A fully discrete approximation of the one-dimensional stochastic heat equation

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Swedish Research Council (VR) project nr. $2013 - 4562$

KAAS seminar, Karlstad University $29.09.2021$

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I. Motivations and problem setting

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The problem

Problem: Consider the one-dimensional nonlinear stochastic heat equation on $[0,1]$:

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x),$$
$$u(t,0) = u(t,1) = 0 \quad \text{for } t \in (0,T),$$
$$u(0,x) = u_0(x) \quad \text{for } x \in [0,1],$$

where W is a Brownian sheet on $[0,T] \times [0,1]$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the solution $u(t,x)$ is a random field (see next slides).

Illustration (S. Cox): Model for population of micro-organisms in a closed tube filled with motionless water. Motion in water only by diffusion. Here, $u(t,x)$ would give the concentration of micro-organisms at time t and position x in the tube. At a random moment a micro-organism may divide itself ("birth"). At random moment it may die. $\frac{\partial^2}{\partial t \partial x}W(t,x)$ models randomness of births and deaths.

Applications: In biology, material science, neurophysiology, fluid dynamics, etc. see for instance the book by Lord, Powell, Shardlow.

Motivations

1. SPDE in Walsh/Dalang framework and not in the Hilbert-space setting.

2. Gyöngy, *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise I & II* $1998-1999$:
   For globally Lipschitz coefficients, $u_0 \in C([0,1])$, every $0<\alpha<1/4$, $0<\beta<1/2$, $p \geq 1$, one has:

   $$\sup_{x\in[0,1]} \mathbb{E}[|u^{M,N}(t,x)-u(t,x)|^{2p}] \leq C(p,\alpha,t)\big(N^{-\alpha p}+M^{-\beta p}\big),$$

   where $u^{M,N}$ numerical solution given by finite differences with $\Delta x = 1/M$ and semi-implicit Euler-Maruyama with $\Delta t = T/N$.
   Additional result for $u_0 \in C^3([0,1])$ and almost-sure convergence results.
   Convergence in probability for non-globally Lipschitz continuous coefficients.

3. C., Quer-Sardanyons, *A fully discrete approximation of the one-dimensional stochastic wave equation* 2016:
   Convergence results for an explicit time integrator (glob. Lip. coeff.).

Problem setting (I)

Recall: $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$.

Definition: $G(t,x)$ is a Gaussian random field if

the vector $(G(t_1,x_1),\dots,G(t_M,x_M))$ is a Gaussian random variable

for any $(t_1,x_1),\dots,(t_M,x_M)$ and $M \in \mathbb{N}$.
Furthermore, $\mu(t,x) := \mathbb{E}[G(t,x)]$ is the mean function and
$\Sigma(t,x) :=$ Cov$(G(t,x),G(s,y))$ is the covariance function.

A Brownian sheet (multi-parameter version of Brownian mot. $\beta(t) \sim N(0,t)$)
$\{W(t,x) : (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is a Gaussian random field with mean zero ($\mu = 0$) and covariance function

$$\mathbb{E}[W(t,x)W(s,y)] = (s \wedge t)(x \wedge y).$$

A space-time white noise $\dot{W}(t,x) \sim \frac{\partial^2 W(t,x)}{\partial t \partial x}$ is (formally) a mean zero Gaussian noise with

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\delta_0(x-y),$$

where δ_0 is a Dirac delta function at the origin.

Problem setting (II)

Green function: The linear deterministic heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with homogeneous Dirichlet b.c. has the Green function (i. e. $Lu = f$ has solution $u = G * f$)

$$G(t,x,y) = \sum_{j=1}^{\infty} \mathrm{e}^{-j^2\pi^2 t} \varphi_j(x)\varphi_j(y),$$

where $\big(\varphi_j(x)\big)_{j\geq 1} := \big(\sqrt{2}\sin(j\pi x)\big)_{j\geq 1}$ forms a complete orthonormal system of $L^2(0,1)$.

Mild solution: Unique solution to the stochastic heat equation $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$ (Duhamel's formula/ variation-of-const.):

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y) f(u(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t \int_0^1 G(t-s,x,y)\sigma(u(s,y))\,W(\mathrm{d}s,\mathrm{d}y).$$

Problem setting (III)

Hilbert-space setting (Da Prato's school): Another interpretation of the random forcing term is given by

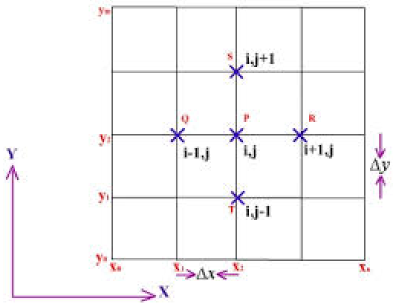
$$W(t,x) = \sum_{j=1}^{\infty} \varphi_j(x)\beta_j(t),$$

where $(\varphi_j)_{j\geq 1}$ is a complete orthonormal basis of $L^2(0,1)$, and $\beta_j(t)$ are i.i.d. standard Brownian motion, i. e. $\beta_j(t) \sim N(0,t)$ for all j.

In this setting, solutions to SPDEs $u(t)$ are seen as function-valued process with a single parameter, time t.

Walsh and Da Prato theories are essentially equivalant, see
Gyöngy, Krylov: On stochastic equations with respect to semimartingales, 1980
or
Dalang, Quer-Sardanyons: Stochastic integrals for spde's: a comparison, 2011.

II. Finite difference approximation in space



Thanks to www.images.google.com

Finite difference (I) (from *Gyöngy, Lattice approximations for ...* 1998)

Recall: $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$.

Uniform grid: Let an integer $M \geq 1$ and the partition $x_m = m/M$, for $m = 1, \ldots, M-1$, of the unit interval $(0,1)$ with equidistant (spatial) mesh size $\Delta x = 1/M$.

A discretisation of the SPDE by standard FD gives the system of SDEs:

$$\begin{aligned} \mathrm{d}u^M(t,x_m) &= M^2\left(u^M(t,x_{m+1}) - 2u^M(t,x_m) + u^M(t,x_{m-1})\right)\mathrm{d}t \\ &\quad + f(u^M(t,x_m))\,\mathrm{d}t \\ &\quad + M\sigma(u^M(t,x_m))\,\mathrm{d}(W(t,x_{m+1}) - W(t,x_m)) \end{aligned}$$

with Dirichlet boundary conditions $u^M(t,0) = u^M(t,1) = 0$, and initial value $u^M(0,x_m) = u_0(x_m)$, for $m = 1, \ldots, M-1$.
For $x \in [x_m, x_{m+1})$, define

$$u^M(t,x) := u^M(t,x_m) + (Mx - m)(u^M(t,x_{m+1}) - u^M(t,x_m)).$$

Set notations $u^M_m(t) := u^M(t,x_m)$ and $W^M_m(t) := \sqrt{M}(W(t,x_{m+1}) - W(t,x_m))$, for $m = 1, \ldots, M-1$.

Finite difference (II)

Reformulation of the FD problem in a more compact notation:

$$\mathsf{d}u_m^M(t) = M^2 \sum_{i=1}^{M-1} D_{mi} u_i^M(t)\,\mathrm{d}t + f(u_m^M(t))\,\mathrm{d}t + \sqrt{M}\sigma(u_m^M(t))\,\mathrm{d}W_m^M(t),$$

with initial value $u_m^M(0) = u_0(x_m)$, for $m = 1, \dots, M-1$.

Here $D = (D_{mi})_{m,i}$ square tridiagonal matrix of size $M-1$,
$W^M(t) := (W_m^M(t))_{m=1}^{M-1}$ is an $M-1$ dimensional Wiener process.

(Not essential for the rest) The matrix $M^2 D$ has eigenvalues

$$\lambda_j^M := -4\sin^2\left(\frac{j\pi}{2M}\right) M^2 = -j^2\pi^2 c_j^M,$$

where

$$\frac{4}{\pi^2} \le c_j^M := \frac{\sin^2\left(\frac{j\pi}{2M}\right)}{\left(\frac{j\pi}{2M}\right)^2} \le 1,$$

for $j = 1, 2, \dots, M-1$ and every $M \ge 1$.

Finite difference (III)

Recall: The FD problem is given by the system of SDEs ($m = 1, \ldots, M-1$)

$$\mathrm{d}u_m^M(t) = M^2 \sum_{i=1}^{M-1} D_{mi} u_i^M(t)\,\mathrm{d}t + f(u_m^M(t))\,\mathrm{d}t + \sqrt{M}\sigma(u_m^M(t))\,\mathrm{d}W_m^M(t).$$

Mild solution of the FD problem: Variation-of-constants formula gives:

$$\begin{aligned} u^M(t, x_m) &= \frac{1}{M} \sum_{l=1}^{M-1} \sum_{j=1}^{M-1} \exp(\lambda_j^M t) \varphi_j(x_m) \varphi_j(x_l) u_0(x_l) \\ &\quad + \int_0^t \frac{1}{M} \sum_{l=1}^{M-1} \sum_{j=1}^{M-1} \exp(\lambda_j^M (t-s)) \varphi_j(x_m) \varphi_j(x_l) f(u^M(s, x_l))\,\mathrm{d}s \\ &\quad + \int_0^t \frac{1}{\sqrt{M}} \sum_{l=1}^{M-1} \sum_{j=1}^{M-1} \exp(\lambda_j^M (t-s)) \varphi_j(x_m) \varphi_j(x_l) \sigma(u^M(s, x_l))\,\mathrm{d}W_l^M(s), \end{aligned}$$

where $\varphi_j(x) := \sqrt{2}\sin(jx\pi)$ for $j = 1, \ldots, M-1$.

Finite difference (IV)

Continuous version of the FD solution: Written now with a discrete Green function

$$\begin{aligned}u^M(t,x) &= \int_0^1 G^M(t,x,y)u^M(0,\kappa_M(y))\,\mathrm{d}y \\ &+ \int_0^t \int_0^1 G^M(t-s,x,y)f(u^M(s,\kappa_M(y)))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_0^1 G^M(t-s,x,y)\sigma(u^M(s,\kappa_M(y)))\,W(\mathrm{d}s,\mathrm{d}y)\end{aligned}$$

for $x \in (0,1)$ and $t \in (0,T]$.

Here, we use the notation $\kappa_M(y) = [My]/M$ and the discrete Green function $G^M(t,x,y)$ (see next slide).

Recall: Mild formulation of exact solution

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y)f(u(s,y))\,\mathrm{d}y\,\mathrm{d}s + \ldots$$

Finite difference (V)

Discrete Green function: For the mild solution of the FD problem one uses the discrete Green function

$$G^M(t,x,y) := \sum_{j=1}^{M-1} \exp(\lambda_j^M t)\varphi_j^M(x)\varphi_j^M(\kappa_M(y)),$$

where $\kappa_M(y) := \frac{[My]}{M}$, $\varphi_j^M(x) := \varphi_j(x_l)$ for $x = x_l$ and $\varphi_j^M(x) := \varphi_j(x_l) + (Mx - l)\left(\varphi_j(x_{l+1}) - \varphi_j(x_l)\right)$ for $x \in (x_l, x_{l+1}]$.

Precise form of G^M not essential for the rest of the presentation

III. Time discretisation: Stochastic exponential integrator

$$y_{n+1} = e^{-Ah} y_n + \int_0^h e^{-(h-\tau)A} \mathcal{N}\left(y\left(t_n + \tau\right)\right) \, d\tau.$$

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Intermezzo on exponential methods for deterministic problems

Stiff systems of deterministic differential equations: $\dot{u}(t) + Au(t) = f(u(t))$.

Exponential integrators: Numerical methods that integrates the linear part of the problem exactly.

Examples of results:
Stiff order of convergence ($|\text{error}| \leq C\Delta t^p$, with C indep. of stiffness and Δt) for ODEs and PDEs.

Longtime conservation properties for the numerical solutions of ODEs (HOP) and PDEs (semi-linear wave, Schrödinger, Hamiltonian PDE).

Applications: Reaction-advection-diffusion, mathematical finance, molecular dynamics, Maxwell's equations, etc.

Hochbruck, Ostermann 2010, Minchev, Wright 2005, C., Certaine, Deuflhard, Gauckler, Gautschi, Grimm, Hersch, Hairer, Hochbruck, Lawson, Lubich, . . . , Jentzen, Lord, Wang, . . .

Time integration

Recall: The FD problem is given by ($m = 1, \ldots, M-1$)

$$\mathrm{d}u_m^M(t) = M^2 \sum_{i=1}^{M-1} D_{mi} u_i^M(t)\,\mathrm{d}t + f(u_m^M(t))\,\mathrm{d}t + \sqrt{M}\sigma(u_m^M(t))\,\mathrm{d}W_m^M(t).$$

For an integer $N \geq 1$ and some fixed final time $T > 0$, let $\Delta t = \frac{T}{N}$ and define the discrete times $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$. Set $A := M^2 D$,

Mild sol. of the FD problem: Variation-of-constants formula on $[t_n, t_{n+1}]$

$$u^M(t_{n+1}) = e^{A\Delta t} u^M(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)} F(u^M(s))\,\mathrm{d}s$$
$$+ \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)} \Sigma(u^M(s))\,\mathrm{d}W^M(s).$$

Stochastic exponential integrator in time: With ΔW^n Wiener incr.

$$\mathcal{U}^0 := u^M(0),$$
$$\mathcal{U}^{n+1} := e^{A\Delta t}\Big(\mathcal{U}^n + F(\mathcal{U}^n)\Delta t + \Sigma(\mathcal{U}^n)\Delta W^n\Big).$$

Fully-discrete solution

Recall: $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$.

Discretisation: FD in space and exponential integrator in time.

As for the exact and semi-discrete solutions, one can find a continuous version of the fully-discrete solution $u^{M,N}(t,x)$ to the stochastic heat equation.

Mild solution of the fully-discrete problem: The process $\{u^{M,N}(t,x),\ (t,x) \in [0,T]\times[0,1]\}$ satisfies the following integral equation:

$$
\begin{aligned}
u^{M,N}(t,x) := \int_0^1 & G^M(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y \\
& + \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) f(u^{M,N}(\kappa_N^T(s),\kappa_M(y)))\,\mathrm{d}y\,\mathrm{d}s \\
& + \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) \sigma(u^{M,N}(\kappa_N^T(s),\kappa_M(y)))\,W(\mathrm{d}s,\mathrm{d}y),
\end{aligned}
$$

where $\kappa_N^T(s) := T\kappa_N(\frac{s}{T})$ (recall $\kappa_N(z) = [Nz]/N$).

Quick recap

Problem:

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$$

with, for the first results, the assumption that

$$|f(u)-f(v)| + |\sigma(u)-\sigma(v)| \leq C\big(|u-v|\big) \quad \text{and} \quad |f(u)|+|\sigma(u)| \leq C(1+|u|)$$

for all $u, v \in \mathbb{R}$.

Mild solution:

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y)f(u(s,y))\,\mathrm{d}y\,\mathrm{d}s + \ldots$$

Fully-discrete numerical solution:

$$u^{M,N}(t,x) = \int_0^1 G^M(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y + \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y)\ldots$$

For $\beta \geq 0$, define $H^\beta([0,1])$ as the set of measurable functions $g\colon [0,1] \to \mathbb{R}$ such that

$$\|g\|_\beta = \left(\sum_{j=1}^{\infty}(1+j^2)^\beta |\langle g, \varphi_j\rangle|^2\right)^{1/2} < \infty,$$

where $\varphi_j(x) = \sqrt{2}\sin(jx\pi)$, for $j = 1, 2, \ldots$, ON basis of $L^2([0,1])$.

Main results (I)

Theorem (*Anton, C., Quer-Sardanyons* $2017^* - 2018$).

1. Assume f and σ satisfy a globally Lipschitz and linear growth condition and $u_0 \in C([0,1])$. Then, for every $p \geq 1$, $t \in (0,T]$, $0 < \alpha_1 < \frac{1}{4}$ and $0 < \alpha_2 < \frac{1}{4}$, there are constants $C_i = C_i(t)$, $i = 1, 2$, such that

$$\sup_{x\in[0,1]} \left(\mathbb{E}[|u^{M,N}(t,x) - u(t,x)|^{2p}]\right)^{\frac{1}{2p}} \leq C_1(\Delta x)^{\alpha_1} + C_2(\Delta t)^{\alpha_2}.$$

2. Let $\beta > \frac{1}{2}$ and assume that $u_0 \in H^\beta([0,1])$ with $u_0(0) = u_0(1) = 0$. Assume f and σ satisfy a globally Lipschitz and linear growth condition. Then, for every $p \geq 1$, $t \in (0,T]$, $0 < \alpha_1 < \frac{1}{4}$, there are constants $C_1 = C_1(t)$ and C_2 such that

$$\sup_{x\in[0,1]} \left(\mathbb{E}[|u^{M,N}(t,x) - u(t,x)|^{2p}]\right)^{\frac{1}{2p}} \leq C_1(\Delta x)^{\alpha_1} + C_2(\Delta t)^{\tau},$$

where $\tau = \frac{1}{4} \wedge (\frac{\beta}{2} - \frac{1}{4})$.

If u_0 is sufficiently smooth (e.g. $u_0 \in C^3([0,1])$), then the estimates above hold with $\alpha_1 = 1$ and uniformly in $t \in [0,T]$.

Moreover, $u^{M,N}(t,x)$ converges to $u(t,x)$ $\mathbb{P}$-a.s., as M and N tend to infinity, uniformly with respect to $(t,x) \in [0,T] \times [0,1]$.

The spatial error estimates are results from *Gyöngy, Lattice approx. ...* 1998.

Main results (II)

Problem:

$$\frac{\partial}{\partial t}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)+f(u(t,x))+\sigma(u(t,x))\frac{\partial^2}{\partial t\partial x}W(t,x).$$

Assumptions:

- Pathwise uniqueness of solutions to the stochastic heat equation.
- f and σ are continuous.
- f and σ satisfy a linear growth condition.
- Initial value $u_0 \in H^\beta([0,1])$ for some $\beta > \frac{1}{2}$.

Theorem (*Anton, C., Quer-Sardanyons* $2017^* - 2018$). Convergence in probability: For every $\varepsilon > 0$, one has

$$\lim_{k\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]}\sup_{x\in[0,1]}|u^{M_k,N_k}(t,x)-u(t,x)|\geq\varepsilon\Big)=0$$

for all sequences of positive integers $(M_k,N_k)_{k\geq 1}$ such that $M_k, N_k \to \infty$ as $k \to \infty$.

Results on time discretisations from the literature

- *Gyöngy, Nualart* 1995: Convergence in probability of semi-implicit EM (SPDE with additive noise)
- *Gyöngy, Nualart* 1997: Strong conv. of semi-implicit EM with rate (add. results for non-smooth coeff.)
- *Shardlow* 1999*: Strong conv. of θ-scheme with rate (additive noise)
- *Gyöngy* 1999: Strong conv. of semi-implicit EM with rate (add. results for non-smooth coeff.)
- *Printems* 2001*: Strong conv. of θ-scheme with rate (add. results for non-smooth coeff.)
- *Hausenblas* 2003*: Strong conv. with rates for EM, semi-implicit EM and Crank-Nicolson
- *Walsh* 2005: Strong conv. of θ-scheme with rate
- *Jentzen* 2009*: Pathwise estimates (additive noise, non-smooth coeff.)
- *Lord, Tambue* 2013*: Strong conv. of exponential methods
- *Davie, Gaines, Pettersson, Signahl, Millet, Morian, Müller-Gronbach, Ritter, Jentzen, Kloeden, Cox, Van Neerven, Wang, Barth, Lang, Röckner, Prohl, Bréhier, Cui, Hong, Liu, Qiao, etc.*

References* deal with convergence in $L^p(0,1)$, i. e. $\mathbb{E}[\|\text{error}\|_{L^p}^2]^{1/2}$.

Main ingredients of the proofs (glob. Lipschitz case)

Recall:

$$
\begin{aligned}
u^{M,N}(t,x) := & \int_0^1 G^M(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y \\
& + \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) f(u^{M,N}(\kappa_N^T(s),\kappa_M(y)))\,\mathrm{d}y\,\mathrm{d}s \\
& + \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) \sigma(u^{M,N}(\kappa_N^T(s),\kappa_M(y)))\,W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}
$$

1. Write the error as $u^{M,N}(t,x) - u(t,x) = u^{M,N}(t,x) - u^M(t,x) + u^M(t,x) - u(t,x)$ and use the spatial error estimate from *Gyöngy, Lattice approximations . . .* 1998.

2. Analyse $u^{M,N}(t,x) - u^M(t,x)$ and use a Gronwall-type argument.

3. To do this: Use properties of $G^M(t,x,y)$, of $u^M(t,x)$, of $u^{M,N}(t,x)$, assumptions on f and σ, Hölder's inequality, Burkholder–Davis–Gundy's inequality and finally Gronwall-type inequality to bound the error of the fully-discrete solution.

Main steps for the proofs (I)

Using mild forms of u^M and $u^{M,N}$, one obtains

$$\begin{aligned} u^{M,N}(t,x) - u^M(t,x) = \int_0^t\!\!\int_0^1 \Big\{ & G^M(t-\kappa_N^T(s),x,y) f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) \\ & - G^M(t-s,x,y) f\big(u^M(s,\kappa_M(y))\big) \Big\} \,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t\!\!\int_0^1 \Big\{ & G^M(t-\kappa_N^T(s),x,y) \sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) \\ & - G^M(t-s,x,y) \sigma\big(u^M(s,\kappa_M(y))\big) \Big\} W(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

Next, add and subtract some terms in order to be able to use properties of f and σ.

Main steps for the proofs (II)

The stochastic integral in

$$u^{M,N}(t,x) - u^M(t,x) = \int_0^t\!\!\int_0^1 \Big\{ G^M(t-\kappa_N^T(s),x,y)\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) - G^M(t-s,x,y)\sigma\big(u^M(s,\kappa_M(y))\big)\Big\}\, W(\mathrm{d}s,\mathrm{d}y) + \int_0^t\!\!\int_0^1 \text{blabla terms}$$

can be decomposed as the sum of 2 terms:

$$B_1 = \int_0^t \int_0^1 (G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y))\sigma(u^{M,N}(\kappa_N^T(s),\kappa_M(y)))\, W(\mathrm{d}s,\mathrm{d}y)$$

$$B_2 = \int_0^t \int_0^1 G^M(t-s,x,y)(\sigma(u^{M,N}(\kappa_N^T(s),\kappa_M(y))) - \sigma(u^M(s,\kappa_M(y))))\, W(\mathrm{d}s,\mathrm{d}y).$$

Similarly for the deterministic integral.

Main steps for the proofs (III)

Estimates for B_1: By Burkholder's and Minkowski's inequality one has

$$
\begin{aligned}
(\mathbb{E}[|B_1|^{2p}])^{1/p} &= \left(\mathbb{E}\left[\left|\int_0^t\int_0^1 (G^M(t-\kappa_N^T(s)) - G^M(t-s))\sigma(u^{M,N}(\cdot)\,W(\mathrm{d}s,\mathrm{d}y)\right|^{2p}\right]\right)^{1/p} \\
&\leq C\left(\mathbb{E}\left[\left(\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s)) - G^M(t-s)|^2|\sigma(u^{M,N}(\cdot))|^2\,\mathrm{d}y\mathrm{d}s\right)^p\right]\right)^{1/p} \\
&=: C|||\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s)) - G^M(t-s)|^2|\sigma(u^{M,N}(\cdot))|^2\,\mathrm{d}y\mathrm{d}s|||_p \\
&\leq C\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s)) - G^M(t-s)|^2|||\sigma(u^{M,N}(\cdot))|||_{2p}^2\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

Use linear growth assumption for σ, boundedness of the numerical solution, and estimates for G^M* to get

$$
\begin{aligned}
(\mathbb{E}[|B_1|^{2p}])^{1/p} &\leq \sup_{(s,y)\in[0,T]\times[0,1]} |||\sigma(u^{M,N}(s,y))|||_{2p}^2 \\
&\quad\times\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leq C(\Delta t)^{1/2}.
\end{aligned}
$$

$* \sup_{M\geq 1}\sup_{x\in[0,1]}\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)|^2\,\mathrm{d}y\,\mathrm{d}s \leq C(\Delta t)^{1/2}$

Main steps for the proofs (IV)

Estimates for B_2: Similarly, one gets

$$(\mathbb{E}[|B_2|^{2p}])^{1/p} \leq C \int_0^t \int_0^1 |G^M(t-s,x,y)|^2 \, \mathrm{d}y \times \sup_{y\in[0,1]} |||\sigma(u^{M,N}(\kappa_N^T(s),y)) - \sigma(u^M(s,y))|||_{2p}^2 \, \mathrm{d}s.$$

Use Lipschitz assumption on σ and estimates for G^M to get

$$\begin{aligned}(\mathbb{E}[|B_2|^{2p}])^{1/p} &\leq C \int_0^t \int_0^1 |G^M(t-s)|^2 \, \mathrm{d}y \sup_{x\in[0,1]} |||u^{M,N}(\kappa_N^T(s),x) - u^M(s,x)|||_{2p}^2 \, \mathrm{d}s \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \Bigg(\sup_{x\in[0,1]} |||u^{M,N}(\kappa_N^T(s),x) - u^{M,N}(s,x)|||_{2p}^2 \\ &\quad + \sup_{x\in[0,1]} |||u^{M,N}(s,x) - u^M(s,x)|||_{2p}^2 \Bigg) \, \mathrm{d}s.\end{aligned}$$

Now distinguish between the two different cases for the initial value u_0.

Main steps for the proofs (V)

Recall:

$$(\mathbb{E}[|B_2|^{2p}])^{1/p} \leq C \int_0^t \int_0^1 |G^M(t-s)|^2 \,\mathrm{d}y \sup_{x\in[0,1]} |||u^{M,N}(\kappa_N^T(s),x) - u^M(s,x)|||_{2p}^2 \,\mathrm{d}s$$

$$\leq C \int_0^t \frac{1}{\sqrt{t-s}} \left(\sup_{x\in[0,1]} |||u^{M,N}(\kappa_N^T(s),x) - u^{M,N}(s,x)|||_{2p}^2 + \sup_{x\in[0,1]} |||u^{M,N}(s,x) - u^M(s,x)|||_{2p}^2 \right) \mathrm{d}s.$$

If $u_0 \in H^\beta([0,1])$ for some $\beta > \frac{1}{2}$, use regularity estimates for $u^{M,N}(t,x)$ to get

$$(\mathbb{E}[|B_2|^{2p}])^{1/p} \leq C(\Delta t)^\tau + C \int_0^t \frac{1}{\sqrt{t-s}} z(s) \,\mathrm{d}s,$$

where $\tau = \frac{1}{2} \wedge (\beta - \frac{1}{2})$ and

$$z(s) := \sup_{(r,x)\in[0,s]\times[0,1]} |||u^{M,N}(r,x) - u^M(r,x)|||_{2p}^2.$$

If $u_0 \in C([0,1])$: more technical.

Main steps for the proofs (VI)

Recall: $u^{M,N}(t,x) - u^{M}(t,x) = B_1 + B_2 + A_1 + A_2$, where A_1 and A_2 come from the deterministic integral.

If $u_0 \in H^{\beta}([0,1])$ for some $\beta > \frac{1}{2}$, we end up with (recall $z(s) := \sup_{(r,x)\in[0,s]\times[0,1]} |||u^{M,N}(r,x) - u^{M}(r,x)|||_{2p}^2$):

$$\begin{aligned} z(t) &\leq C\left(\mathbb{E}[|A_1|^{2p}] + \mathbb{E}[|A_2|^{2p}] + \mathbb{E}[|B_1|^{2p}] + \mathbb{E}[|B_2|^{2p}]\right) \\ &\leq C_1(\Delta t)^{\tau} + C_2 \int_0^t \frac{1}{\sqrt{t-s}} z(s)\,\mathrm{d}s, \end{aligned}$$

for $\tau = \frac{1}{2} \wedge (\beta - \frac{1}{2})$.

A version of Gronwall's lemma concludes the proof of the mean-square error estimates:

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} \left(\mathbb{E}[|u^{M,N}(t,x) - u^{M}(t,x)|^{2p}]\right)^{\frac{1}{2p}} \leq C(\Delta t)^{\frac{\tau}{2}}.$$

A similar analysis provides the result for $u_0 \in C([0,1])$.

(Skip?) Almost sure convergence: main ideas (I)

Recall: Mean-square error estimates for the time discretisation

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}\left(\mathbb{E}\left[|u^{M,N}(t,x)-u^{M}(t,x)|^{2p}\right]\right)^{\frac{1}{2p}}\leq C(\Delta t)^{1/4}.$$

Let $N>0$ and use Markov's inequality (after some work)

$$\mathbb{P}\left(\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}|u^{M,N}(t,x)-u^{M}(t,x)|^{2p}>\left(\frac{1}{N}\right)^{2}\right)\leq C\left(\frac{1}{N}\right)^{2p\min(\delta,\mu)-4}$$

for $\delta\in(0,1/4)$ and $\mu\in(0,1/4)$.

This is summable if p is large enough.

Use Borel–Cantelli to get

$$\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}|w^{M,N}(t,x)-w^{M}(t,x)|^{2p}\leq\frac{1}{N^{2}}$$

with probability one.

(Skip?) Almost sure convergence: main ideas (II)

Recall: Borel–Cantelli gave us ($\mathbb{P}$-a.s.)

$$\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}|w^{M,N}(t,x)-w^{M}(t,x)|^{2p}\leq \frac{1}{N^2}\quad \text{for all large enough}\quad N.$$

This provides almost sure convergence of the numerical solution

$$\sup_{x\in[0,1]}\sup_{t\in[0,T]}|u^{M,N}(t,x)-u(t,x)|\underset{a.s.}{\longrightarrow}0\quad \text{as}\quad M,N\to\infty$$

using a result by Gyöngy for the a.s. convergence of the spatial approximation $u^{M}(t,x)$.

Conv. in probability for non-glob. Lip. coeff. (I)

1. Show that $\{u^{M,N}(t,x)\}_{M,N\geq 1}$ is tight in $C([0,T]\times[0,1])$:

   To do so:
   Use Hölder regularity results for the numerical solutions: For $0\leq s,t\leq T$, $0\leq x,y\leq 1$ and any $p\geq 1$ one has

   $$\mathbb{E}[|u^{M,N}(t,x)-u^{M,N}(s,y)|^{2p}]\leq C\big(|t-s|^{\frac{\tau p}{2}}+|x-y|^{\tau p}\big),$$

   where $\tau=1\wedge(2\beta-1)$ with a constant C independent of M and N.
   Use a tightness criterium on the plane from *Bardina, Jolis, Quer-Sardanyons* 2010.

   Need: Linear growth condition.

2. Prokhorov's theorem implies that $\{u^{M,N}(t,x)\}_{M,N\geq 1}$ is relatively compact in $C([0,T]\times[0,1])$.

3. Now, fix any pair of sequences $(M_k,N_k)_{k\geq 1}$ such that $M_k,N_k\to\infty$, as $k\to\infty$. Then, the laws of $v_k:=u^{M_k,N_k}$, $k\geq 1$, form a tight family in the space $C([0,T]\times[0,1])$.

Conv. in probability for non-glob. Lip. coeff. (II)

4. Switch from weak convergence to a.s. convergence:

   Consider two subsequences $(v_j^1)_{j\geq 1}$ and $(v_\ell^2)_{\ell\geq 1}$ of $(v_k)_{k\geq 1}$.
   By Skorokhod's Representation Theorem, there exists subsequences of positive integers $(j_r)_{r\geq 1}$ and $(\ell_r)_{r\geq 1}$ of the indices j and ℓ, a prob. space $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}_t})_{t\geq 1}, \widehat{\mathbb{P}})$, and a sequence of cont. rdm fields $(z_r)_{r\geq 1}$ with $z_r := \left(\widetilde{u}_r, \overline{u}_r, \widehat{W}_r\right)$, $r \geq 1$, such that

   $$z_r \underset{r\to\infty}{\longrightarrow} z := (\widetilde{u}, \overline{u}, \widehat{W}) \text{ a.s. in } C([0,T]\times[0,1], \mathbb{R}^3)$$

   and the finite dimensional distributions of z_r and $\zeta_r := \left(v_{j_r}^1, v_{\ell_r}^2, W\right)$ are the same for $r = 1, 2, \ldots$.

5. Show that $\widetilde{u}$ and $\overline{u}$ are mild solutions to the heat equation:

   Since law$(z_r) =$ law(ζ_r), for $r = 1, 2, \ldots$, and components of ζ_r satisfy weak form, so do the components of z_r for $r = 1, 2, \ldots$.
   Take $r \to \infty$ in the weak forms for $\widetilde{u}_r$ and $\overline{u}_r$ to show that $\widetilde{u}$ and $\overline{u}$ are solutions to the stochastic heat eq.

   Need: Continuity of f and σ and results from *Gyöngy* 1998.

Conv. in probability for non-glob. Lip. coeff. (III)

5. Show that $\widetilde{u} = \overline{u}$:

   One gets $\widetilde{u} = \overline{u} \quad \widehat{\mathbb{P}}$-a.s.

   Need: Pathwise uniqueness.

6. Show uniform convergence in probability of $\{u^{M_k,N_k}\}_{k\geq 1}$ to u, solution to the stochastic heat equation.

   Need: Result from *Gyöngy* 1998 for a criterium for convergence in probability.

IV. Numerical experiments

Thanks to www.images.google.com

Settings (I)

Problem: Stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$$
$$u(t,0) = u(t,1) = 0$$
$$u(0,x) = u_0(x) \quad \text{for } \; x \in [0,1].$$

Spatial discretisation: Finite difference on a uniform grid ($m = 1, \ldots, M-1$)

$$\mathsf{d}u_m^M(t) = M^2 \sum_{i=1}^{M-1} D_{mi} u_i^M(t)\,\mathrm{d}t + f(u_m^M(t))\,\mathrm{d}t + \sqrt{M}\sigma(u_m^M(t))\,\mathrm{d}W_m^M(t)$$

or in the more compact form

$$u^M(t) = e^{A\Delta t}u^M(0) + \int_0^t e^{A(t-s)}F(u^M(s))\,\mathrm{d}s + \int_0^t e^{A(t-s)}\Sigma(u^M(s))\,\mathrm{d}W^M(s),$$

with the finite difference matrix $A := M^2 D$.

Settings (II)

Recall: System resulting from spatial discretisation:

$$u^M(t) = e^{A\Delta t} u^M(0) + \int_0^t e^{A(t-s)} F(u^M(s))\,\mathrm{d}s + \int_0^t e^{A(t-s)} \Sigma(u^M(s))\,\mathrm{d}W^M(s).$$

Time discretisations: Done with the following numerical schemes:

Stochastic exponential integrator (SEXP)

$$\mathcal{U}^0 := u^M(0),$$
$$\mathcal{U}^{n+1} := e^{A\Delta t}\big(\mathcal{U}^n + F(\mathcal{U}^n)\Delta t + \Sigma(\mathcal{U}^n)\Delta W^n\big),$$

with the $(M-1)$-dimensional Wiener increments
$\Delta W^n := W^M(t_{n+1}) - W^M(t_n)$.

Semi-implicit Euler-Maruyama scheme (SEM)

$$\mathcal{U}^{n+1} = \mathcal{U}^n + \Delta t A \mathcal{U}^{n+1} + \Delta t F(\mathcal{U}^n) + \Sigma(\mathcal{U}^n)\Delta W^n.$$

Semi-implicit Crank-Nicolson-Maruyama scheme (CNM)

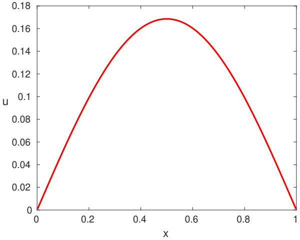
$$\mathcal{U}^{n+1} = \mathcal{U}^n + \frac{\Delta t}{2} A\big(\mathcal{U}^{n+1} + \mathcal{U}^n\big) + \Delta t F(\mathcal{U}^n) + \Sigma(\mathcal{U}^n)\Delta W^n.$$

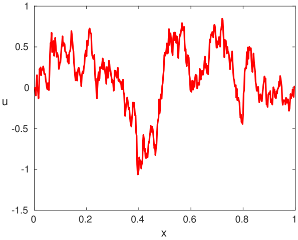
Profile of the numerical solutions

Problem: Consider

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x)$$
$$u(t,0) = u(t,1) = 0$$
$$u(0,x) = u_0 \quad \text{for } x \in [0,1],$$

with $u_0(x) = \sin(\pi x)$, $f(u) = \sin(u)$, $\sigma(u) = 3 - 0.1u$, and $T = 0.2$.



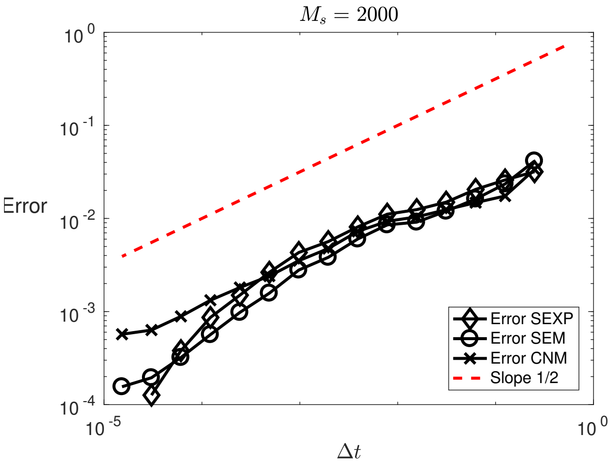


Movie Heat

Temporal rates of convergence

Temporal rates of convergence ($\Delta x_{\text{ref}} = 2^{-9}$ and $\Delta t_{\text{ref}} = 2^{-16}$)

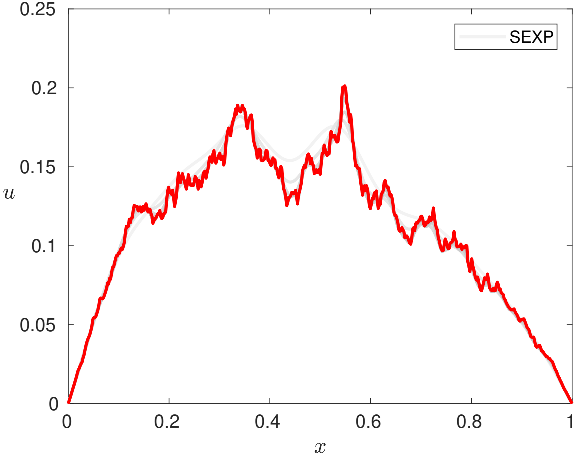
$$\sup_{(t,x)\in[0,0.5]\times[0,1]} \mathbb{E}[|u^{M,N}(t,x) - u^M(t,x)|^2]$$



Data $u_0(x) = \cos(\pi(x - 1/2))$, $f(u) = u/2$, $\sigma(u) = 1 - u$.

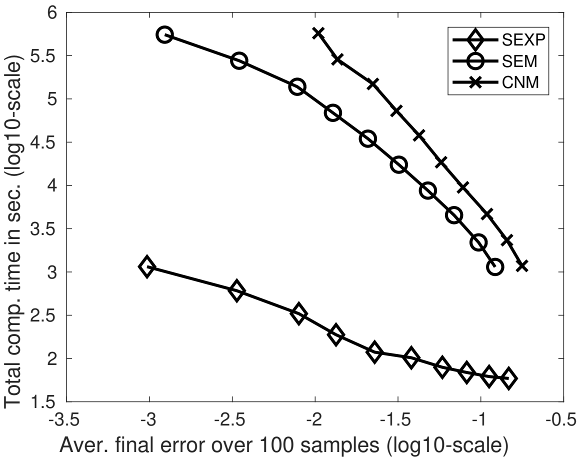
Almost sure convergence

Reference solution in red computed with $\Delta x_{\text{ref}} = 2^{-9}$ and $\Delta t_{\text{ref}} = 2^{-20}$.
Numerical solutions with $\Delta t = 2^{-10}$ to $\Delta t = 2^{-20}$ from light to dark grey plots.



Data $u_0(x) = \cos(\pi(x - 1/2))$, $f(u) = 1 - u$, $\sigma(u) = \sin(u)$, $\Delta x = 2^{-9}$, and $T = 0.5$.

Computational cost



Data $u_0(x) = \cos(\pi(x - 1/2))$, $f(u) = 1 - u$, $\sigma(u) = \sin(u)$, $\Delta x = 2^{-9}$, $\Delta t_{\text{ref}} = 2^{-16}$ and $T = 1$.

V. Ongoing work

Thanks to www.images.google.com

Stoch. heat equations in higher dimension

With Lluís Quer-Sardanyons (UAB) and Johan Ulander (Chalmers), we are extending the presented results to the SHE in higher dimension:

Let $d \geq 1$ be an integer and set $Q = [0,1]^d$. Let $\alpha = \in (0, 2 \wedge d)$. The noise F is defined by means of a family of centered Gaussian random variables $F = \{F(\varphi), \, \varphi \in \mathcal{D}(\mathbb{R}_+ \times Q)\}$ with covariance structure

$$\mathbb{E}\left[F(\varphi)F(\psi)\right] = \int_0^\infty \int_Q \int_Q \varphi(t,x)|x-y|^{-\alpha}\psi(t,y)\,\mathsf{d}x\,\mathsf{d}y\,\mathsf{d}t.$$

The SPDE is then given by

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\frac{\partial^2 F}{\partial t \partial x}(t,x)$$

$$u(t,x) = 0 \quad \text{for } x \text{ on the boundary of } Q.$$

Difficulties: The noise has some covariance structure in space. This adds technical difficulties in the theoretical analysis as well as implementation issues.

Thanks for your attention!!

Rikard Anton, David Cohen, Lluis Quer-Sardanyons *A fully discrete approximation of the one-dimensional stochastic heat equation*, arXiv (2017), IMAJNA (2018).

Thanks to www.images.google.com