Strong contrasting diffusivity in general oscillating domains: Homogenization of optimal control problems

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Joint work with Prof. A. K. Nandakumaran

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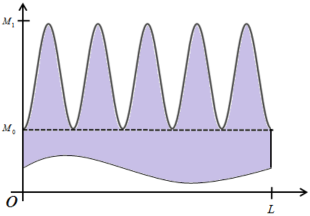
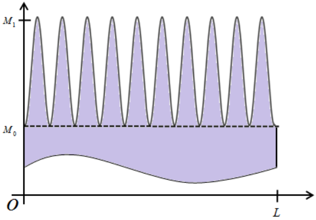
Oscillating domain

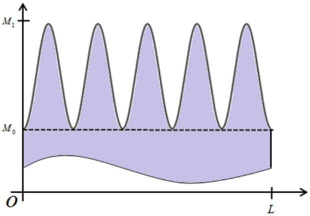


$\varepsilon = \frac{1}{5}$



$\varepsilon = \frac{1}{10}$

Oscillating domain



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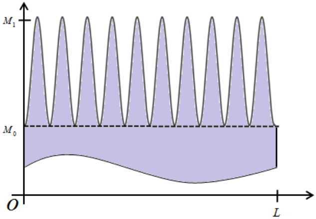
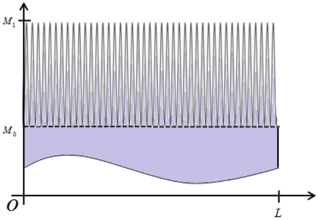
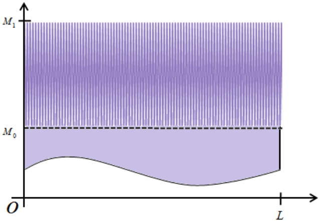


$\varepsilon = \frac{1}{10}$



$\varepsilon = \frac{1}{50}$



$\varepsilon = \frac{1}{100}$

Introduction: Strong contrasting diffusivity

- Partial differential equations (PDEs) with strong contrasting diffusivity are appeared in several context such as: modeling of several multi-scale physical problems such as the double porosity model, effective properties of composite material with soft and hard core, effective conductivity of composites made of materials having high and low conductivities, etc.

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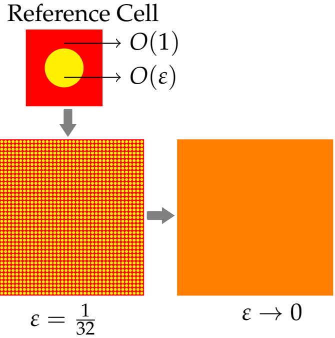

Figure 1: Composite material

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To be more precised, they have considered domain like the following;

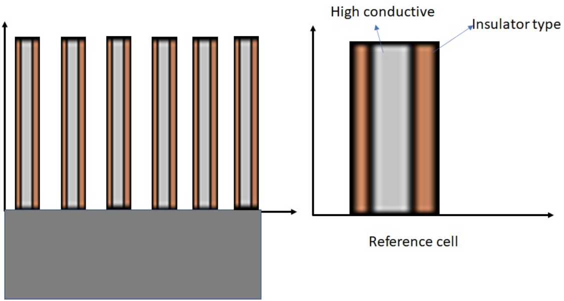

Figure 2: Pillar type oscillating domain

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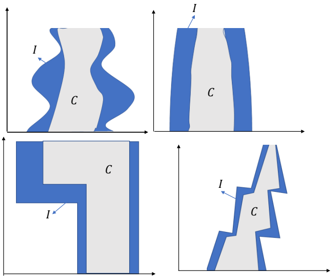

Figure 3: Typical example of reference cells

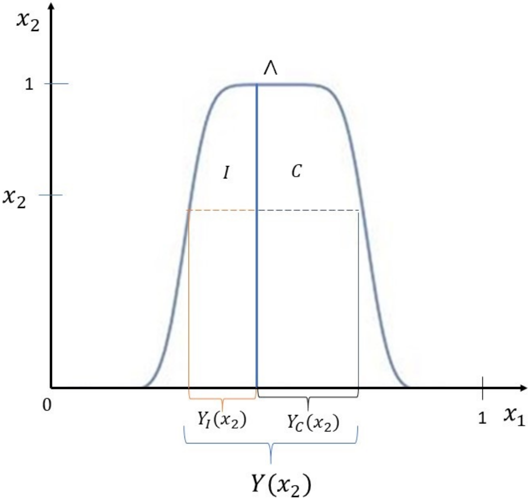

Figure 4: Reference cell

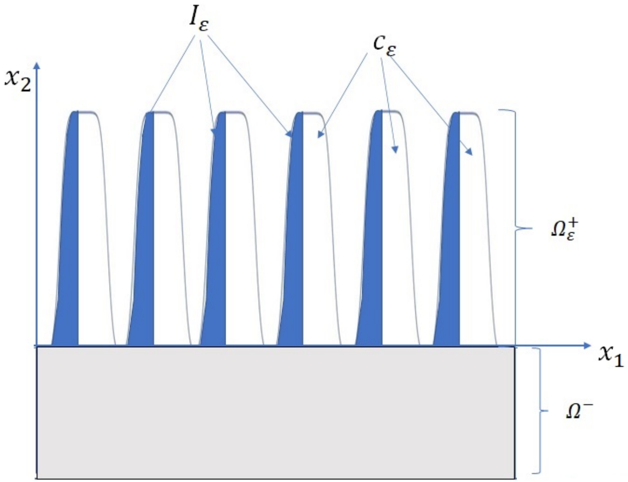

Figure 5: Oscillating domain

Assumptions on the reference cell

- Let $\Lambda \subset (0,1) \times (0,1)$ (it is just for simplicity, one can consider $(0,L) \times (0,L)$ for any $L > 0$) and $C, I \subset \Lambda$. We divide Λ into two components C and I, that is $\overline{\Lambda} = \overline{C} \cup \overline{I}$, $C \cap I = \phi$ (empty set)and satisfies the following properties:

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    We assume that there exists $\delta > 0$, such that the $|Y(x_2)|$, $|Y_C(x_2)|$,$|Y_I(x_2)| > \delta$ for all $x_2 \in (0,1)$.

Oscillating domain

For $\varepsilon = \frac{1}{m}$ where $m \in \mathbb{Z}^+$, (in fact, one can take any $\varepsilon \to 0$) define

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- The interface between Ω^+ and Ω^- is demoted by γ, which is given by $\gamma = \{(x_1, 0) : \ x_1 \in (0,1)\}$.

Variational problem

- We want to consider the following ε dependent variational problem,

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{\mathrm{C}_\varepsilon} + \varepsilon^2 \chi_{\mathrm{I}_\varepsilon} \right) \nabla u_\varepsilon \nabla \phi + \int_{\Omega_\varepsilon} u_\varepsilon \phi = \int_{\Omega_\varepsilon} f \phi, \end{array} \right. \tag{1}$$

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  for all $\phi \in H^1(\Omega_\varepsilon)$, where $f \in L^2(\Omega)$. The Lax-Milgram theorem ensures the existence and the uniqueness of the solution u_ε of the problem (1).
- Our aim is to analyze the asymptotic behavior of the above variational form as the oscillating parameter $\varepsilon \to 0$.

Unfolding operator and properties

The unfolded domain corresponding to the upper part Ω_ε^+ is given by

- $\Omega^u = \{(x_1, x_2, y_1) : (x_1, x_2) \in \Omega^+, y_1 \in Y(x_2)\}$.

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Definition.

(The unfolding operator) Let $\phi^\varepsilon : \Omega^u \to \Omega_\varepsilon^+$ be defined as $\phi^\varepsilon(x_1, x_2, y_1) = \left(\varepsilon \left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right)$. The ε- unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega^u \to \mathbb{R}$. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator is denoted by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon u : \Omega^u \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon u(x_1, x_2, y_1) = u\left(\varepsilon \left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right) \textit{ for all } (x_1, x_2, y_1) \in \Omega^u.$$

We denote $T^{\varepsilon}|_{\Omega^{u}_{C}}$ by T^{ε}_{C} .

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For each $\varepsilon > 0$,

- *for* $u \in L^{2}(\Omega_{\varepsilon}^{+})$. *Then,* $\|T^{\varepsilon}(u)\|_{L^{2}(\Omega^{u})} = \|u\|_{L^{2}(\Omega_{\varepsilon}^{+})}$

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- *For* $u \in H^1(C_{\varepsilon})$, *we have* $T_C^{\varepsilon}u, \dfrac{\partial}{\partial x_2}(T_C^{\varepsilon}u) \in L^2(\Omega_C^u)$. *Moreover,* $\dfrac{\partial}{\partial x_2}T_C^{\varepsilon}u = T_C^{\varepsilon}\dfrac{\partial u}{\partial x_2}$ *and* $\dfrac{\partial}{\partial y_1}T_C^{\varepsilon}u = \varepsilon T_C^{\varepsilon}\dfrac{\partial u}{\partial x_1}$.

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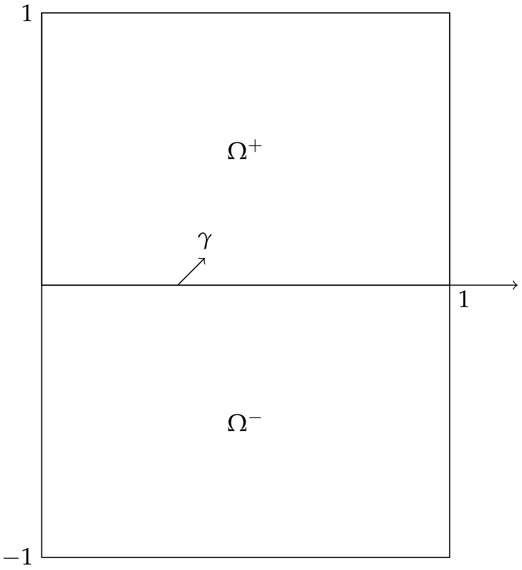
- *For* $u \in H^1(C_\varepsilon)$*, we have* $T^\varepsilon_C u, \dfrac{\partial}{\partial x_2}(T^\varepsilon_C u) \in L^2(\Omega^u_C)$*. Moreover,* $\dfrac{\partial}{\partial x_2} T^\varepsilon_C u = T^\varepsilon_C \dfrac{\partial u}{\partial x_2}$ *and* $\dfrac{\partial}{\partial y_1} T^\varepsilon_C u = \varepsilon T^\varepsilon_C \dfrac{\partial u}{\partial x_1}$.
- *Let, for every* $\varepsilon > 0$, $u_\varepsilon \in L^2(\Omega^+_\varepsilon)$ *be such that* $T^\varepsilon u_\varepsilon \rightharpoonup u$ *weakly in* $L^2(\Omega^u)$*. Then,*
$$\tilde{u}_\varepsilon \rightharpoonup \int_{Y(x_2)} u(x_1, x_2, y_1) dy_1 \ \text{ weakly in } \ L^2(\Omega^+).$$

Limit domain



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- Define
  $H(\Omega)$=$\{\phi : \phi^+ \in L^2((0,1); H^1(0,1)), \phi^- \in H^1(\Omega^-), \phi^+ = \phi^- on\ \gamma\}$ with the following norm

$$\|\phi\|_{H(\Omega)} = \|\phi^-\|_{H^1(\Omega^-)} + \|\phi^+\|_{L^2(\Omega^+)} + \left\|\frac{\partial\phi^+}{\partial x_2}\right\|_{L^2(\Omega^+)}.$$

Limit function spaces

For any function ϕ defined on Ω, we may write $\phi = \phi^+\chi_{\Omega^+} + \phi^-\chi_{\Omega^-} = (\phi^+, \phi^-)$ throughout the presentation.

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$$\|\psi\|_{V(\Omega)} = \|\psi\|_{L^2(\Omega^u)} + \left\| \frac{\partial \psi}{\partial y_1} \right\|_{L^2(\Omega^u)}.$$

Limit problem

The limit variational problem :

$$\left\{\begin{array}{l} \text{find } u = (u^+, u^-) \in H(\Omega) \text{ such that} \\ \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u^- \phi \\ \qquad\qquad\qquad \displaystyle + \int_{\Omega^-} \nabla u^- \nabla \phi = \int_{\Omega^+} \alpha(x) f \phi + \int_{\Omega^-} f \phi, \end{array}\right.$$

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for all $\phi \in H(\Omega)$, here $\alpha(x) = \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \, dy_1 \right)$,where

$$\begin{cases} \xi(x_2, \cdot) \in V^{x_2} \\ \int_{Y(x_2)} \frac{\partial \xi(x_2, y_1)}{\partial y_1} \frac{\partial w(y_1)}{\partial y_1} + \int_{Y(x_2)} \xi(x_2, y_1) w(y_1) = \int_{Y(x_2)} w(y_1), \end{cases}$$

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Theorem.

For every $\varepsilon > 0$*, let* u_ε *be the unique solution to the considered variational problem. Let* $H(\Omega)$ *and* V^{x_2} *be defined as earlier and* $u = (u^+, u^-) \in H(\Omega)$ *be the unique solution of the limit variational form.*

Theorem.

For every $\varepsilon > 0$, let u_ε be the unique solution to the considered variational problem. Let $H(\Omega)$ and V^{x_2} be defined as earlier and $u = (u^+, u^-) \in H(\Omega)$ be the unique solution of the limit variational form. Then

$$u_\varepsilon^- \rightharpoonup u^- \text{ weakly in } H^1(\Omega^-),$$

$$\begin{cases} \widetilde{u_\varepsilon^+} \rightharpoonup |Y(x_2)|u^+ + \int_{Y_I(x_2)} (f - u^+)\xi(x_2, y_1)dy_1 \\ \chi_{C_\varepsilon}^+ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup 0, \quad \chi_{C_\varepsilon}^+ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \\ \varepsilon\chi_{I_\varepsilon}^+ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup (f - u^+) \int_{Y_I(x_2)} \frac{\partial \xi}{\partial y_1} dy_1, \quad \varepsilon\chi_{I_\varepsilon}^+ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup 0 \\ \text{weakly in } L^2(\Omega^+) \end{cases}$$

as $\varepsilon \to 0$.

Sketch of the proof

- By taking $\phi = u_\varepsilon$ as a test function to get

$$\|\chi_{C_\varepsilon^+} \nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon \|\chi_{I_\varepsilon^+} \nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^-)} \\ + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|u_\varepsilon\|_{L^2(\Omega^-)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)}$$

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- Observe that

$$\|\nabla u_\varepsilon\|_{L^2(C_\varepsilon^+)} \leq k, \ \|\nabla u_\varepsilon\|_{L^2(I_\varepsilon^+)} \leq k\varepsilon^{-1},$$

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- By the properties of unfolding operator and weak compactness of $H^1(\Omega^-)$ and $L^2(\Omega^u)$ there exist $u^- \in H^1(\Omega^-)$, $u_0(x, y_1) \in L^2(\Omega^u)$, $\eta(x, y_1) = (\eta_1, \eta_2)$ and , $z(x, y_1) = (z_1, z_2) \in (L^2(\Omega^u))^2$ such that
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In the remaining steps, we identify $u_0, \eta_1, \eta_2, z_1, z_2$ and get properties enjoyed by these functions.

Identification

- u_0 is independent of y_1 in Ω_C^u and the existence of $u^+ \in L^2(\Omega^+)$, $u_1 \in L^2(\Omega^u)$ with $u_1 = 0$ a.e. in Ω_C^u

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This follows from $T_C^\varepsilon u_\varepsilon \rightharpoonup u_0(x,y_1)|_{\Omega_C^u}$ weakly in $L^2(\Omega_C^u)$, and $\dfrac{\partial}{\partial y_1} T_C^\varepsilon u_\varepsilon^+ = \varepsilon T_C^\varepsilon \left(\dfrac{\partial u_\varepsilon}{\partial x_1}\right)$.

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$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} \varepsilon\chi_{I_\varepsilon}\frac{\partial u_\varepsilon^+}{\partial x_1}\psi_\varepsilon &= \lim_{\varepsilon\to 0}\varepsilon\int_{\Omega^u} T^\varepsilon\chi_{I_\varepsilon}T^\varepsilon\left(\frac{\partial u_\varepsilon^+}{\partial x_1}\right)T^\varepsilon\psi_\varepsilon \\
&= \lim_{\varepsilon\to 0}\int_{\Omega^u}\chi_I\left(T^\varepsilon u_\varepsilon^+\right)\frac{\partial}{\partial y_1}T^\varepsilon\psi_\varepsilon \\
&= -\int_{\Omega^u}(u^+ + u_1(x,y_1))\chi_I(y_1,x_2)\frac{\partial\psi}{\partial y_1}.
\end{aligned}$$

Identification

Hence we have,

$$\int_{\Omega^u} \chi_{\mathrm{I}}(y_1, x_2) z_1(x, y_1) \psi(x, y_1) = -\int_{\Omega^u} \chi_{\mathrm{I}}(y_1, x_2) (u^+ + u_1(x, y_1)) \frac{\partial \psi}{\partial y_1}$$

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$$\int_{\Omega^u_{\rm C}} T^\varepsilon \left(\frac{\partial u^+_\varepsilon}{\partial x_2}\right) \phi dx dy_1 = -\int_{\Omega^u_{\rm C}} T^\varepsilon u^+_\varepsilon \frac{\partial \phi}{\partial x_2} dx dy_1 + \int_{\gamma^u_{\rm C}} T^\varepsilon (u^+_\varepsilon)\phi.$$

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Hence we have $u^+ = u^-$ on γ^u_{C}. Since u^+ and u^- are independent of y_1, we have $u^+ = u^-$ on γ.

Limit variational form

- Let $\phi_\varepsilon(x) = \phi(x) + \phi_1\left(x, \frac{x_1}{\varepsilon}\right)$ where $\phi \in C^1(\bar{\Omega})$ and $\phi_1 \in C^\infty(\overline{\Omega^u})$ with 1 periodic in y_1 variable and $\phi_1 = 0$ on Ω^u_C.

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- Now using ϕ_ε as a test function in (1), applying unfolding operator both side and letting $\varepsilon \to 0$ to get

$$\int_{\Omega^u_C} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_I} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int_{\Omega^-} \nabla u^- \nabla \phi + \int_{\Omega^-} u^- \phi = \int_{\Omega^u} f(\phi + \phi_1) + \int_{\Omega^-} f\phi$$

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- Hence, using the cell problem we get,

$$u_1(x, y_1) = (f(x) - u^+(x))\xi(x_2, y_1).$$

- Now if we put $\phi_1 = 0$ and substitute the definition of u_1 to get

$$\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi dy_1 \right) u^+ \phi +$$
$$\int_{\Omega^-} (\nabla u^- \nabla \phi + u^- \phi) = \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi dy_1 \right) f \phi + \int_{\Omega^-} f \phi,$$

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- Now we will show $\alpha(x) = \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \right) > 0$. By taking $w = \xi$ in the cell problem, we get

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$$\int_{Y(x_2)} \left(\left| \frac{\partial \xi}{\partial y_1} \right|^2 + \xi^2 \right) = \int_{Y(x_2)} \xi \Rightarrow \|\xi\|_{L^2(Y_I(x_2))} \leqslant |Y_I(x_2)|^{\frac{1}{2}}$$

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$$\left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \right) \geqslant (|Y(x_2)| - |Y_I(x_2)|^{1/2} \|\xi\|_{L^2(Y_I(x_2))})$$
$$\geqslant |Y(x_2)| - |Y_I(x_2)| = |Y_C(x_2)| > \delta.$$

Control on C_ε

For $\theta_\varepsilon \in L^2(C_\varepsilon)$ consider the cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2} \int_{C_\varepsilon} |\theta_\varepsilon|^2$$

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$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\Omega_\varepsilon} \chi_{C_\varepsilon} \theta_\varepsilon \phi, \\ \text{for all } \phi \in H^1(\Omega_\varepsilon). \end{cases}$$

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The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(C_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon)\}. \tag{2}$$

- We will use the characterization of optimal control $\bar{\theta}_\varepsilon$ by introducing the adjoin state $\bar{v}_\varepsilon$ which is the solution of the following variational form

$$\left\{ \begin{array}{l} \text{find } \bar{v}_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon} \right) \nabla \bar{v}_\varepsilon \nabla \phi + \bar{v}_\varepsilon \phi = \int_{\Omega_\varepsilon} (\bar{u}_\varepsilon - u_d) \phi, \end{array} \right. \tag{3}$$

for all $\phi \in H^1(\Omega_\varepsilon)$.

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for all $\phi \in H^1(\Omega_\varepsilon)$.

Theorem.

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (2) *and $\bar{v}_\varepsilon$ be the unique solution of* (3)*. Then $\bar{\theta}_\varepsilon$ is characterized by*

$$\bar{\theta}_\varepsilon = -\chi_{C_\varepsilon} \frac{1}{\beta} \bar{v}_\varepsilon. \tag{4}$$

Limit optimal control

Cost functional: For $\theta \in L^2(\Omega^+)$

$$J(u,\theta) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} \left|(1-\xi)u^+ + f\xi - u_d\right|^2 + \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2}\int_{\Omega^+} |Y_C(x_2)||\theta|^2$$

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Limit state equation:

$$\begin{cases} \text{find } u \in H(\Omega), \text{ such that,} \\ \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u\phi + \int_{\Omega^-} \nabla u^- \nabla \phi \\ \displaystyle\qquad = \int_{\Omega^+} \alpha(x) f\phi + \int_{\Omega^-} f\phi + \int_{\Omega^+} |Y_C(x_2)|\theta\phi, \\ \text{for all } \phi \in H(\Omega). \end{cases}$$

- $J(\bar{u},\bar{\theta}) = \inf\{J(u,\theta)\}$

Adjoint equation:

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial \bar{v}^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x)\bar{v}^+\phi + \int_{\Omega^-} \bar{v}^-\phi + \int_{\Omega^-} (\nabla \bar{v}^- \nabla \phi \\ \displaystyle = \int_{\Omega^+} \left[\left(\int_{Y(x_2)} (1-\xi)^2 dy_1 \right) \bar{u}^+ - \alpha(x) u_d + \left(\int_{Y_I(x_2)} (\xi - \xi^2) dy_1 \right) f \right] \phi \\ \displaystyle \qquad + \int_{\Omega^-} (\bar{u}^- - u_d)\phi. \end{cases}$$

Adjoint equation:

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- Optimal control is given by $\bar{\theta} = -\frac{1}{\beta}\bar{v}^+$

Control on I_ε

For $\theta_\varepsilon \in L^2(I_\varepsilon)$, consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\mathrm{I}_\varepsilon} |\theta_\varepsilon|^2,$$

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where u_ε is the unique solution of the following variational problem: for $f \in L^2(\Omega)$

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{\mathrm{C}_\varepsilon} + \varepsilon^2 \chi_{\mathrm{I}_\varepsilon} \right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f \phi + \int_{\Omega_\varepsilon} \chi_{\mathrm{I}_\varepsilon} \theta_\varepsilon \phi, \end{cases} \tag{5}$$

for all $\phi \in H^1(\Omega_\varepsilon)$.

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for all $\phi \in H^1(\Omega_\varepsilon)$. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(I_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : (u_\varepsilon, \theta_\varepsilon) \text{ satisfies (5)}\}. \tag{6}$$

Theorem (Characterization).

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ *be the optimal solution to the optimal control problem* (6) *and* $\bar{v}_\varepsilon$ *be the unique solution of the adjoint state. Then* $\bar{\theta}_\varepsilon$ *can be written as* $\bar{\theta}_\varepsilon = -\chi_{I_\varepsilon} \dfrac{1}{\beta} \bar{v}_\varepsilon$.

Partial scales separation

Reduced cost functional: The L^2-cost functional reduces to

$$J(u, u_{11}, \theta, \theta_1) = \frac{1}{2} \int_{\Omega^+} \int_{Y(x_2)} ((1-\xi)u^+ + \xi f + u_{11} - u_d)^2 + \int_{\Omega^-} (u^- - u_d)^2 + \frac{\beta}{2} \int_{\Omega^+} \int_{Y(x_2)} (\theta + \theta_1)^2$$

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Reduced state equation: The state $(\bar{u}, \bar{u}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+}\alpha(x)u^+\phi^+ + \int_{\Omega^-}\nabla u^-\nabla\phi^- + \int_{\Omega^-} u^-\phi \\ \displaystyle\quad = \int_{\Omega^+}\int_{Y(x_2)}((1-\xi)f + (1-\xi)(\theta+\theta_1))\phi^+ + \int_{\Omega^-} f\phi^-, \\ \displaystyle\int_{\Omega^u}\frac{\partial u_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{11}\phi_1 = \int_{\Omega^u}(\theta+\theta_1)\phi_1, \end{cases}$$

- $J(\bar{u}, \bar{u}_{11}, \bar{\theta}, \bar{\theta}_1) = \inf\{J(u, u_{11}, \theta, \theta_1)\}$

- **Reduced adjoint state:** The adjoint state $(\bar{v}, \bar{v}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial \bar{v}^+}{\partial x_2} \frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} \alpha(x) \bar{v}^+ \phi^+ + \int_{\Omega^-} \nabla v^- \nabla \phi^- + \int_{\Omega^-} u^- \phi^- \\ \qquad = \displaystyle\int_{\Omega^-} (\bar{u}^- - u_d)\phi^- + \int_{\Omega^+} \int_{Y(x_2)} \left[(1-\xi)^2 \bar{u}^+ + \xi(1-\xi) f\right] \phi^+ \\ \qquad\qquad + \displaystyle\int_{\Omega^+} \int_{Y(x_2)} [(1-\xi)\bar{u}_{11} - (1-\xi) u_d] \, \phi^+, \\ \displaystyle\int_{\Omega^u} \frac{\partial \bar{v}_{11}}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} \bar{v}_{11} \phi_1 = \int_{\Omega^u} [(1-\xi)\bar{u}^+ + \xi f + \bar{u}_{11} - u_d] \phi_1, \end{cases}$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.

- **Reduced adjoint state:** The adjoint state $(\bar{v}, \bar{v}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

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  for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.
- The optimal control is given by $\bar{\theta} + \bar{\theta}_1 = -\frac{1}{\beta}[(1-\xi)\bar{v}^+ + \bar{v}_{11}]$ in Ω_{I}^u.

Remark

In the above variational problem we have considered the contrasting diffusive coefficients as 1 and ε^2. In fact, we can consider the coefficient of the form $O(1)$ and α_ε^2, where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. According to the limit $k = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\varepsilon}$, we will get three different limit problems for, $k = 0, k = \infty$ and $k \in (0, \infty)$.

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Thank you for your attention!