# Heuristic methods for rank minimization with applications

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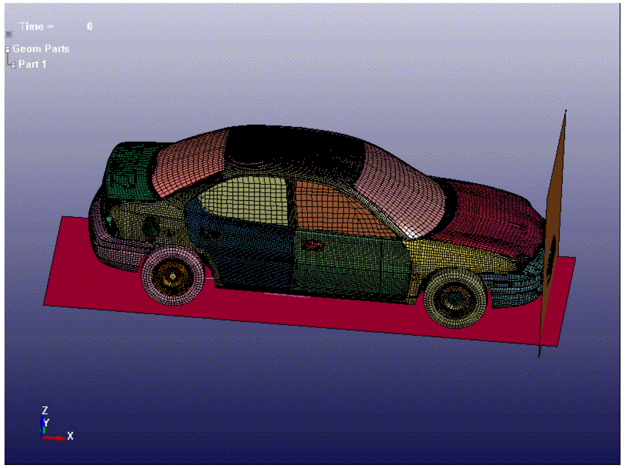
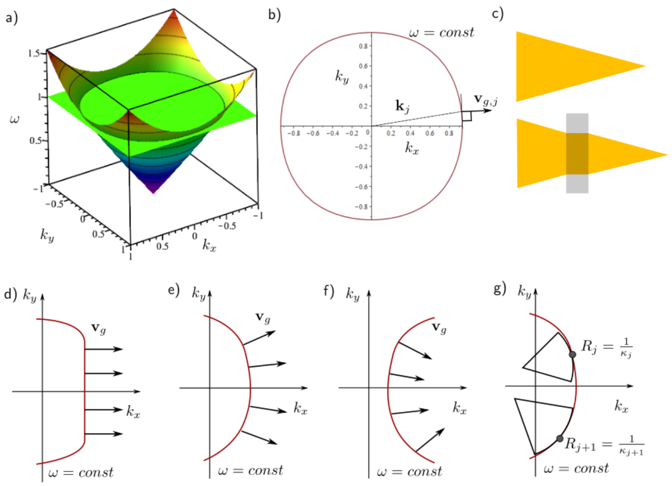
# Bio

- Born in Kharkiv, Ukraine in 1986
- Study of mechanical engineering at Kharkiv (2002-2007) and computational mechanics at Stuttgart (2007-2009)
- Dr.-Ing. on finite element method with Manfred Bischoff in 2013
- PostDoc with Kurt Maute at Colorado on 4D printing (2014) and on dynamics Stuttgart (2015-2020)
- Senior lecturer in mechanical engineering at Karlstat from 2021
- Future focus on dynamics, simulation-based material design, heuristics in optimization and damage/fatigue mechanics



# Outline

- Origins of rank minimization formulations
  - Minimal Euclidean embedding
  - Matrix completion
- Exact method for a special case
- Rank minimization heuristics
  - Trace and Log-Det
- Applications of rank minimization formulations
  - Numerical dispersion for finite difference stencils
  - Minimization of numerical dispersion for FE stencils
  - Dispersion design of periodic systems
- Conclusions

# Origins of rank minimization formulations

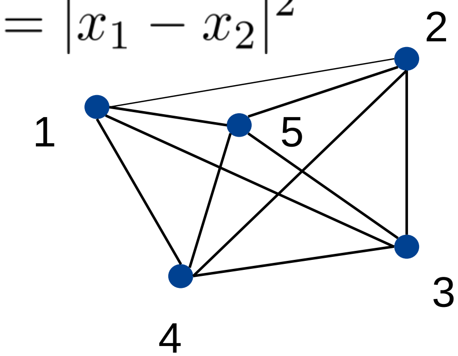
## Minimal Euclidean embedding. Known information

Euclidean distance matrix

$$\mathbf{D} \in \mathbb{R}^{n \times n} \quad D_{ij} = |x_i - x_j|^2$$

$$x_i \in \mathbb{R}^r$$

$$D_{12} = |x_1 - x_2|^2$$




Minimal possible *r* is called **embedding dimension**

Theorem by Schoenberg’1935: $\mathbf{D}$ is EDM with embedding dimension *r* iff

$$\begin{aligned} D_{ii} &= 0 \\ \mathbf{VDV} &\prec 0 \quad \mathbf{V} = \mathrm{eye}(n) - \mathrm{ones}(n)/n \\ \mathrm{rank}\,(\mathbf{VDV}) &\leq r \end{aligned}$$

$$\mathbf{V}|_{n=3} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

First two conditions are equivalent to distance properties

$$D_{ii} = 0 \quad D_{ij} = D_{ji} \geq 0 \quad D_{ik} + D_{ki} \geq D_{ij}$$

# Example

## Minimal Euclidean embedding. Known information

Euclidean distance matrix with uncertainty in the measurement

$$n = 5 \quad \mathbf{D} \in \mathbb{R}^{n \times n}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 18.86 & 15.89 & 57.72 \pm 10 & 78.61 \pm 10 \\ 18.86 & 0 & 15.21 & 15.46 & 112.77 \pm 10 \\ 15.89 & 15.21 & 0 & 48.61 & 45.48 \\ 57.72 \pm 10 & 15.46 & 48.61 & 0 & 182.93 \\ 78.61 \pm 10 & 112.77 \pm 10 & 45.48 & 182.93 & 0 \end{bmatrix}$$

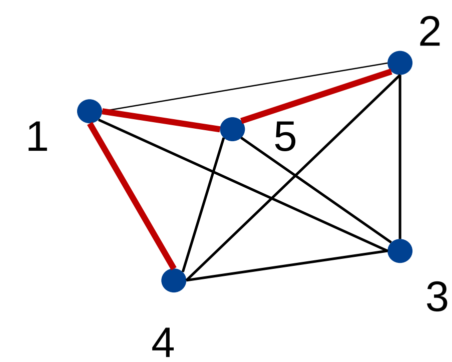
Find the possible embedding dimension r by a problem

$$\min \operatorname{rank}(\mathbf{VDV})$$

$$D_{ii} = 0$$

$$-\mathbf{VDV} \succ 0 \quad \mathbf{V} = \mathrm{eye}(n) - \mathrm{ones}(n)/n$$

$$L_{ij} \leq D_{ij} \leq U_{ij}, \{ij\} \in Q$$



# Example

## Minimal Euclidean embedding. Solution

Euclidean distance matrix with uncertainty in the measurement

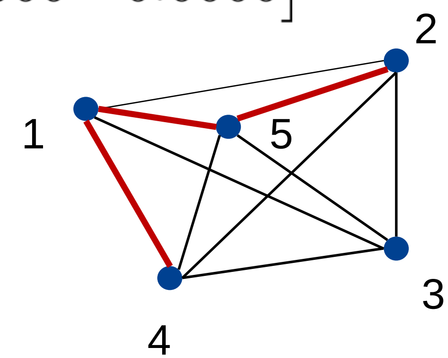
$$n = 5 \quad \mathbf{D} \in \mathbb{R}^{n \times n}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 18.86 & 15.89 & 57.72 + x_1 & 78.61 + x_2 \\ 18.86 & 0 & 15.21 & 15.46 & 112.77 + x_3 \\ 15.89 & 15.21 & 0 & 48.61 & 45.48 \\ 57.72 + x_1 & 15.46 & 48.61 & 0 & 182.93 \\ 78.61 + x_2 & 112.77 + x_3 & 45.48 & 182.93 & 0 \end{bmatrix}$$

$$\mathbf{x}^* = \begin{bmatrix} 9.9947 & -9.9266 & -4.8421 \end{bmatrix}$$

$$\operatorname{rank}\left(\mathbf{V}\mathbf{D}(\mathbf{x}^*)\mathbf{V}, tol = 10^{-6}\right) = 3$$

$$\operatorname{eig}\left(-\mathbf{V}\mathbf{D}(\mathbf{x}^*)\mathbf{V}\right) = \begin{bmatrix} 202.0116 & 31.4276 & 1.2671 & 0.0000 & 0.0000 \end{bmatrix}$$



# Origins of rank minimization formulations

## Matrix completion

Partial information about a matrix

$$\mathbf{X} \in \mathbb{R}^{n\times m}$$

$$\Phi : \mathbb{R}^{n\times m} \to \mathbb{R}^P \quad \Phi(\mathbf{X}) = (x_{ij})_{(ij)\in I} =: \mathbf{y}$$

$$P = \mathrm{round}(C(n+m)\log(n+m))$$

Low rank matrix completion

$$\mathbf{X}^* = \arg\min \mathrm{rank}\,(\mathbf{X})$$
$$\Phi(\mathbf{X}) = \mathbf{y}$$

**Netflix prize ($\$10^6$)**

Predict preferences of users from few rating values

Columns ($10^6$): ratings (100) of a user from 1 to 5
Rows ($10^5$): films

| | U1 | U2 | U3 | |
|---|---|---|---|---|
| | * | 3 | * | Film 1 |
| | 5 | * | 2 | Film 2 |
| $\mathbf{X} =$ | 4 | 5 | * | Film 3 |
| | * | 2 | 5 | Film 4 |

[Matrix Completion with Nuclear Norm Minimization (numerical-tours.com)]
[Matrix Completion and the Netflix Prize - YouTube]

# Exact method for a special case

## Rank minimization problem with constraint on the norm difference

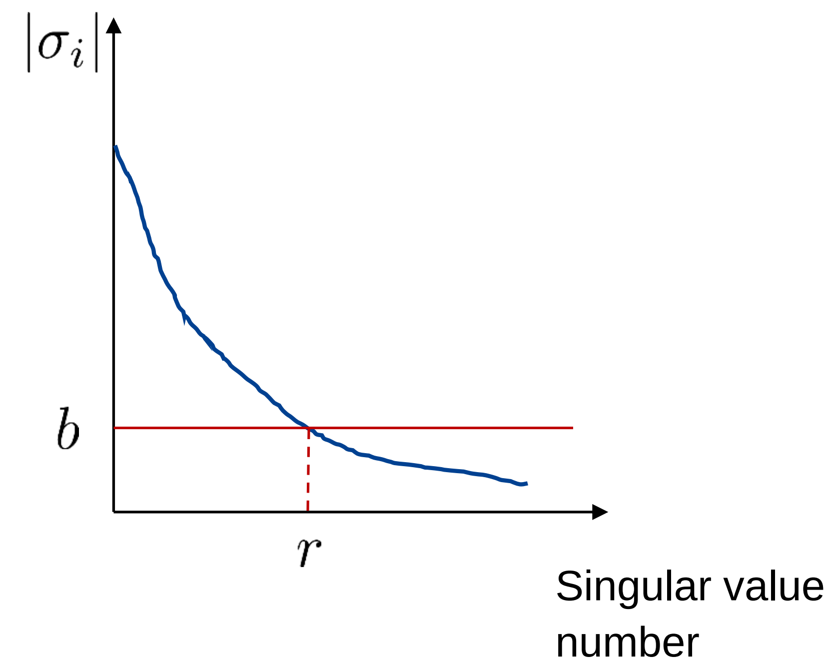
Problem definition

$$\mathbf{X}^* = \arg\ \min \operatorname{rank}(\mathbf{X})$$
$$||\mathbf{X} - \mathbf{A}||_{2,F} \leq b$$

Solution

$$[\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}] = \operatorname{svd}(\mathbf{A})$$
$$\tilde{\mathbf{\Sigma}} = \operatorname{diag}(f(\sigma_i))$$
$$f(x) = \begin{cases} x & |x| > b \\ 0 & |x| \leq b \end{cases}$$
$$\mathbf{X}^* = \mathbf{U}\tilde{\mathbf{\Sigma}}\mathbf{V}^T$$

Thresholding of singular values



# Rank minimization heuristics

## Requirements to RMP algorithms

- General, e.g. handle any symmetric positive semi-definite matrix and convex constraints $\mathbf{X} \in \mathcal{C}$
- No strict reqirements on initialization
- Numerically efficient, e.g., reduce to a quadratic programing problem
- Depend on few interpretable parameters
- Effective for practical problems

Herein, two heuristics from simple to more complex are presented.

Test example with square matrix $\boldsymbol{X}$ depending on parameters $a$ and $b$

$$\mathbf{A} = \begin{bmatrix} \frac{5}{4} - 2b & \frac{1}{3} - a & \frac{1}{4} - 2b \\ \frac{1}{3} - a & \frac{1}{3} - a & \frac{1}{3} - a \\ \frac{1}{4} - 2b & \frac{1}{3} - a & \frac{1}{4} - 2b \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$$

Known rank-1 solution: $$\mathbf{X}^*\left(a = \frac{1}{3}, b = \frac{1}{8}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1$$

# Rank minimization heuristics

## Trace heuristics

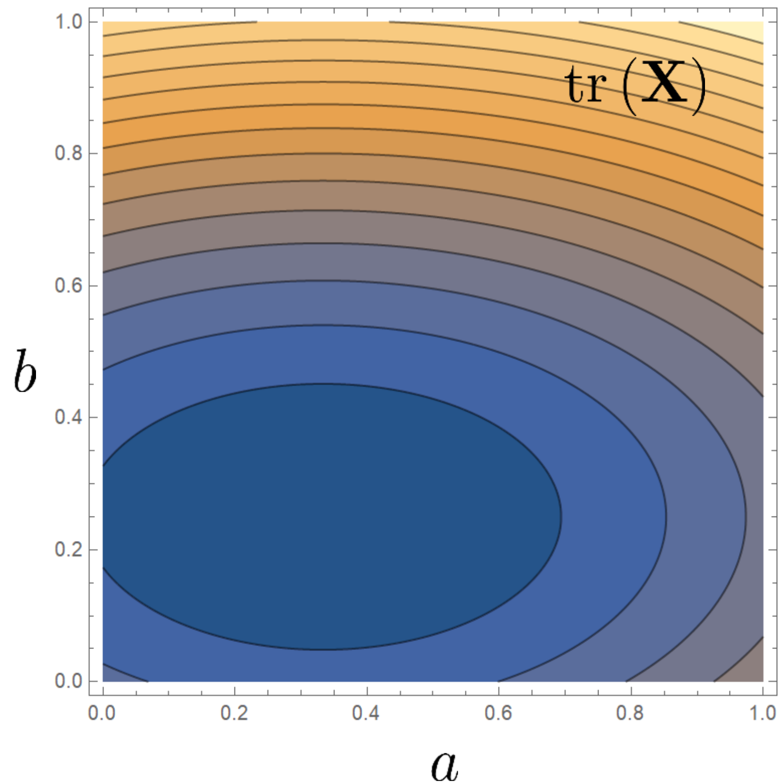
Observation for symmetric semi-positive matrices: minimizing trace gives low-rank solutions

Smooth surrogate problem

$$\mathbf{X}^* = \arg\min_{\mathbf{X} \in \mathcal{C}} \operatorname{trace}(\mathbf{X})$$

Obtained rank-2 solution:

$$\mathbf{X}^*\left(a = \frac{1}{3}, b = \frac{1}{4}\right) = \begin{bmatrix} \frac{5}{8} & 0 & -\frac{1}{8} \\ 0 & 0 & 0 \\ -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}$$




MESBAHI, M., & PAPVASSILOUPOULOS, G. P. (1997, JUNE). SOLVING A CLASS OF RANK MINIMIZATION PROBLEMS VIA SEMI-DEFINITE PROGRAMS, WITH APPLICATIONS TO THE FIXED ORDER OUTPUT FEEDBACK SYNTHESIS. IN PROCEEDINGS OF THE 1997 AMERICAN CONTROL CONFERENCE (CAT. NO. 97CH36041) (VOL. 1, PP. 77-80). IEEE..

# Rank minimization heuristics

## Log-det heuristics

Smooth surrogate problem

$$\mathbf{X}^* = \arg\min \left(\log\det\left(\mathbf{X} + \delta \mathbf{I}\right)\right)$$
$$\mathbf{X} \in \mathcal{C}$$

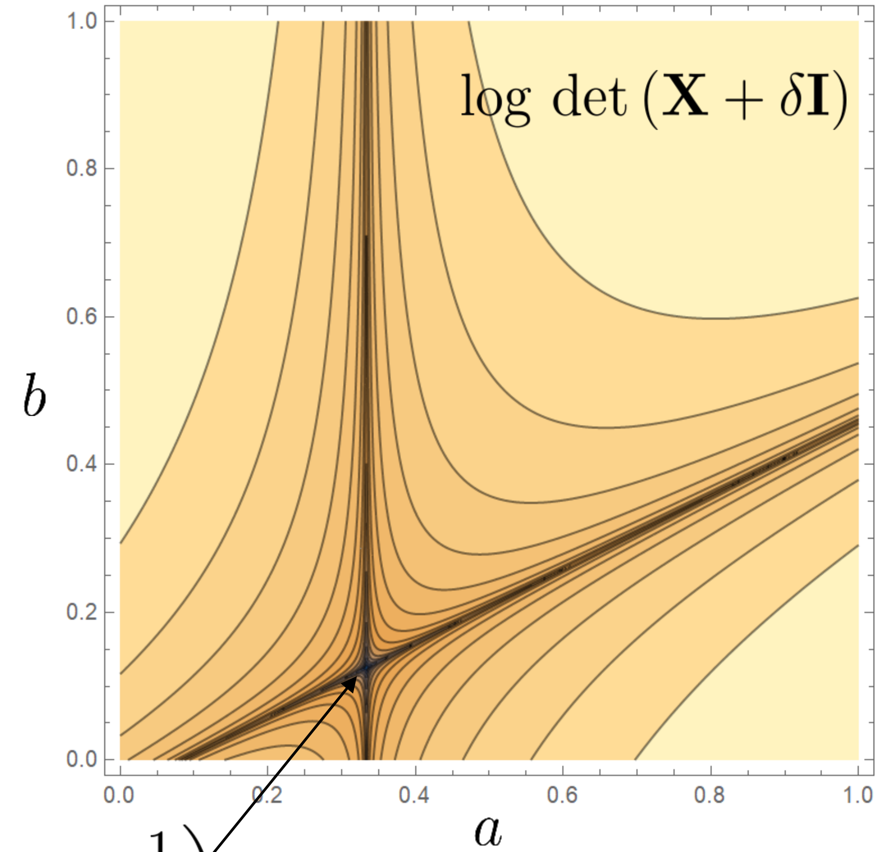
$0 < \delta \ll 1$ regularization parameter

Obtained rank-1 solution in 10 iterations:

$$\mathbf{X}^* \left(a = 0.333331, b = 12503\right) =$$

$$\begin{bmatrix} 0.998787 & 3 \cdot 10^{-6} & -0.00607 \\ 3 \cdot 10^{-6} & 3 \cdot 10^{-6} & -3 \cdot 10^{-6} \\ -0.00607 & 2 \cdot 10^{-11} & 7 \cdot 10^{-7} \end{bmatrix}$$

$$\delta = 10^{-6}$$




$$\left(a = \frac{1}{3}, b = \frac{1}{8}\right)$$


FAZEL, M., HAITHAM H., AND S.P. BOYD. "LOG-DET HEURISTIC FOR MATRIX RANK MINIMIZATION WITH APPLICATIONS TO HANKEL AND EUCLIDEAN DISTANCE MATRICES." AMERICAN CONTROL CONFERENCE, 2003. PROCEEDINGS OF THE 2003. VOL. 3. IEEE, 2003.

# Rank minimization heuristics

## Log-det heuristics

Local approximation via Taylor expansion

weighted trace norm of increment, quadratic in parameters

$$\log\det(\mathbf{X} + \delta\mathbf{I}) \approx \log\det(\mathbf{X}_k + \delta\mathbf{I}) + \mathrm{trace}\left(\overbrace{(\mathbf{X}_k + \delta\mathbf{I})^{-1}(\mathbf{X} - \mathbf{X}_k)}\right)$$

$$\mathbf{W}_k = (\mathbf{X}_k + \delta\mathbf{I})^{-1}$$ - weight for trace norm at k-th step

$$\log\det(\mathbf{X} + \delta\mathbf{I}) \approx \log\det(\mathbf{X}_k + \delta\mathbf{I}) + \mathrm{trace}\left(\mathbf{W}_k(\mathbf{X} - \mathbf{X}_k)\right)$$

**Algorithm**

1. Initialize matrix $\mathbf{X}_0 \in \mathcal{C}$, regularization $\delta > 0$ and tolerance $\epsilon > 0$
2. Repeat until $||\mathbf{X}_{k+1} - \mathbf{X}_k||_2 > \epsilon$ otherwise goto end
3. Compute weighting $\mathbf{W}_k = (\mathbf{X}_k + \delta\mathbf{I})^{-1}$
4. Solve QP $\mathbf{X}_{k+1} = \arg\min_{\mathbf{X}\in\mathcal{C}} \mathrm{trace}\left(\mathbf{W}_k(\mathbf{X} - \mathbf{X}_k)\right)$
5. goto 2.

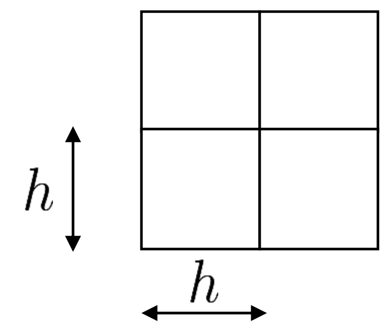
# Applications of rank minimization formulations

## Minimization of numerical dispersion for stencil of finite difference method

Wave equation in 2D

$$\ddot{u} = c^2 \Delta u$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$




Two discrete Laplacians

$$\Delta_h^1 = \frac{1}{h^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \Delta_h^2 = \frac{1}{2h^2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

A parametric family of stencils (9-point explicit)

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = c^2((1-\alpha)\Delta_h^1 + \alpha\Delta_h^2)u$$

# Applications of rank minimization formulations

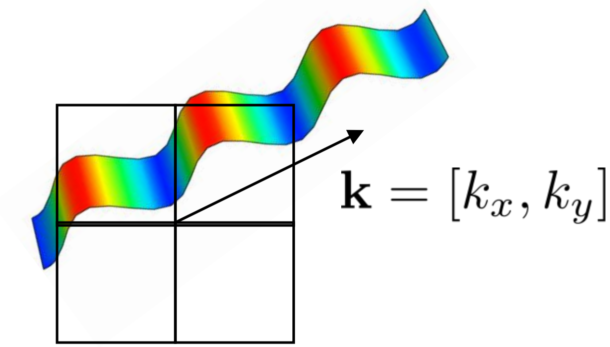
## Minimization of numerical dispersion for stencil of finite difference method

Plane wave assumption for analytical solution and the dispersion relation

$$u = e^{-i(\omega t - k_x x - k_y y)}$$

$$\omega^2 = c^2(k_x^2 + k_y^2)$$

$$c = \frac{\omega}{\sqrt{k_x^2 + k_y^2}}$$




Discrete dispersion relation for 9-point explicit stencil

$$u = e^{-i(\omega^h \Delta t - k_x \Delta x - k_y \Delta y)}$$

$$1 - \cos\left(\omega^h \Delta t\right) = \frac{c^2 \Delta t^2}{h^2}[(1-\alpha)(\cos(k_x h) + \cos(k_y h) - 2) + \alpha(\cos(k_x h)\cos(k_y h) - 1)]$$

Relative numerical dispersion error in phase speed

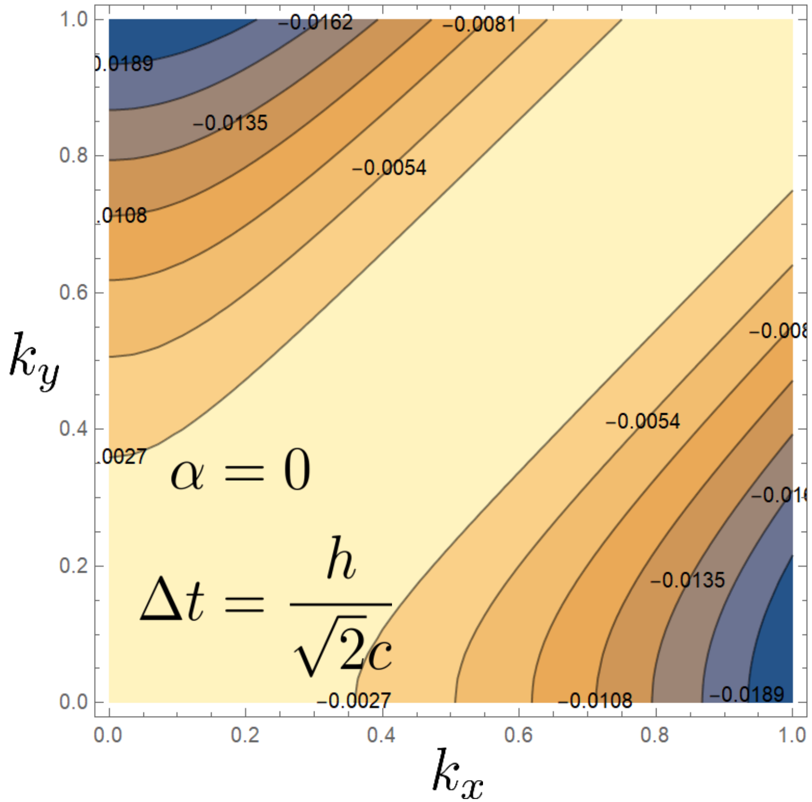
$$error = \frac{\omega^h}{c\sqrt{k_x^2 + k_y^2}} - 1$$

# Applications of rank minimization formulations

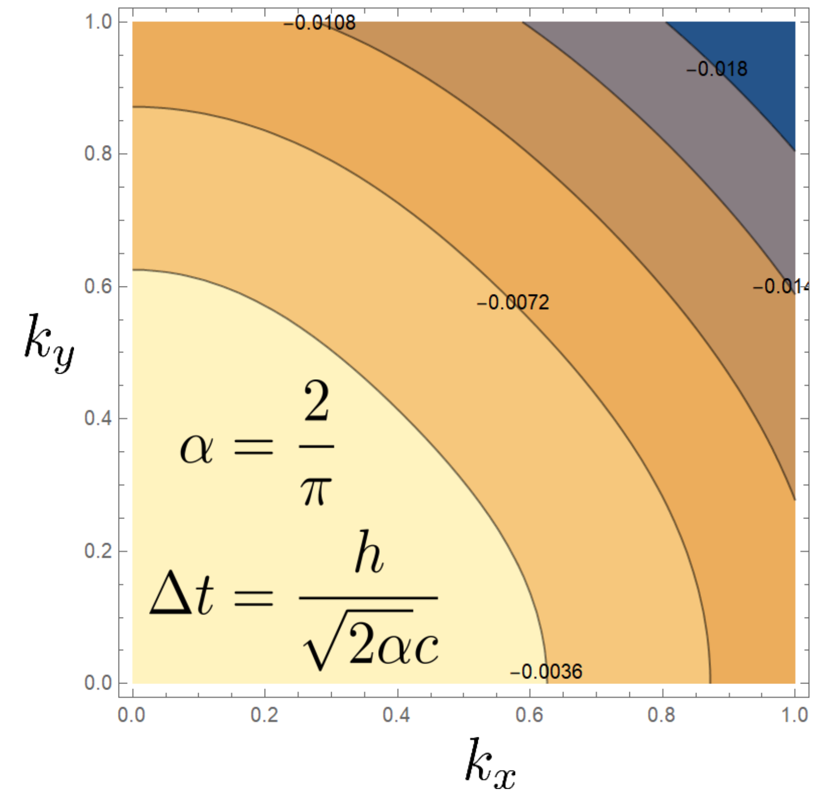
## Minimization of numerical dispersion for stencil of finite difference method

Comparison of two schemes

$$error = \frac{\omega^h}{c\sqrt{k_x^2 + k_y^2}} - 1$$




Standard 5-point stencil



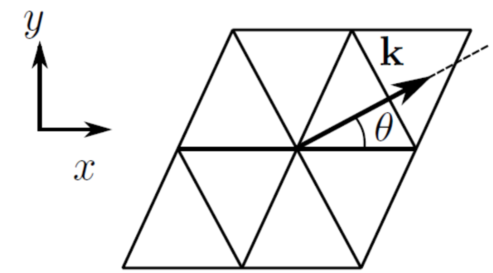

Optimized 9-point stencil

# Applications of rank minimization formulations

## Minimization of numerical dispersion for stencil of finite element method

Case of reciprocal mass matrices

$$\ddot{\mathbf{U}} = \mathbf{C}(\mathbf{F}^{\text{ext}} - \mathbf{K}\mathbf{U})$$




- equilibrium on representative patch level for plane wave ansatz $u(\mathbf{x}, t) = \tilde{\mathbf{U}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$

$$\left(\mathbf{C}^{\circ}_{\text{rep}}(\mathbf{k}, C_{2i})\mathbf{K}_{\text{rep}}(\mathbf{k}) - \omega^2 \mathbf{I}_{\text{rep}}\right)\tilde{\mathbf{U}} = \mathbf{0} \qquad \text{rank}\left(\mathbf{C}^{\circ}_{\text{rep}}(\mathbf{k}, C_{2i})\mathbf{K}_{\text{rep}}(\mathbf{k}) - \omega^2 \mathbf{I}_{\text{rep}}\right) < n_{\text{rep}}$$

$\mathbf{k}, \omega$ - wavevector and angular frequency
$C_{2i}$ - customization parameters
$\tilde{\mathbf{U}} \in \mathbb{C}^{n_{\text{rep}}}$ - discrete amplitudes
$n_{\text{rep}}$ - size of representative patch

- heuristic objective

$$C_{2i} = \arg\min_{C_{2i}} \sum_j \text{rank}\left(\mathbf{C}^{\circ}_{\text{rep}}(\mathbf{k}_j, C_{2i})\mathbf{K}_{\text{rep}}(\mathbf{k}_j) - \omega_j^2 \mathbf{I}_{\text{rep}}\right)$$


TKACHUK, ANTON. "CUSTOMIZATION OF RECIPROCAL MASS MATRICES VIA LOG-DET HEURISTIC." INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING 121.4 (2020): 690-711..

# Applications of rank minimization formulations

## Minimization of numerical dispersion for stencil of finite element method

### 2D elastic-dynamics discretized with 6-node FE, hexagonal mesh

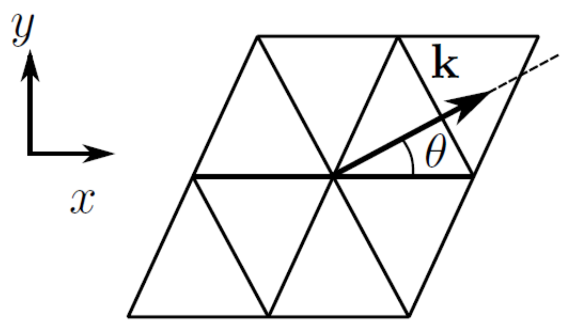
$$\left(\mathbf{C}^{\circ}_{\mathrm{rep}}(k, C_{21}, C_{22})\mathbf{K}_{\mathrm{rep}}(k) - \omega^2 \mathbf{I}_{\mathrm{rep}}\right)\tilde{\mathbf{U}} = \mathbf{0}$$

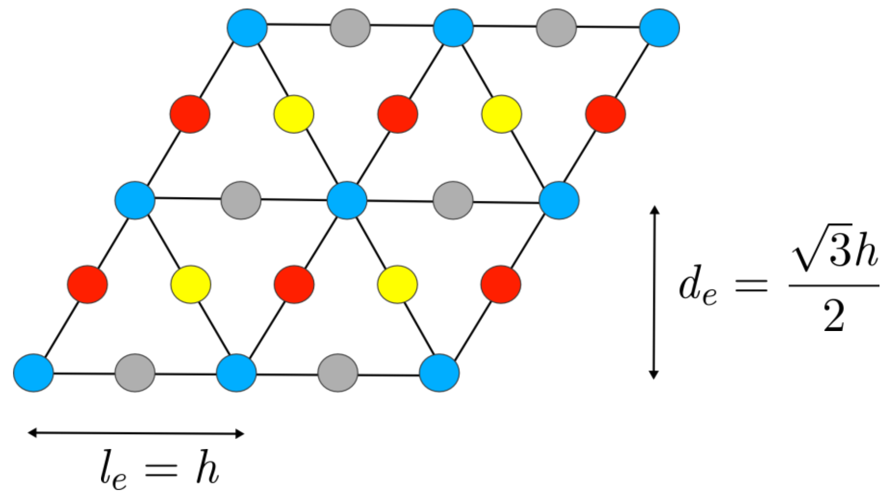
$\mathbf{k}, \omega$ - wavevector and angular frequency

$C_{21}, C_{22}$ - customization parameters

$\tilde{\mathbf{U}} = [\tilde{U}_{xc}, \tilde{U}_{yc}, \tilde{U}_{xm1}, \tilde{U}_{xm2}, \tilde{U}_{xm3}, \tilde{U}_{ym1}, \tilde{U}_{ym2}, \tilde{U}_{ym3}] \in \mathbb{C}^{n_{\mathrm{rep}}}$ - 2 corner and 6 mid-node amplitudes

$n_{\mathrm{rep}} = 8$ - size of representative patch



# Applications to reciprocal mass matrix

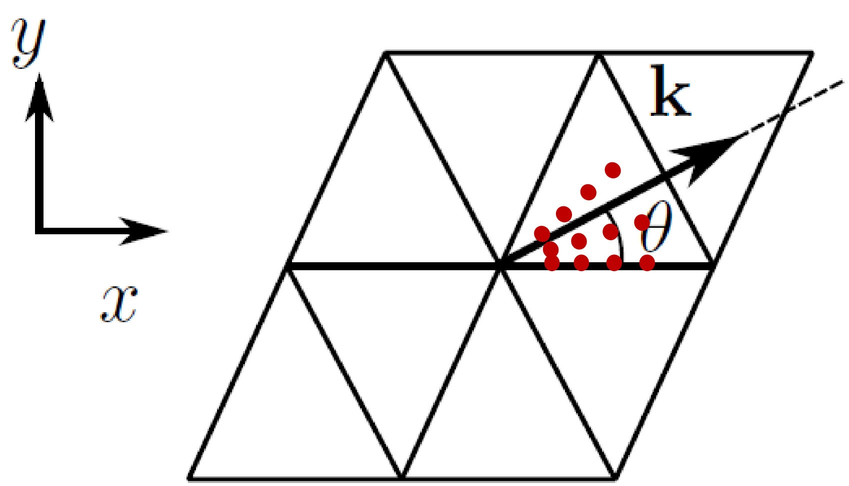
## Minimization of numerical dispersion for stencil of finite element method

### Reformulate for reduce the problem to Hermitian semi-positive definite

$$\tilde{\mathbf{A}}(\mathbf{k},\omega,C_{21},C_{22}) = \left(\mathbf{C}^{\circ}_{\mathrm{rep}}(\mathbf{k},C_{21},C_{22})\mathbf{K}_{\mathrm{rep}}(\mathbf{k}) - \omega^2\mathbf{I}_{\mathrm{rep}}\right)$$

$$\mathbf{X}(\mathbf{k},\omega,C_{21},C_{22}) = \tilde{\mathbf{A}}^{\mathrm{H}}\tilde{\mathbf{A}}$$

### Surrogate problem




•••• $\mathbf{k} = k[\cos(\theta);\sin(\theta)]$

- sampling array for wavevecktor

### Two waves types (plane stress assumption)

- P-wave

$$\omega_{\mathrm{P}}^2 = \frac{E}{(1-\nu^2)\rho}\mathbf{k}^2$$

- S-wave

$$\omega_{\mathrm{S}}^2 = \frac{E}{2(1+\nu)\rho}\mathbf{k}^2$$

# Applications of rank minimization formulations

## RMM example 3: 2D elastic-dynamics discretized with 6-node FE, hexagonal mesh

### Best results from log-det heuristics

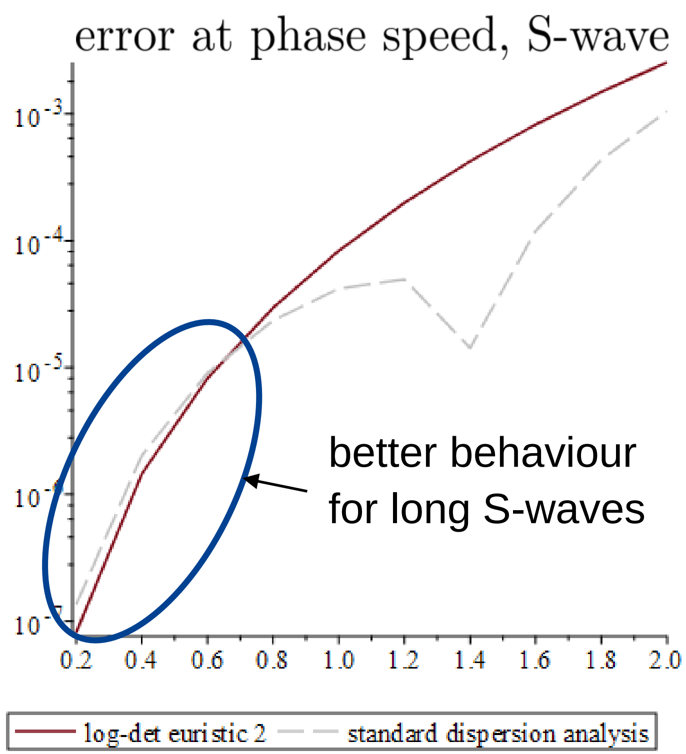
$$n_k = 7 \quad k_j = [0.025; 0.05; 0.1; 0.2; 0.3; 0.4; 0.5]\, h$$

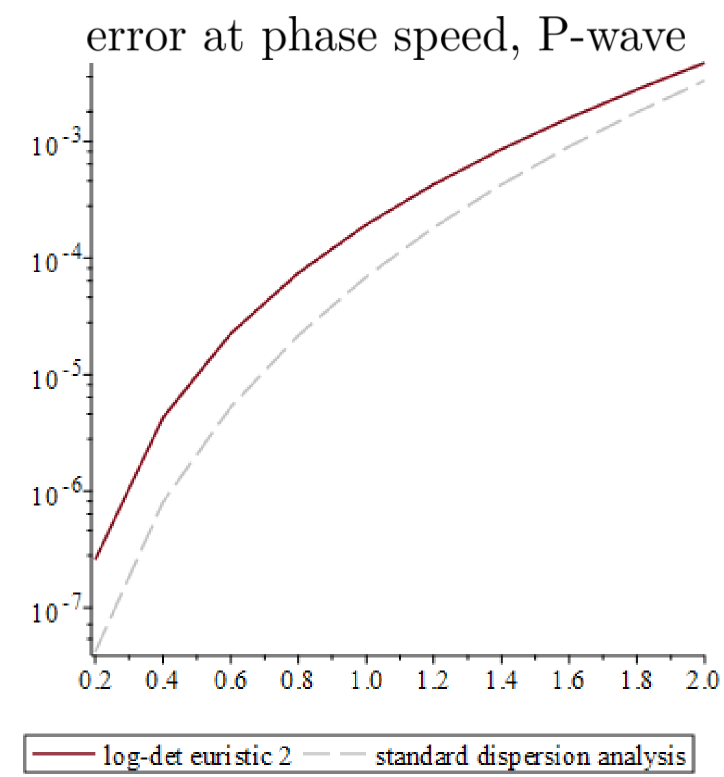
$$n_\theta = 5 \quad \theta_s = [0; \pi/30; \pi/15; \pi/10; \pi/6]$$

$$\omega_{j,s} = c_0 k_j \quad \delta = 3 \cdot 10^{-42}$$

$$C_{21} = 0.35999923115819346922334586255,$$

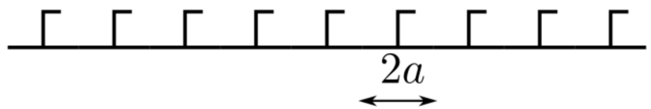
$$C_{22} = 0.42181367522022318508140913666$$

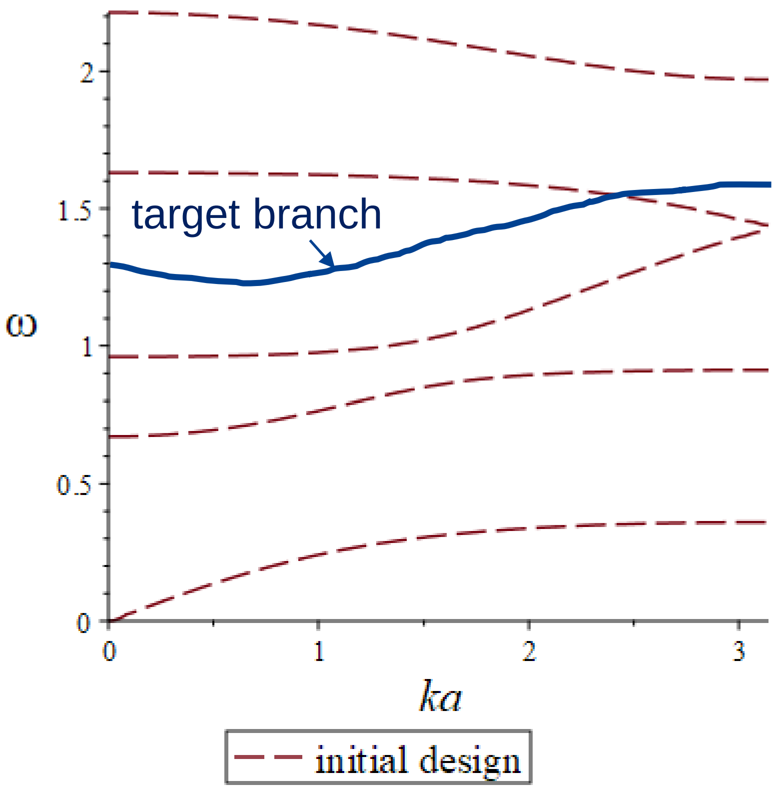


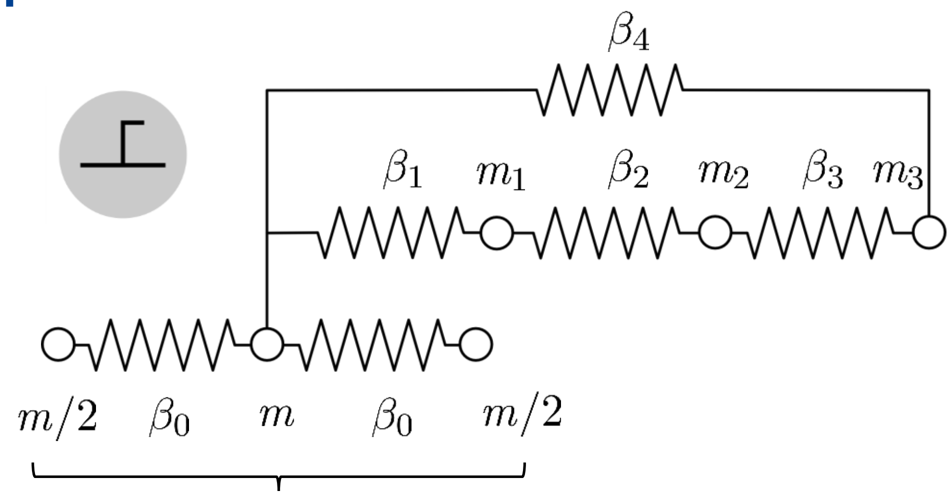



# Applications of rank minimization formulations

## Problem statement for a dispersion design problem



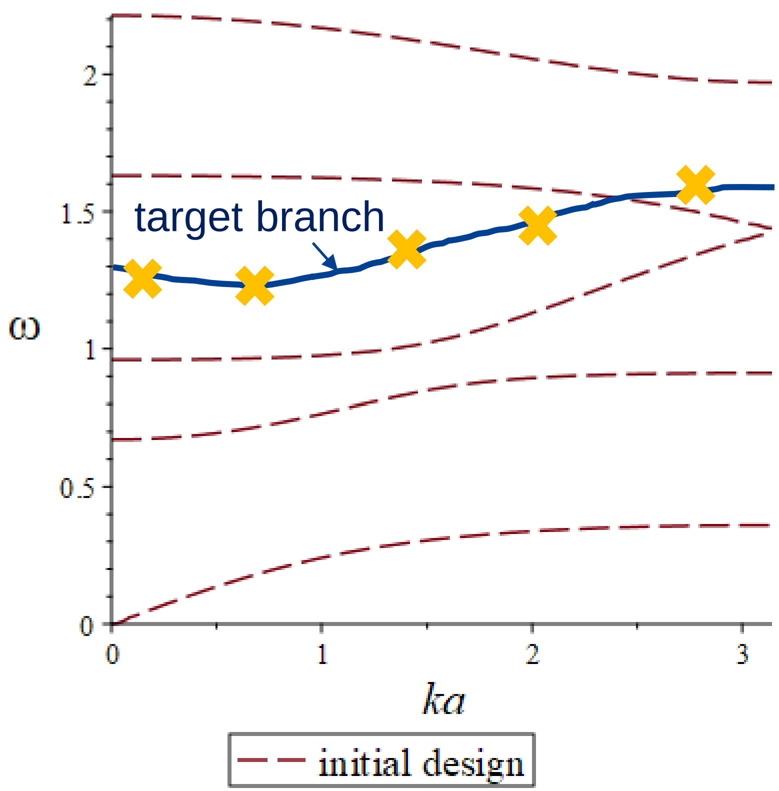

*Given:* a periodic system depending on a set of parameters **s** and a target branch with a dispersion relation $k = f(\omega)$.

*Find:* a design whose dispersion relation matches the given dispersion relation best

## Two desired features of algorithm

- performance function depends explicitly on the design parameters
- ordering of eigenfrequencies should be avoided

# Dispersion design as matrix rank minimization problem




## A possible formalization and relaxation

- equilibrium on representative patch level for plane wave ansatz

$$u(\mathbf{x},t) = \tilde{\mathbf{U}}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\left(\mathbf{K}_{\mathrm{rep}}(k,\mathbf{s}) - \omega^2\, \mathbf{M}_{\mathrm{rep}}(\mathbf{s})\right) \tilde{\mathbf{U}} = \mathbf{0}$$

$\tilde{\mathbf{U}} \in \mathbb{C}^{n_{\mathrm{rep}}}$ - discrete amplitudes

$n_{\mathrm{rep}}$ - size of representative patch

- require not full rank at discrete pairs $(k_j,\omega_j)$ ✖

$$\mathrm{rank}\,\left(\mathbf{K}_{\mathrm{rep}}(k_j,\mathbf{s}) - \omega_j^2\, \mathbf{M}_{\mathrm{rep}}(\mathbf{s})\right) < n_{\mathrm{rep}}$$

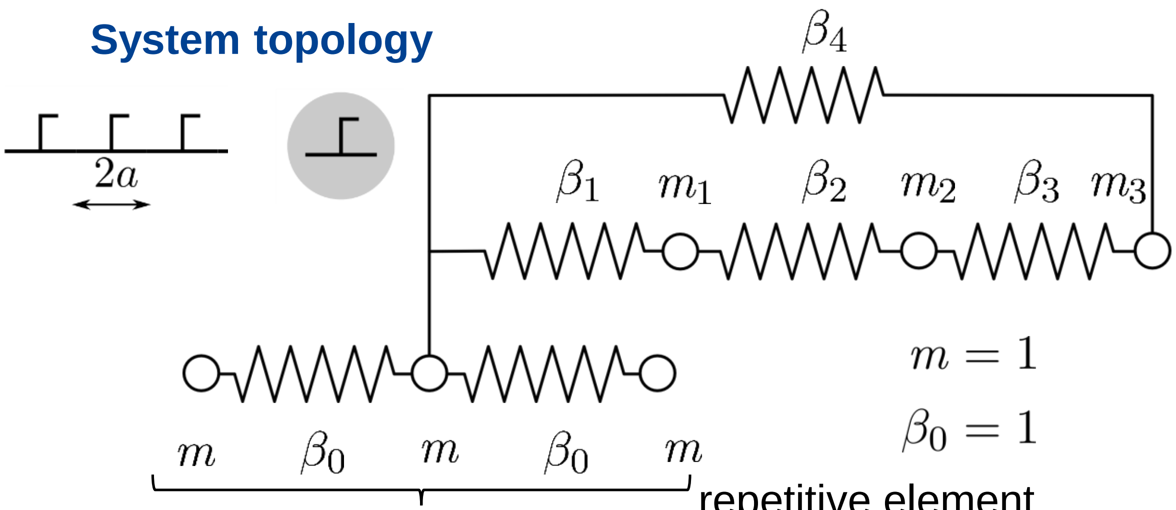
- set up a heuristic objective function

$$J(\mathbf{s}) = \sum_j \mathrm{rank}\,\left(\mathbf{K}_{\mathrm{rep}}(k_j,\mathbf{s}) - \omega_j^2\, \mathbf{M}_{\mathrm{rep}}(\mathbf{s})\right)$$

# Applications

## Mass-spring system 1: targeting several points on two acoustic branches

### System topology




$m = 1$

$\beta_0 = 1$

$m_j$ - nodal masses $\quad a$ - distance between nodes

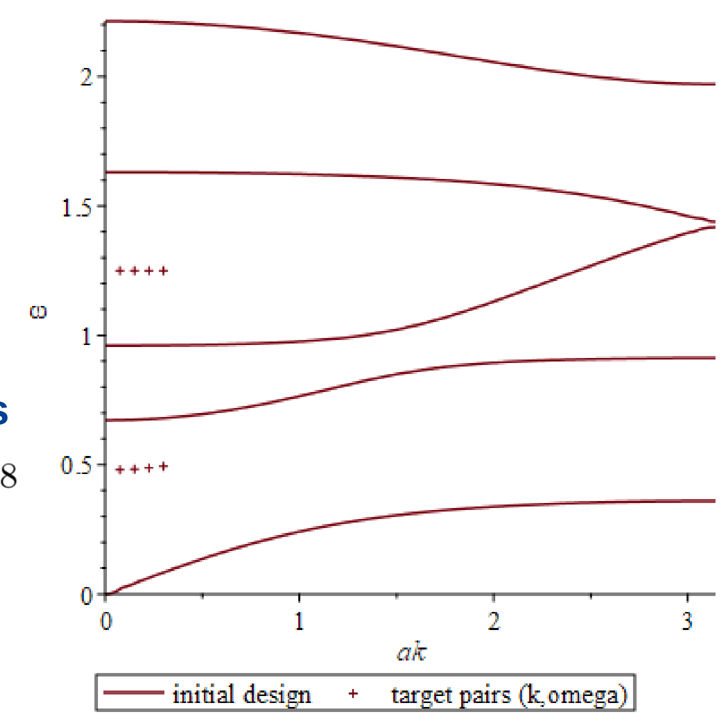
$\beta_j$ - spring stiffness $\quad n_{\text{rep}} = 4$ - size of the representative patch

### Target wave number-frequency pairs (8)

$ak_1 = 0.075 \quad \omega_1 = 0.4797$

$ak_2 = 0.150 \quad \omega_2 = 0.4824$

$\dots$




### Admissible range of parameters

$m_i \in [0.2{,}2] \quad \beta_{1,2,3} \in [0.3{,}4] \quad \beta_4 \in [0.005{,}4]$

### Algorithm parameters

$\delta = 3 \cdot 10^{-8}$

$Digits = double$

$\#ite = 21$

### Initial design

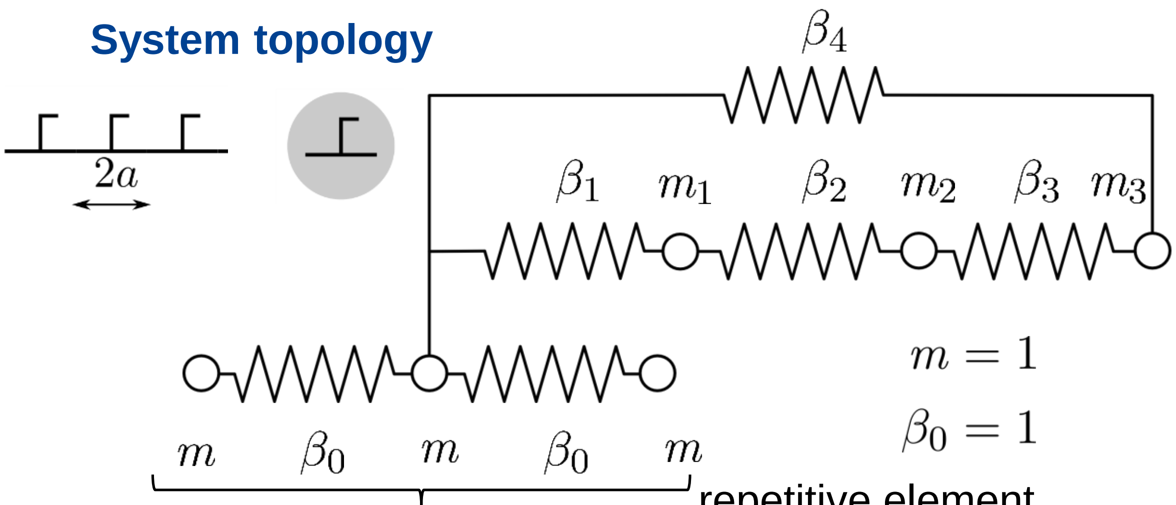
$m_1 = 1 \quad m_2 = 2 \quad m_3 = 1.2$

$\beta_1 = 1 \quad \beta_2 = 0.5 \quad \beta_3 = 0.6 \quad \beta_4 = 0.1$

# Applications

## Mass-spring system 1: targeting several points on two acoustic branches

### System topology




$m = 1$

$\beta_0 = 1$

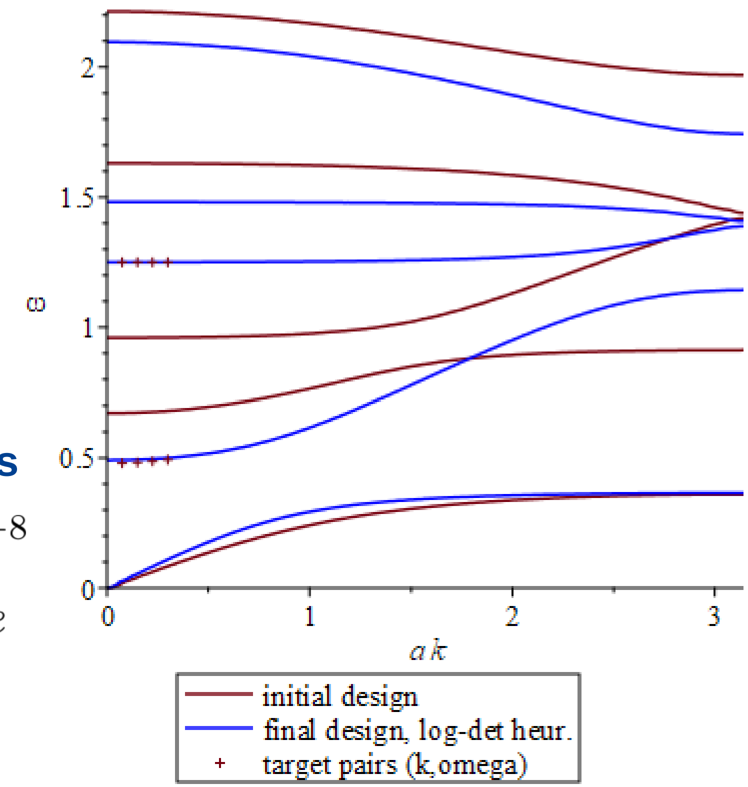
$m_j$ - nodal masses $\quad a$ - distance between nodes

$\beta_j$ - spring stiffness $\quad n_{\text{rep}} = 4$ - size of the representative patch

### Final design

$m_1 = 0.42 \quad m_2 = 0.78 \quad m_3 = 0.37$

$\beta_1 = 0.49 \quad \beta_2 = 0.3 \quad \beta_3 = 0.41 \quad \beta_4 = 0.005$




### Admissible range of parameters

$m_i \in [0.2,2] \quad \beta_{1,2,3} \in [0.3,4] \quad \beta_4 \in [0.005,4]$

### Algorithm parameters

$\delta = 3 \cdot 10^{-8}$

$Digits = double$

$\#ite = 21$

### Initial design

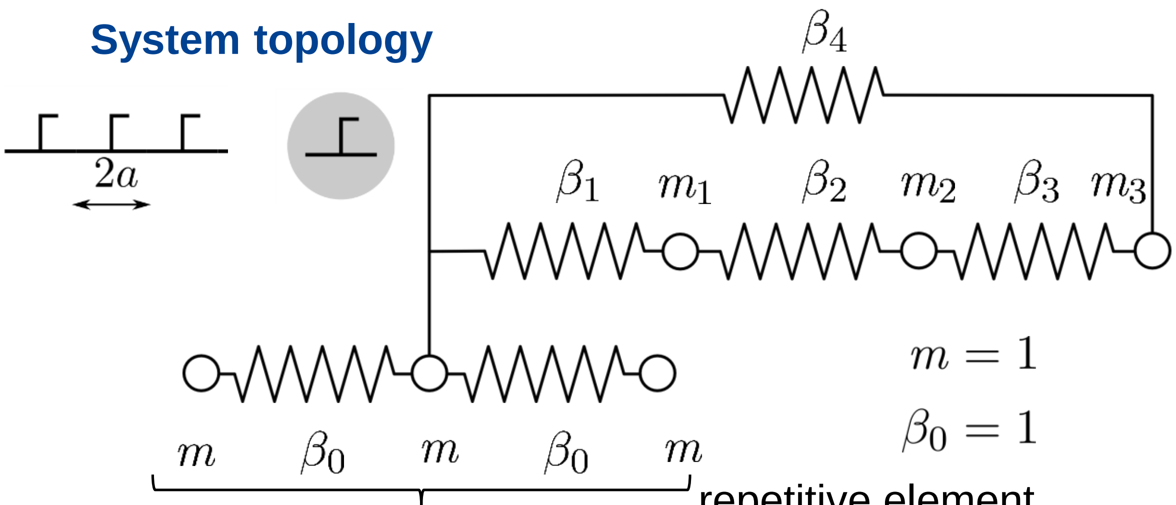
$m_1 = 1 \quad m_2 = 2 \quad m_3 = 1.2$

$\beta_1 = 1 \quad \beta_2 = 0.5 \quad \beta_3 = 0.6 \quad \beta_4 = 0.1$

# Applications

## Mass-spring system 1: targeting a constant optical branch

### System topology




$m = 1$

$\beta_0 = 1$

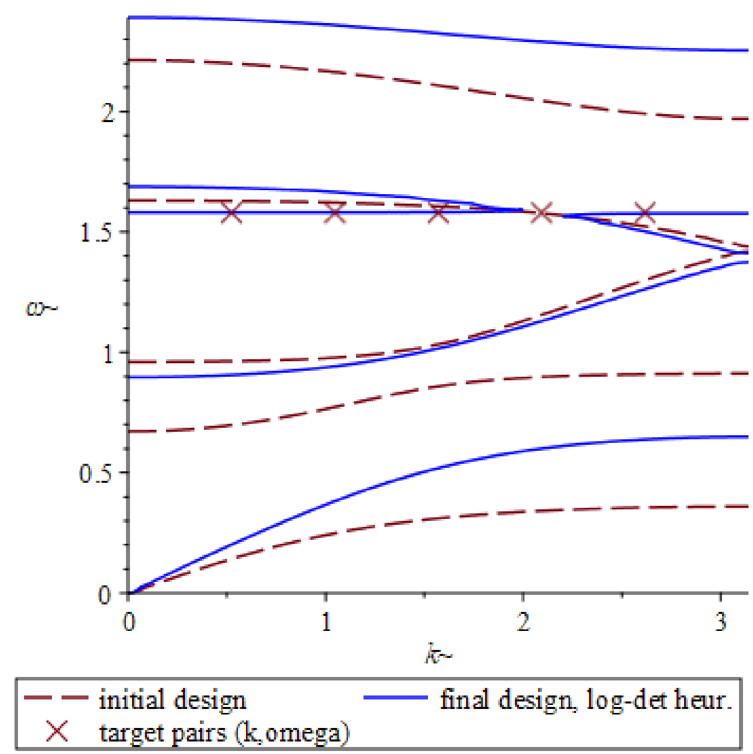
$m_j$ - nodal masses  $a$ - distance between nodes

$\beta_j$ - spring stiffness  $n_{\text{rep}} = 4$ - size of the representative patch

### Final design

$m_1 = 0.4443 \quad m_2 = 0.5395 \quad m_3 = 0.2848$

$\beta_1 = 1.098 \quad \beta_2 = 0.3 \quad \beta_3 = 0.3009 \quad \beta_4 = 0.3245$




### Admissible range of parameters

$m_i \in [0.2,2] \quad \beta_{1,2,3} \in [0.3,4] \quad \beta_4 \in [0.005,4]$

### Algorithm parameters

$\delta = 3 \cdot 10^{-8}$

$Digits = double$

$\#ite = 21$

### Initial design

$m_1 = 1 \quad m_2 = 2 \quad m_3 = 1.2$

$\beta_1 = 1 \quad \beta_2 = 0.5 \quad \beta_3 = 0.6 \quad \beta_4 = 0.1$

## Conclusions

- Rank minimization formulation appear in several geometrical, data analysis and mechanical applications
- Sevaral tractable relaxation are known and usable
- Instances of reciprocal mass matrices with reduced numerical dispersion are obtained via RMP
- Mechanical systems with prescribed dispersion are obtainable

## Future work

- Larger systems and combination with reduced-ordering systems
- Dispersion customization for acoustic metamaterials
- Using other rank minimization heuristics

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# Tack så mycket! Thank you! Vielen Dank!

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