Equidistribution, Discrepancy, Pseudorandom numbers

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KAAS

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The full story

M. Lind, "A sharp estimate of the discrepancy of a concatenation sequence of inversive pseudorandom numbers with consecutive primes", Int. J. Number Theory, to appear
("A sharp estimate of the discrepancy of a certain numerical sequence", 2021, arxiv)

Equidistribution

Definition (H. Weyl, 1916)

The sequence $\xi = \{\xi_n\}_{n=1}^{\infty} \subseteq [0,1]$ is called **equidistributed in [0, 1]** if

$$\lim_{N\to\infty} \frac{\sharp(\{\xi_1, \xi_2, \dots, \xi_N\} \cap J)}{N} = \mathrm{length}(J)$$

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Equidistributed $\approx$ deterministic version of "uniformly distributed" from probability.

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Equidistribution as a concept appear in much more general settings than sequences in $[0, 1]$.

It seems to be one of the fundamental concepts of mathematics.

Weyl's criterion

$$\{\xi_n\}_{n=1}^{\infty} \text{ equidistributed in } [0,1] \Leftrightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \xi_n} = 0$$

Weyl's Criterion, number theoretic applications

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Exercise $\{[n\alpha]\}_{n=1}^{\infty}$ equidistributed in $[0,1] \Leftrightarrow \alpha \notin \mathbb{Q}$.
($[x] = x - \lfloor x \rfloor$ is the *fractional part* of x.)

Measure-theoretic formulation

Empirical measure of $\{\xi_n\}_{n=1}^N$

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$\xi = \{\xi_n\}_{n=1}^\infty$ equidistributed in $[0,1] \Leftrightarrow$

$$\lim_{N\to\infty}\mu_N(J) = m(J), \quad \text{for every interval} \quad J \subseteq [0,1]$$

($m =$ Lebesgue measure on $[0,1]$).

Discrepancy

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Faster convergence rate of $D_N^*(\xi) \Rightarrow \xi$ "more equidistributed".

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Another look at discrepancy as a deviation:

$$D_N^*(\xi) = \sup_{0 \le r \le 1} |\mu_N([0,r]) - m([0,r])|$$

"Total variation-like" distance between μ_N and m.

An example

Exercise Construct an **easy** example of a equidistributed sequence.

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Solution: put together (concatenate) **blocks** of equidistant rational numbers. Denominators increases from one block to another.

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Exercise (nice) Prove that $D_N^*(\omega) = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$, and that the convergence rate $N^{-1/2}$ is sharp:

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Shall return to ω later!

Recall the main character!

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To answer this, we need randomness!

Pseudorandom numbers

Random numbers are useful!

Problem: What/how/where is "random"?

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Generating **good** pseudorandom numbers is a serious scientific problem!

R. R. Coveyou: "Random number generation is too important to be left to chance."

D. E. Knuth: "Random numbers should not be generated with a method chosen at random."

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Any other distribution (e.g. normal, Poisson,...) can be obtained from $U[0,1]$ by **inverse transform sampling**.

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Getting the "uniformly unpredictable" integers is of course the hard part.

They are often generated arithmetically.

Simple but powerful example

Let p be a (large) prime and consider the map

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Take a "small chunk" $\{k, k+1, \ldots, k+m\} \subset \mathbb{Z}_p$ and consider its image under the inverse map. Will look rather unpredictable!

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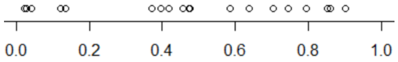

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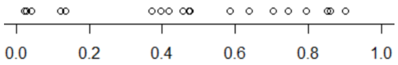

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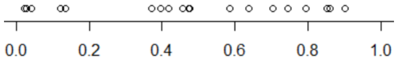

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Can also perform a statistical test.

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(Observe that $F(t) = t \quad (0 \leq t \leq 1)$ is the distribution function of a random variable $X \sim U[0,1]$.)

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Reject H_0 if T is larger than tabulated critical value.

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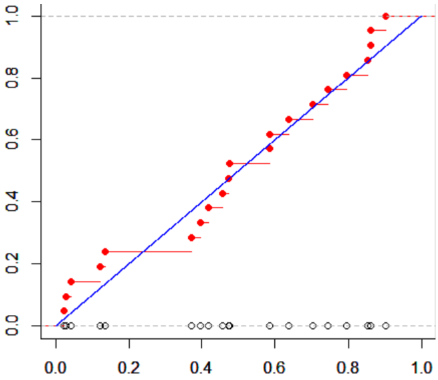

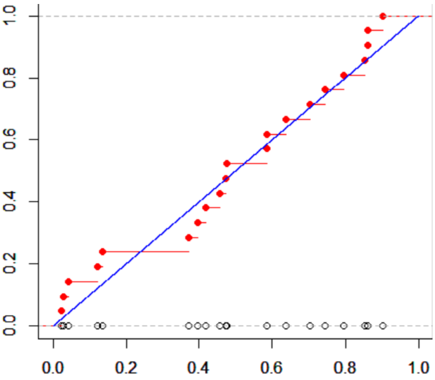

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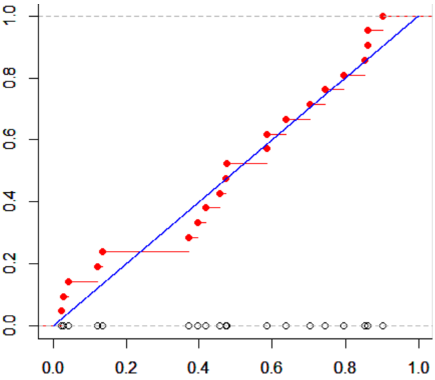

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Additional testing necessary to guarantee quality of pseudorandom numbers.

Back to ω and η

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Two possible factors that slow down the convergence rate:

1. ω contains many "duplicates", e.g. $1/2 = 2/4 = 3/6$ etc.;
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$$\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \underbrace{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}}, \dots \right\}$$

Issue with order within each block remains!

More about η

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The result is

$$\eta = \left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \underbrace{\frac{1}{7}, \frac{4}{7}, \frac{5}{7}, \frac{2}{7}, \frac{3}{7}, \frac{6}{7}}, \frac{1}{11}, \frac{6}{11}, \frac{4}{11}, \frac{3}{11}, \frac{9}{11}, \frac{2}{11}, \frac{8}{11}, \frac{7}{11}, \frac{5}{11}, \frac{10}{11}, \dots\right\}$$

The main problem

Now that we know the construction of η, I formulate the main problem that I solved

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Problem

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Motivation: silly curiosity

The main result

Theorem (The Skarphyttan Theorem, 2021)

For $N \geq 3$

$$D_N^*(\eta) \leq \frac{2}{\sqrt{N \ln(N)}}.$$

Moreover, the rate is sharp:

$$\liminf_{N\to\infty} \sqrt{N \ln(N)} D_N^*(\eta) \geq \frac{1}{2}$$

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Who/what is Skarphyttan?

The main result

Main result, some remarks

The improvement in rate of $D_N^*(\eta) = \mathcal{O}\left(\frac{1}{\sqrt{N\ln(N)}}\right)$ compared to $D_N^*(\omega) = \mathcal{O}(N^{-1/2})$ is due to the pseudorandom ordering of the elements in η.

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On the other hand, it is interesting to note the following.

(**Law of the iterated logarithm for** D_N^*)
If $\xi = \{\xi_n\}_{n=1}^{\infty}$ is a random sequence (i.e. $\xi_n \sim U[0,1]$), then almost surely

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$$D_N^*(\xi) = \mathcal{O}\left(\sqrt{\frac{\ln(\ln(N))}{N}}\right).$$

So our notion of "pseudorandom" is quite far away from "really random"!

Proof technique

If time permits: I want to say something about the proof.

Mainly to illustrate the last ingredient of the argument: asymptotics for prime numbers.

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2. the first term can be estimated as $I \leq \sum_{n=1}^{m-1} p_n$;
3. using general discrepancy estimates due to Niederreiter for inversive congruential generators (i.e. for the map $\zeta \mapsto \zeta^{-1}$ on $\mathbb{Z}_p^*$ for fixed p), the second term can be estimated as $II \leq C\sqrt{p_m}\ln^2(p_m)$.

Proof technique

I thus arrive (essentially) at

$$ND_N^*(\eta) \le \sum_{n=1}^{m-1} p_n + C\sqrt{p_m}\ln^2(p_m)$$

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$$ND_N^*(\eta) \leq \sum_{n=1}^{m-1} p_n + C\sqrt{p_m}\ln^2(p_m)$$

The above can be massaged into the desired estimate **if** I have some knowledge of the asymptotics of primes.

Last ingredient: size of primes

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I also need the asymptotic behaviour of $\sum_{n=1}^{m} p_n$.

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Heuristic argument (Skarphyttan has no Internet)

$$\begin{aligned}\sum_{n=1}^{m} p_n &\approx \sum_{n=1}^{m} n\ln(n) \approx \int_1^m x\ln(x)dx \\ &= \frac{m^2}{2}\ln(m) - \int_1^m \frac{x}{2}dx\end{aligned}$$

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&= \frac{m^2}{2}\ln(m) - \int_1^m \frac{x}{2}dx \\
&= \frac{m^2}{2}\ln(m)\left(1 - \frac{C_1}{\ln(m)} + \frac{C_2}{m^2\ln(m)}\right)
\end{aligned}$$

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Fantastically, the above heuristic actually works!

$$\sum_{n=1}^{m} p_n = \frac{m^2}{2}\ln(m)(1+o(1))$$

(see e.g. Landau's "Handbuch")