, x_1, x_2) = \begin{bmatrix} 1 & a'(x_2)\tau \\ 0 & 1 \end{bmatrix}$$

- Not uniformly bounded in τ
  main difficulty: $M(J) \not< \infty$.
- Lagrangian stretching.



- Compute $J(\tau, x)^\top J(\tau, x)$

$$= \begin{bmatrix} 1 & a'(x_2)\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a'(x_2)\tau & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + |a'(x_2)|^2\tau^2 & a'(x_2)\tau \\ a'(x_2)\tau & 1 \end{bmatrix}$$

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