A system of PDEs modelling noisy grid cells

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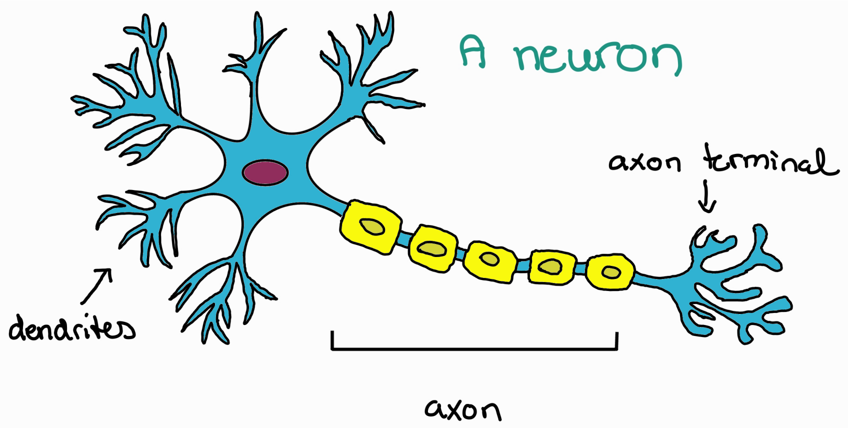
Based on works with José A. Carrillo, Andrea Clini,
and Pierre Roux (Oxford), and Helge Holden (NTNU, Trondheim).

Outline

- What are grid cells?
- What is so special about them?
- Derivation of a PDE model for grid cells.
- Response to noise?

Neurons

- Neurons are electrical excitable cells.
- They communicate via action potentials and synapses.
- A neuron is said to fire when it generates an action potential.



Grid cells

- Discovered (in rats) by Hafting, Fyhn, Molden, Moser and Moser at NTNU in 2005.
- A grid cell fires in a hexagonal pattern as a rat (or mammal) traverses an open space. **(GIF)**
- Play a pivotal role in spatial representation.

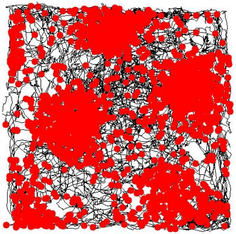
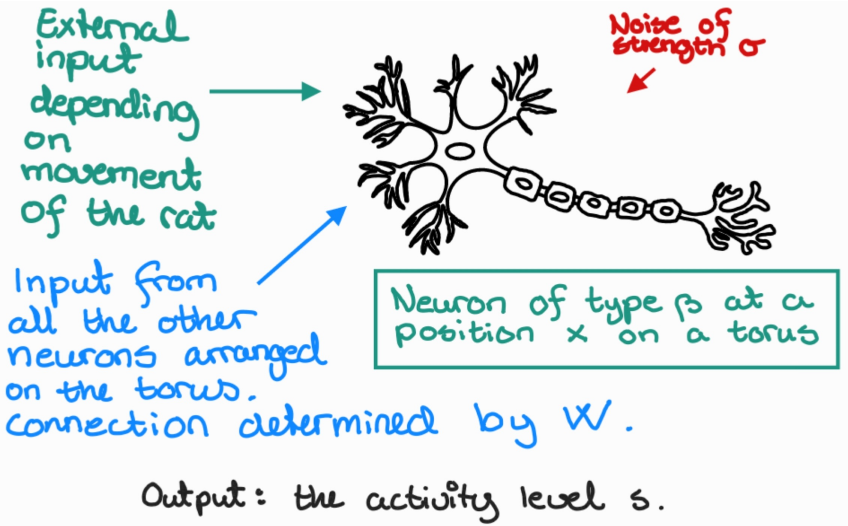


Figure: Khardcastle 2017

Set-up[1]



[1]Based on Burak and Fiete 2009, Couey et al. 2013

SDE system

Assume neurons are stacked in N columns at locations x_i on the torus, with M neurons each. The k^{th} neuron of type β at location x_i has the activity level s_{ik}^{β} described by

$$\tau ds_{ik}^{\beta} + s_{ik}^{\beta}dt = \Phi(x_i)dt + \sqrt{2\sigma}d\mathcal{W}_{ik}^{\beta} - d\ell_{ik}^{\beta},$$

$$\ell_{ik}^{\beta}(t) = -\left|\ell_{ik}^{\beta}\right|(t), \quad \left|\ell_{ik}^{\beta}\right|(t) = \int_0^t 1_{\{s_{ik}^{\beta}(r)=0\}} d\left|\ell_{ik}^{\beta}\right|(r),$$

$\beta = 1, 2, 3, 4$, where

$$\Phi(x_i) = \Phi\left(\frac{1}{4NM}\sum_{\beta'=1}^{4}\sum_{j=1}^{N}\sum_{m=1}^{M} W^{\beta'}(x_i - x_j)s_{jm}^{\beta'} + B^{\beta}(t)\right).$$

The reflection term ℓ_{ik}^{β} prevents s_{ik}^{β} from becoming negative.

Let the number of grid cells go to ∞.[2]

[2] Heuristic: Carrillo, Holden, S., to appear in JOMB 2022
Rigorous: Carrillo, Clini, S., arXiv 2021

The PDE model

For all $\beta \in \{1,2,3,4\}$, $x \in \mathbb{T}^2$, $s \geqslant 0$,

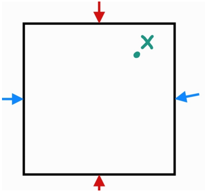
$$\tau \frac{\partial \rho^\beta}{\partial t} = -\frac{\partial}{\partial s}\left(\left[\Phi^\beta(x) - s\right]\rho^\beta\right) + \sigma \frac{\partial^2 \rho^\beta}{\partial s^2},$$

where $\Phi^\beta(\mathbf{x})$ is given by

$$\Phi^\beta(x) = \Phi\left(\frac{1}{4}\sum_{\beta'=1}^{4}\int_{\mathbb{T}^2} W^{\beta'}(x-y)\bar{\rho}^{\beta'}(t,\mathbf{y})\,\mathrm{d}\mathbf{y} + B^\beta(t)\right),$$

with $\bar{\rho}^\beta(t,x) = \int_0^\infty s\rho^\beta(t,x,s)\,\mathrm{d}s$, and

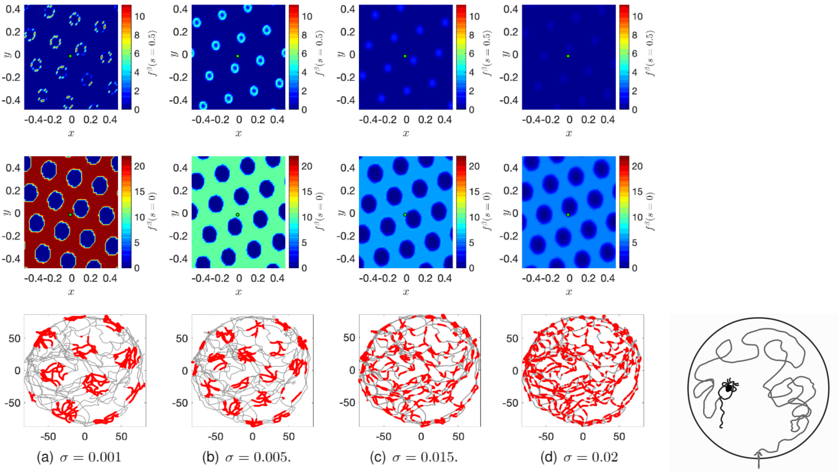
$$\Phi^\beta(x)\rho^\beta(t,x,0) - \sigma\frac{\partial \rho^\beta}{\partial s}(t,x,0) = 0.$$



$\rho(t,x,s)$ is the probability density at time t of finding a neuron at x with activity level s.

(GIF)

Idea: vary the noise parameter in the model



(a) $\sigma = 0.001$ (b) $\sigma = 0.005.$ (c) $\sigma = 0.015.$ (d) $\sigma = 0.02$

Local bifurcation analysis of the homogeneous in $\mathbf{x}$ stationary states ρ_∞.

A one population model

$$\tau \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\Big[\Phi_{\bar{\rho}}(x) - s\Big]\rho\right) + \sigma \frac{\partial^2 \rho}{\partial s^2},$$

where $\Phi^{\beta}(x)$ is given by

$$\Phi_{\bar{\rho}}(x) = \Phi\left(W * \bar{\rho}\,(t,x) + B\right),$$

where $\mathbf{B}$ **is constant** and

$$\bar{\rho}(t,x) = \int_0^\infty s\rho(t,x,s)\ \mathrm{d}s, \qquad \Phi_{\bar{\rho}}(x)\rho(t,x,0) - \sigma \frac{\partial \rho}{\partial s}(t,x,0) = 0.$$

Stationary states

Stationary states satisfy

$$\sigma \partial_s \rho(x,s) = -\left(s - \Phi_{\bar{\rho}}\right)\rho(x,s).$$

Hence, they solve

$$\mathcal{G}(\rho,\sigma) = 0, \quad \text{where} \quad \mathcal{G}(\rho,\sigma) = \rho - \frac{1}{Z}\,\mathrm{e}^{-\frac{\left(s-\Phi_{\bar{\rho}}\right)^2}{2\sigma}}\,.$$

With conservation of unit mass we have

$$Z = \int_0^{+\infty} \mathrm{e}^{-\frac{\left(s-\Phi_{\bar{\rho}}\right)^2}{2\sigma}}\,\mathrm{d}s.$$

If $B > 0$ and $W_0 = \int_{\mathbb{T}^d} W(x)\,\mathrm{d}x < 0$ (and some assumptions on Φ), the homogeneous in space stationary states ρ_∞ are unique.[3]

[3] Carrillo, Holden, S., to appear in JOMB 2022

Procedure[4]

Step 1: Simplify. Let $\kappa = \frac{1}{\sigma}$. The average in s stationary states satisfy $\bar{\mathcal{G}}(\bar{\rho}, \kappa) = 0$, where

$$\bar{\mathcal{G}}(\bar{\rho}, \kappa) = \bar{\rho} - \frac{1}{Z} \int_0^{+\infty} s\, \mathrm{e}^{-\kappa \frac{(s-\Phi_{\bar{\rho}})^2}{2}} \, \mathrm{d}s.$$

Step 2: Define the functional setting: $L^2_S(\mathbb{T}^d)$, with Hilbert basis

$$\omega_k(x) = \Theta(k) \prod_{i=1}^{d} \cos(2\pi k_i x_i), \quad k \in \mathbb{N}^d.$$

[4]Based on techniques from Carrillo, Gvalani, Pavliotis, Schlichting, ARMA 2020

Procedure[4]

Step 3: Utilise the Crandall–Rabinowitz theorem:

Grant some technical assumptions. Assume that

$$\ker\big(D_{\bar\rho}\mathcal{H}(0,\kappa_0)\big) = \operatorname{span}(\omega_0), \qquad \|\omega_0\| = 1,$$

and

$$D^2_{\bar\rho\kappa}\mathcal{H}(0,\kappa_0)[\omega_0] \notin \operatorname{range}\big(D_{\bar\rho}\mathcal{H}(0,\kappa_0)\big).$$

Then $(0,\kappa_0)$ is a bifurcation point.

Step 4: Painfully compute the needed Fréchet derivatives of the functional

$$\mathcal{H}(\bar\rho,\kappa) = \bar{\mathcal{G}}(\bar\rho + \bar\rho^{\kappa}_{\infty},\kappa).$$

[4]Based on techniques from Carrillo, Gvalani, Pavliotis, Schlichting, ARMA 2020

Result[5]

Theorem (Carrillo, Roux, S.)

Let $\kappa_0 > \frac{2|W_0|^2}{\pi B^2}$, *and assume* $\Phi'' > -C_{\kappa_0}$ *and* $\exists!\, k^* \in \mathbb{N}^d$ *s.t.*

$$\frac{\tilde{W}(k^*)}{\Theta(k^*)} = \Big[\Phi'\big(W_0\bar\rho_\infty + B\big)\left(1 - \bar\rho_\infty\left(\bar\rho_\infty - \Phi\big(W_0\bar\rho_\infty + B\big)\right)\kappa_0\right)\Big]^{-1}.$$

Then, in a neighbourhood of $(\bar\rho_\infty^{\kappa_0}, \kappa_0)$ *in* $L^2_S(\mathbb{T}^d) \times \mathbb{R}^*_+$, *the stationary states are either of the form* $(\bar\rho_\infty^{\kappa}, \kappa)$ *or on the curve*

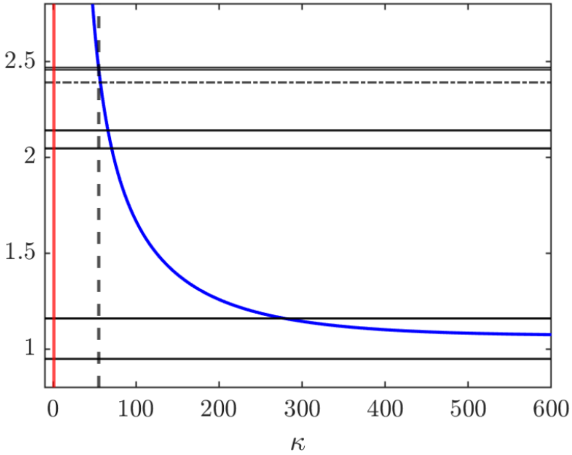
$$\Big\{\ (\bar\rho_{\kappa(z)}, \kappa(z)) \mid z \in (-\delta, \delta),\ (\bar\rho_{\kappa(0)}, \kappa(0)) = (\rho_\infty^{\kappa_0}, \kappa_0),\ \delta > 0\ \Big\},$$

defined by,

$$\bar\rho_{\kappa(z)}(x) = \bar\rho_\infty^{\kappa(z)} + z\omega_{k^*}(x) + o(z), \qquad x \in \mathbb{T}^d.$$

[5] Carrillo, Roux, S., arXiv 2022

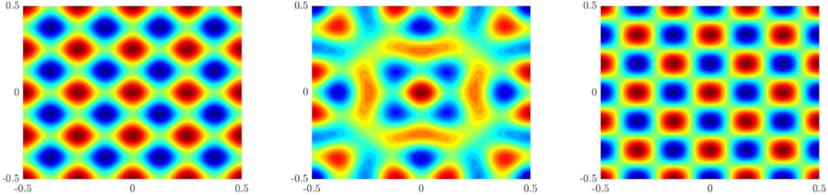
Illustration of the bifurcation condition



$$B = 3, \quad \Phi(x) = 0.5x\left(1 + \frac{x}{\sqrt{x^2+0.1}}\right)^{+}$$

$$W(x,y) = -0.005 \cdot 2^{14}\left(1 + \tanh\left(10 - 50\sqrt{x^2+y^2}\right)\right)$$

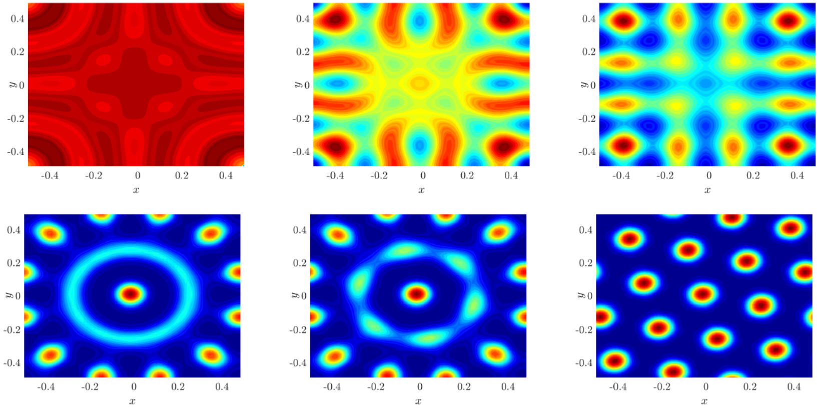
Patterns at the first three bifurcation points



Left to right (first to third bifurcation point):

$k^* = (0, 4)$ and $(4, 0)$,
$k^* = (1, 4)$ and $(4, 1)$, and
$k^* = (3, 3)$.

What about the hexagonal pattern?



Top left to bottom right: A time transient pattern for a specific σ.

Thank you!

I appreciate that you used your neurons on this presentation.