Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

BOUNDARY LAYERS FOR DISCRETE KINETIC MODELS: MULTICOMPONENT MIXTURES, POLYATOMIC MOLECULES, BIMOLECULAR REACTIONS, AND QUANTUM KINETIC EQUATIONS

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(Communicated by the associate editor name)

Abstract. We consider some extensions of the classical discrete Boltzmann equation to the cases of multicomponent mixtures, polyatomic molecules (with a finite number of different internal energies), and chemical reactions, but also general discrete quantum kinetic Boltzmann-like equations; discrete versions of the Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation for bosons and fermions and a kinetic equation for excitations in a Bose gas interacting with a Bose-Einstein condensate. In each case we have an H-theorem and so for the planar stationary half-space problem, we have convergence to an equilibrium distribution at infinity (or at least a manifold of equilibrium distributions). In particular, we consider the nonlinear half-space problem of condensation and evaporation for these discrete Boltzmann-like equations. We assume that the flow tends to a stationary point at infinity and that the outgoing flow is known at the wall, maybe also partly linearly depending on the incoming flow. We find that the systems we obtain are of similar structures as for the classical discrete Boltzmann equation (for single species), and that previously obtained results for the discrete Boltzmann equation can be applied after being generalized. Then the number of conditions on the assigned data at the wall needed for existence of a unique solution is found. The number of parameters to be specified in the boundary conditions depends on if we have subsonic or supersonic condensation or evaporation. All our results are valid for any finite number of velocities.

1. **Introduction.**

The Boltzmann equation is a fundamental equation in kinetic theory [22, 23]. Half-space problems for Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers [40, 41], and have been extensively studied both for the full Boltzmann equation, see for example [4, 29, 44, 45], and for the discrete Boltzmann equation [3, 7, 32, 33, 43]. The half-space problems provide the boundary conditions for the fluid-dynamic-type equations and

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2010 *Mathematics Subject Classification.* Primary: 82C40; Secondary: 35Q20,76P05.

Key words and phrases. Nordheim-Boltzmann equation, discrete velocity models, Boltzmann equation, mixtures, polyatomic molecules, chemical reactions, boundary layers, half-space problems.

The first author is supported by NSF grant xx-xxxx.

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Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary.

The Boltzmann equation can be approximated by discrete velocity models (DVMs) up to any order [18, 27, 38], and these discrete velocity approximations can be used for numerical methods [35] (and references therein). The studies in this paper is a continuation of the studies in the papers [6, 7, 8, 16]. We also want to point out the reference [9] for the case of a condensing vapor in the presence of a non-condensable gas.

In the present paper we consider some extensions of the classical discrete Boltzmann equation (DBE). We extend the DBE to the cases of multicomponent mixtures, using a more efficient approach than the one used for binary mixtures in [8]. In fact we add to each velocity an index (each corresponding to a different species) and assume that the set of indexes stays fixed under collisions. By using a similar approach we also consider DVMs for polyatomic molecules. Here polyatomic molecules means that each molecule has one of a finite number of different internal energies, which can change, or not, during a collision. The approach includes that we to each velocity also add of the finite number of internal energies, cf. [20]. Combining these two approaches for multicomponent mixtures and polyatomic molecules, we obtain DVMs multicomponent mixtures with a finite number of internal energies. Then we can also include bimolecular chemical reactions. We also consider some general discrete quantum kinetic Boltzmann-like equations; discrete versions of the Nordheim-Boltzmann [37] (or Uehling-Uhlenbeck [42]) equation for bosons and fermions and a kinetic equation for excitations in a Bose gas interacting with a Bose-Einstein condensate [2], see [10]. In each case we have an H-theorem and so for the planar stationary half-space problem, we have convergence to an equilibrium distribution at infinity (or at least a manifold of equilibrium distributions), by arguments in [30], see also [13].

Existence and uniqueness of solutions of half-space problems for a general discrete kinetic model (DKM) of the Boltzmann equation are studied. The number of conditions, on the assigned data for the outgoing particles at the boundary, needed for the existence of a unique (in a neighborhood of an assigned equilibrium distribution at infinity) solution of the problem are found. The distribution for the outgoing particles at the boundary might, under some restrictions on the dependence, be partly linearly depending on the distribution of the incoming particles. We improve previously obtained results for the classical discrete Boltzmann equation in the degenerate cases, which in the continuous case are corresponding to the cases when the Mach number of the Maxwellian at infinity is 0 or ± 1, in the way that some restrictive conditions on the quadratic part are made superfluous. This improvement is made possible by some improvements of the proof in [7]. Even if the proof is similar to the one in [7], we still present it for the sake of completeness and clarity. To our knowledge no similar results exist in the continuous case (except for single species [4, 29, 44, 45] or binary mixtures, with equal masses, [5]).

The remaining part of the paper is organized as follows. In Section 2 we present the general system of partial differential equations (PDEs) of our interest. We also review a fundamental result [16] for our studies of boundary layers. In Section 3 we present several examples of specific systems of the type in Section 2: the discrete quantum Boltzmann equation, including the classical DBE (subsection 3.1); extensions to multicomponent mixtures (subsection 3.2), polyatomic molecules (subsection 3.3), and bimolecular chemical reactions (subsection 3.4); and a discrete model for excitations in a Bose gas interacting with a Bose-Einstein condensate (subsection 3.5).

The main results on the boundary layers are presented in Section 4 (Theorem 4.2). In Section 5 we calculate the characteristic values, where the number of conditions on the assigned data changes, for the different systems in the particular case when the discrete sets are symmetric around the axis. The proof of Theorem 4.2 in Section 6, is based on the corresponding proof for the DBE in [7].

2. **System of partial differential equations.** We study a system of partial differential equations

$$\frac{\partial F_i}{\partial t} + \mathbf{p}_i \cdot \nabla_{\mathbf{x}} F_i = Q_i(F), \text{ with } t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{R}^d, \text{ for } i = 1, ..., N, \tag{1}$$

where $\mathbf{p}_1, ..., \mathbf{p}_N \in \mathbb{R}^d$ are given, $F_1 = F_1(t, \mathbf{x}), ..., F_N = F_N(t, \mathbf{x})$, and $F = (F_1, ..., F_N)$. If we denote by $\mathbf{B}$ the matrix where the rows are the transposes of $\mathbf{p}_1, ..., \mathbf{p}_N$, respectively, and $Q(F) = (Q_1(F), ..., Q_N(F))$, then the system (1) can be rewritten as

$$\frac{\partial F}{\partial t} + (\mathbf{B}\,\nabla_{\mathbf{x}}) \cdot F = Q(F), \text{ with } t, x \in \mathbb{R}_+. \tag{2}$$

We assume that there exist positive equilibrium points P of the system (2), i.e. points P such that

$$Q(P) = 0 \text{ and } P_i > 0 \text{ for } i = 1, ..., N.$$

Given a positive equilibrium point P we denote

$$F = P + Rf, \tag{3}$$

where $R = R(P)$ is an operator of P, and obtain the new system

$$\frac{\partial f}{\partial t} + (\mathbf{B}\,\nabla_{\mathbf{x}}) \cdot F + Lf = S(f), \tag{4}$$

where L is the linearized collision operator ($N \times N$ matrix) and S is the nonlinear part.

We assume that we can choose $R = R(P)$ such that the matrix L is symmetric and semi-positive, and that $\dim(N(L)) = \rho > 0$ for the null-space $N(L)$ of L. Furthermore, the nonlinear part $S(f)$ is assumed to belong to the orthogonal complement of $N(L)$, i.e.

$$S(f) \in N(L)^{\perp},$$

and to fulfill

$$|S(f) - S(g)| \leq \widetilde{K} G(|f|, |g|)\,|f - g|,$$

for some positive constant $\widetilde{K} > 0$ and differentiable function $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with positive partial derivatives and $G(0, 0) = 0$.

In particular, we will consider the planar stationary systems

$$B\frac{df}{dx} + Lf = S(f), \text{ with } f = f(x) \text{ and } B = \operatorname{diag}(p_1^1, ..., p_N^1), \tag{5}$$

where

$$\mathbf{x} = (x = x^1, x^2, ..., x^d) \text{ and } \mathbf{p}_i = (p_i^1, ..., p_i^d),\ i = 1, ..., N,$$

in more details. We will assume below that $p_i^1 \neq 0$ for $i = 1, ..., N$.

We denote by $n^{\pm}$, where $n^+ + n^- = N$, and $m^{\pm}$, with $m^+ + m^- = q$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices B and $B^{-1}L$ respectively, and by m^0 the number of zero eigenvalues of

$B^{-1}L$. Moreover, we denote by k^+, k^- and l, with $k^+ + k^- = k$, where $k + l = \rho$, the numbers of positive, negative and zero eigenvalues of the $\rho \times \rho$ matrix K, with entries $k_{ij} = \langle y_i, y_j \rangle_B = \langle y_i, By_j \rangle$, such that $\{y_1, ..., y_\rho\}$ is a basis of the null-space of L. Here and below, we denote

$$\langle \cdot, \cdot \rangle_B = \langle \cdot, B \cdot \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product in $\mathbb{R}^N$.

We remind the following result by Bobylev and Bernhoff in [16] (also proved in [6]).

Theorem 2.1. *The numbers of positive, negative and zero eigenvalues of $B^{-1}L$ are given by*

$$\begin{aligned} m^+ &= n^+ - k^+ - l \\ m^- &= n^- - k^- - \\ m^0 &= \rho + l. \end{aligned}$$

Remark 1. The proof of Theorem 2.1 in [16] is carried out for any real symmetric matrices L and B, such that L is semi-positive and B is invertible.

In the proof of Theorem 2.1 a Jordan basis of $\mathbb{R}^N$, with respect to $B^{-1}L$,

$$u_1, ..., u_q, y_1, ..., y_k, z_1, ..., z_l, w_1, ..., w_l, \tag{6}$$

such that

$$y_i, z_r \in N(L), \;\; B^{-1}Lw_r = z_r \text{ and } B^{-1}Lu_\alpha = \lambda_\alpha u_\alpha, \tag{7}$$

and

$$\begin{aligned} \langle u_\alpha, u_\beta \rangle_B &= \lambda_\alpha \delta_{\alpha\beta}, \text{ with } \lambda_1, ..., \lambda_{m^+} > 0 \text{ and } \lambda_{m^+ +1}, ..., \lambda_q < 0, \\ \langle y_i, y_j \rangle_B &= \gamma_i \delta_{ij}, \text{ with } \gamma_1, ..., \gamma_{k^+} > 0 \text{ and } \gamma_{k^+ +1}, ..., \gamma_k < 0, \\ \langle u_\alpha, z_r \rangle_B &= \langle u_\alpha, w_r \rangle_B = \langle u_\alpha, y_i \rangle_B = \langle w_r, y_i \rangle_B = \langle z_r, y_i \rangle_B = 0, \\ \langle w_r, w_s \rangle_B &= \langle z_r, z_s \rangle_B = 0 \text{ and } \langle w_r, z_s \rangle_B = \delta_{rs}, \end{aligned} \tag{8}$$

is constructed.

The Jordan Normal form of $B^{-1}L$ (with respect to the basis (6)) is (see also [6, 7])

$$\begin{pmatrix} \lambda_1 &&&&&&&&&&& \\ & \ddots &&&&&&&&&& \\ && \lambda_q &&&&&&&&& \\ &&& 0 &&&&&&&& \\ &&&& \ddots &&&&&&& \\ &&&&& 0 &&&&&& \\ &&&&&& 0 & 1 &&&& \\ &&&&&& 0 & 0 &&&& \\ &&&&&&&& \ddots &&& \\ &&&&&&&&& 0 & 1 \\ &&&&&&&&& 0 & 0 \end{pmatrix},$$

where there are l blocks of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In the non-degenerate case $l = 0$ the matrix $B^{-1}L$ is diagonalizable, in difference to in the degenerate case as $l > 0$.

3. Discrete kinetic models.

3.1. Nordheim-Boltzmann equation.

The discrete Nordheim-Boltzmann equation (or Uehling-Uhlenbeck equation) reads

$$\frac{\partial F_i}{\partial t} + \mathbf{p}_i \cdot \nabla_{\mathbf{x}} F_i = Q_i^{\varepsilon}(F),\ i = 1, ..., N, \tag{9}$$

where $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set and $F = (F_1, ..., F_N)$, $F_i = F_i(x,t) = F(x,t,\mathbf{p}_i)$, is the distribution function, with $F_i > 0$ and, if $\varepsilon = -1$, $F_i < 1$.

Remark 2. For a function $g = g(\mathbf{p})$ (possibly depending on more variables than $\mathbf{p}$), we will identify g with its restrictions to the points $\mathbf{p} \in \mathcal{P}$, i.e.

$$g = (g_1, ..., g_N),\ \text{ with } g_i = g(\mathbf{p}_i)\ \text{ for } i = 1, ..., N.$$

The collision operators $Q_i^{\varepsilon}(F)$, $i = 1, ..., N$, in (9) are given by

$$\begin{aligned} Q_i^{\varepsilon}(F) &= \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} (F_k F_l (1+\varepsilon F_i)(1+\varepsilon F_j) - F_i F_j (1+\varepsilon F_k)(1+\varepsilon F_l)) \\ &= \sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} (1+\varepsilon F_i)(1+\varepsilon F_j)(1+\varepsilon F_k)(1+\varepsilon F_l) \\ &\qquad \left(\frac{F_k}{1+\varepsilon F_k} \frac{F_l}{1+\varepsilon F_l} - \frac{F_i}{1+\varepsilon F_i} \frac{F_j}{1+\varepsilon F_j} \right), \end{aligned} \tag{10}$$

where it is assumed that the collision coefficients Γ_{ij}^{kl}, $1 \leq i,j,k,l \leq N$, satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0, \tag{11}$$

with equality unless the conservation laws

$$\mathbf{p}_i + \mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l \text{ and } |\mathbf{p}_i|^2 + |\mathbf{p}_j|^2 = |\mathbf{p}_k|^2 + |\mathbf{p}_l|^2 \tag{12}$$

are satisfied. The collision operators (10) can be obtained from the expression

$$Q^{\varepsilon}(F) = Q(F,F) + \widehat{Q}(F,F,F) \tag{13}$$

where

$$Q_i(F,G) = \frac{1}{2} \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} ((G_k H_l + H_k G_l) - (G_i H_j + H_j G_i))$$

and

$$\begin{aligned} &\widehat{Q}_i(F,G,H) \\ = \ &\frac{\varepsilon}{2} \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} ((F_i + F_j)(G_k H_l + H_k G_l) - (F_k + F_l)(G_i H_j + H_j G_i)) \end{aligned}$$

for $i = 1, ..., N$. Here $\varepsilon = 0$ corresponds to the classical discrete Boltzmann equation ([21, 28]), and we have $\varepsilon = 1$ for bosons and $\varepsilon = -1$ for fermions.

The collision invariants, i.e. the functions $\phi = \phi(\xi)$, such that

$$\phi_i + \phi_j = \phi_k + \phi_l, \tag{14}$$

for all indices $1 \leq i,j,k,l \leq N$ such that $\Gamma_{ij}^{kl} \neq 0$, are assumed to be on the form

$$\phi = a + \mathbf{b} \cdot \mathbf{p} + c\,|\mathbf{p}|^2 \tag{15}$$

for some constant $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. Then the equation

$$\langle Q^{\varepsilon}(F), \phi \rangle = 0 \tag{16}$$

has the general solution (15). In general, discrete models can also have other, so called spurious (or nonphysical), collision invariants. Models without spurious collision invariants are called normal and methods of their construction are described in e.g. [17, 19]. Our restriction to normal models is not necessary in our general reasoning, but is motivated by the desire to have the same number of collision invariants as in the continuous case.

One can easily obtain that

$$\langle H, Q^{\varepsilon}(F) \rangle = \frac{1}{4} \sum_{i,j,k,l=1}^{N} \Gamma_{ij}^{kl} (1+\varepsilon F_i)(1+\varepsilon F_j)(1+\varepsilon F_k)(1+\varepsilon F_l) (H_i + H_j - H_k - H_l) \left(\frac{F_k}{1+\varepsilon F_k} \frac{F_l}{1+\varepsilon F_l} - \frac{F_i}{1+\varepsilon F_i} \frac{F_j}{1+\varepsilon F_j} \right), \tag{17}$$

and so (substituting $H = \log \frac{F}{1+\varepsilon F}$) that

$$\left\langle \log \frac{F}{1+\varepsilon F}, Q^{\varepsilon}(F) \right\rangle = \frac{1}{4} \sum_{i,j,k=1}^{N} \Gamma_{ij}^{kl} (1+\varepsilon F_i)(1+\varepsilon F_j)(1+\varepsilon F_k)(1+\varepsilon F_l) \left(\log \left(\frac{F_i}{1+\varepsilon F_i} \frac{F_j}{1+\varepsilon F_j} \right) - \log \left(\frac{F_k}{1+\varepsilon F_k} \frac{F_l}{1+\varepsilon F_l} \right) \right) \left(\frac{F_k}{1+\varepsilon F_k} \frac{F_l}{1+\varepsilon F_l} - \frac{F_i}{1+\varepsilon F_i} \frac{F_j}{1+\varepsilon F_j} \right) \leq 0. \tag{18}$$

The inequality in Eq.(18) is obtained by using the relation

$$(z-y) \log \frac{y}{z} \leq 0, \tag{19}$$

with equality if, and only if, $y = z$, which is valid for all $y, z \in \mathbb{R}_+$. Hence, we have equality in Eq.(18) if and only if

$$\frac{F_i}{1+\varepsilon F_i} \frac{F_j}{1+\varepsilon F_j} = \frac{F_k}{1+\varepsilon F_k} \frac{F_l}{1+\varepsilon F_l}, \tag{20}$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$.

A Maxwellian distribution (or just a Maxwellian) is a function $M = M(\xi)$, such that (for normal models)

$$M = e^{\phi} = K e^{\mathbf{b} \cdot \xi + c|\xi|^2}, \text{ with } K = e^a > 0, \tag{21}$$

where ϕ is a collision invariant. There is equality in Eq.(18), if and only if $\log \dfrac{F}{1+\varepsilon F}$ is a collision invariant (take the logarithms of Eq.(20)), or equivalently, if and only if $\dfrac{F}{1+\varepsilon F}$ is a Maxwellian M. That is, if and only if F is a Planckian (or if $\varepsilon = 0$ a Maxwellian)

$$P = \frac{M}{1-\varepsilon M}. \tag{22}$$

We define

$$\mathcal{H}[F] = \mathcal{H}[F](x) = \sum_{i=1}^{n} p_i^1 \mu(F_i(x)),$$

where (cf. [37]), for $\varepsilon \in \{0, \pm 1\}$

$$\mu(y) = \begin{cases} y \log y - \varepsilon (1+\varepsilon y) \log (1+\varepsilon y) \text{ if } y > 0 \\ 0 \text{ if } y = 0 \end{cases}.$$

For the planar stationary system

$$B\frac{dF}{dx} = Q^{\varepsilon}(F), \text{ with } B = \text{diag}(p_1^1, ..., p_N^1), \tag{23}$$

we obtain

$$\frac{d}{dx}\mathcal{H}[F] = \sum_{i=1}^{n} p_i^1 \frac{dF_i}{dx} \log \frac{F_i}{1+\varepsilon F_i} = \left\langle \log \frac{F}{1+\varepsilon F}, Q^{\varepsilon}(F) \right\rangle \leq 0,$$

with equality if, and only if, F is a Planckian. Denote by

$$\begin{aligned} j_1 &= \langle B, F \rangle \\ j_{i+1} &= \left\langle Bp^i, F \right\rangle, \; i = 1, ..., d, \\ j_{d+2} &= \left\langle B \left|\mathbf{p}\right|^2, F \right\rangle. \end{aligned} \tag{24}$$

By Eqs.(23),(16) the numbers $j_1, ..., j_{d+2}$ are independent with respect to x in the planar stationary case. For some fixed numbers $j_1, ..., j_{d+2}$, we denote by $\mathbb{P}$ the manifold of all Planckians $F = P$ (22), such that Eq.(24) is fulfilled. Then we can prove the following theorem by arguments similar to the ones used for the discrete Boltzmann equation in [24] (see also [13]).

Theorem 3.1. *If $F = F(x)$ is a bounded nonnegative solution to Eq.(23), then*

$$\lim_{x \to \infty} \text{dist}(F(x), \mathbb{P}) = 0,$$

where $\mathbb{P}$ is the Planckian manifold associated with the invariants (24) of F. If there are only finitely many Planckians in $\mathbb{P}$, then then there is a Planckian P in $\mathbb{P}$, such that $\lim_{x\to\infty} F(x) = P$.

If we denote (cf. Eq.(3))

$$F = P + R^{1/2} f, \text{ with } R = P(1+\varepsilon P) \text{ and } P = \frac{M}{1-\varepsilon M}, \tag{25}$$

in Eq.(9), we obtain

$$\frac{\partial f_i}{\partial t} + \mathbf{p}_i \cdot \nabla_{\mathbf{x}} f_i + (Lf)_i = S_i(f, f, f)$$

where L is the linearized collision operator ($N \times N$ matrix) given by

$$Lf = -R^{-1/2}\left(2Q(P, R^{1/2}f) + \widehat{Q}(R^{1/2}f, P, P) + 2\widehat{Q}(P, R^{1/2}f, P)\right). \tag{26}$$

and the nonlinear part $S(f, f, f)$ is given by

$$\begin{aligned} S(f,g,h) = R^{-1/2} \Big(& Q(R^{1/2}f, R^{1/2}g) + \widehat{Q}(P + R^{1/2}f, R^{1/2}g, R^{1/2}h) + \\ & \widehat{Q}(R^{1/2}f, P, R^{1/2}h) + \widehat{Q}(R^{1/2}f, R^{1/2}g, P)\Big). \end{aligned} \tag{27}$$

In more explicit forms, the operators (26) and (27) read

$$(Lf)_i = \sum_{j,k,l=1}^{N} \frac{\Gamma_{ij}^{kl}}{R_i^{1/2}} (P_{ij}^{kl} f_i + P_{ji}^{kl} f_j - P_{kl}^{ij} f_k - P_{lk}^{ij} f_l), \; i = 1, ..., N \tag{28}$$

where

$$P_{ij}^{kl} = \left(P_j\left(1+\varepsilon P_k+\varepsilon P_l\right)-\varepsilon P_k P_l\right)R_i^{1/2},$$

and

$$S_i(f,f,f) = \sum_{j,k,l=1}^{N}\frac{\Gamma_{ij}^{kl}}{R_i^{1/2}}\left(S_{ij}^{kl}(f,f,f)-S_{kl}^{ij}(f,f,f)\right),\ i=1,...,N, \tag{29}$$

with

$$S_{ij}^{kl}(f,f,f) = \left(1+\varepsilon P_i+\varepsilon P_j\right)R_k^{1/2}R_l^{1/2}f_k f_l+\varepsilon\left(R_i^{1/2}f_i+R_j^{1/2}f_j\right) \left(P_k R_l^{1/2}f_l+P_l R_k^{1/2}f_k+R_k^{1/2}R_l^{1/2}f_k f_l\right).$$

By Eqs.(12),(11),(28), and the relations

$$\begin{aligned} P_iP_j(1+\varepsilon P_k)(1+\varepsilon P_l) &= P_kP_l(1+\varepsilon P_i)(1+\varepsilon P_j),\\ P_{ij}^{kl} &= P_kP_l(1+\varepsilon P_j)\frac{\sqrt{1+\varepsilon P_i}}{\sqrt{P_i}} \end{aligned}$$

for $\Gamma_{ij}^{kl}\neq 0$, we obtain the equality

$$\langle g,Lf\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{N}\Gamma_{ij}^{kl}P_iP_j(1+\varepsilon P_k)(1+\varepsilon P_l) \left(\frac{f_i}{R_i^{1/2}}+\frac{f_j}{R_j^{1/2}}-\frac{f_k}{R_k^{1/2}}-\frac{f_l}{R_l^{1/2}}\right)\left(\frac{g_i}{R_i^{1/2}}+\frac{g_j}{R_j^{1/2}}-\frac{g_k}{R_k^{1/2}}-\frac{g_l}{R_l^{1/2}}\right).$$

It is easy to see that the matrix L is symmetric and positive semi-definite, i.e.

$$\langle g,Lf\rangle = \langle Lg,f\rangle \text{ and } \langle f,Lf\rangle \geq 0,$$

for all functions $g=g(\xi)$ and $f=f(\xi)$.

Furthermore, $\langle f,Lf\rangle = 0$ if and only if

$$\frac{f_i}{R_i^{1/2}}+\frac{f_j}{R_j^{1/2}} = \frac{f_k}{R_k^{1/2}}+\frac{f_l}{R_l^{1/2}} \tag{30}$$

for all indices satisfying $\Gamma_{ij}^{kl}\neq 0$. We denote $f=R^{1/2}\phi$ in Eq.(30) and obtain Eq.(14). Hence, since L is semi-positive,

$$Lf = 0 \text{ if and only if } f = R^{1/2}\phi,$$

where ϕ is a collision invariant (44). Hence, for normal models the null-space $N(L)$ is

$$\begin{aligned} N(L) &= \operatorname{span}\left(R^{1/2},R^{1/2}p^1,...,R^{1/2}p^d,R^{1/2}\left|\mathbf{p}\right|^2\right)\\ &= \operatorname{span}\left\{\sqrt{P(1+\varepsilon P)},\ \sqrt{P(1+\varepsilon P)}\mathbf{p},\ \sqrt{P(1+\varepsilon P)}\left|\mathbf{p}\right|^2\right\}. \end{aligned}$$

Then also

$$\left\langle S\left(f,f,f\right),R^{1/2}\phi\right\rangle = \left\langle Q^{\varepsilon}\left(F\right),\phi\right\rangle+\left\langle F,LR^{1/2}\phi\right\rangle = 0$$

for all collision invariants ϕ, and for some constant $\widetilde{K}$

$$\begin{aligned}|S(f,f,f)-S(h,h,h)| &= \Big|R^{-1/2}\Big(Q(R^{1/2}(f-h),R^{1/2}(f+h)) \\ &\quad+\widehat{Q}(P,R^{1/2}(f-h),R^{1/2}(f+h)) \\ &\quad+2\Big(\widehat{Q}(R^{1/2}(f-h),P,R^{1/2}f)+\widehat{Q}(R^{1/2}h,P,R^{1/2}(f-h))\Big) \\ &\quad+\widehat{Q}(R^{1/2}f,R^{1/2}(f-h),R^{1/2}(f+h))+\widehat{Q}(R^{1/2}(f-h),R^{1/2}h,R^{1/2}h)\Big)\Big| \\ &\leq \widetilde{K}(|f|+|h|)(1+|f|+|h|)\,|f-h|\,. \qquad (31)\end{aligned}$$

The planar stationary system now reads

$$B\frac{df}{dx}+Lf=S(f,f,f),\text{ with }B=\operatorname{diag}(p_1^1,...,p_N^1),$$

where

$$\mathbf{x}=(x=x^1,x^2,...,x^d)\text{ and }\mathbf{p}=(p^1,...,p^d).$$

We assume that the sets $\mathcal{P}=\{\mathbf{p}_1,...,\mathbf{p}_N\}$ are chosen in such a way that $p_i^1\neq 0$ for $i=1,...,N$.

3.2. **Multicomponent mixtures.** We now consider the case of s different species, labelled with $\alpha_1,...,\alpha_s$, with the masses $m^{\alpha_1},...,m^{\alpha_s}$, respectively. Fixing a set of velocities

$$V^{\alpha_i}=\{\xi_1^{\alpha_i},...,\xi_{n^{\alpha_i}}^{\alpha_i}\}\subset\mathbb{R}^d$$

for each species α_i, and assigning the label α_i to each velocity in V^{α_i} we obtain a set of $N=n^{\alpha_1}+...+n^{\alpha_s}$ pairs (each pair being composed of a velocity and a label)

$$\begin{aligned}\mathcal{P} &= \{(\xi_1^{\alpha_1},\alpha_1),...,(\xi_{n^{\alpha_1}}^{\alpha_1},\alpha_1),...,(\xi_1^{\alpha_s},\alpha_s),...,(\xi_{n^{\alpha_s}}^{\alpha_s},\alpha_s)\} \\ &= \{(\mathbf{p}_1,\alpha(1)),...,(\mathbf{p}_N,\alpha(N))\},\text{ with }N=n^{\alpha_1}+...+n^{\alpha_s}.\end{aligned}$$

Note that the same velocity can be repeated many times, but only for different species.

We consider the system (9) − (10) for $\varepsilon=0$ (even if we in principle don't need to restrict ourselves to the case of the discrete Boltzmann equation) with the collision coefficients

$$\Gamma_{ij}^{kl}=\Gamma_{ji}^{kl}=\Gamma_{kl}^{ij}\geq 0$$

with equality unless

$$\begin{aligned}\{\alpha(i),\alpha(j)\} &= \{\alpha(k),\alpha(l)\} \\ m^{\alpha(i)}\mathbf{p}_i+m^{\alpha(j)}\mathbf{p}_j &= m^{\alpha(k)}\mathbf{p}_k+m^{\alpha(l)}\mathbf{p}_l \\ m^{\alpha(i)}|\mathbf{p}_i|^2+m^{\alpha(j)}|\mathbf{p}_j|^2 &= m^{\alpha(k)}|\mathbf{p}_k|^2+m^{\alpha(l)}|\mathbf{p}_l|^2\,.\end{aligned}$$

The collision invariants include, and for normal models are restricted to

$$\phi=(\phi^{\alpha_1},...,\phi^{\alpha_s}),\text{ with }\phi^{\alpha_i}=\phi^{\alpha_i}(\mathbf{p})=a^{\alpha_i}+m^{\alpha_i}\mathbf{b}\cdot\mathbf{p}+cm^{\alpha_i}|\mathbf{p}|^2 \qquad (32)$$

for some constant $a^{\alpha_1},...,a^{\alpha_s},c\in\mathbb{R}$ and $\mathbf{b}\in\mathbb{R}^d$. For normal models we will have $s+d+1$ linearly independent collision invariants. How to construct such normal models is considered in [14].

The Maxwellians are

$$M=e^{\phi},\text{ i.e. }M=(M^{\alpha_1},...,M^{\alpha_s}),\text{ with }M^{\alpha_i}=e^{\phi^{\alpha_i}}, \qquad (33)$$

where (for normal models) ϕ is given by Eq.(32).

The notion of supernormal models was introduced for binary mixtures by Bobylev and Vinerean in [19], and denotes a normal discrete velocity model, which is normal also considering the sets of velocities for the different species separately. It was later extended to the case of mixtures of several species in [14].

Definition 3.2. [14] A DVM $\{\mathrm{V}^{\alpha_1},\ldots,\mathrm{V}^{\alpha_s}\}$ for mixtures of s species is called supernormal if the restriction to each collection

$$\{\mathrm{V}_1,\ldots,\mathrm{V}_i\} \subseteq \{\mathrm{V}^{\alpha_1},\ldots,\mathrm{V}^{\alpha_s}\},\ 1 \leq i \leq s,$$

of velocity sets is a normal DVM for mixtures of i species.

Theorem 3.3. [14] *A DVM $\{\mathrm{V}^{\alpha_1},\ldots,\mathrm{V}^{\alpha_s}\}$ for mixtures of s species is supernormal if and only if the restriction to each pair $\{\mathrm{V}^{\alpha_i},\mathrm{V}^{\alpha_j}\}$, $1 \leq i < j \leq s$, of velocity sets is a supernormal DVM for binary mixtures.*

Theorem 3.4. [14] *Let $d = 2$ or $d = 3$. For any given number s of species with given rational masses $m^{\alpha_1},...,m^{\alpha_s}$ there is a supernormal DVM for the mixture.*

Assume that $d = 2$, $s = 3$, the mass ratios 2, $\dfrac{3}{2}$, and 3, and let

$$\mathrm{V}^{\alpha_i} = \frac{h}{m_{\alpha_i}}\mathrm{V},\ i = 1, 2, 3\mathrm{V} = \{(\pm 1, \pm 1), (3, \pm 1), (1, 3), (3, 3), (5, 1)\},$$

which is a normal DVM, then we obtain a 27-velocity supernormal DVM (see figure 1)

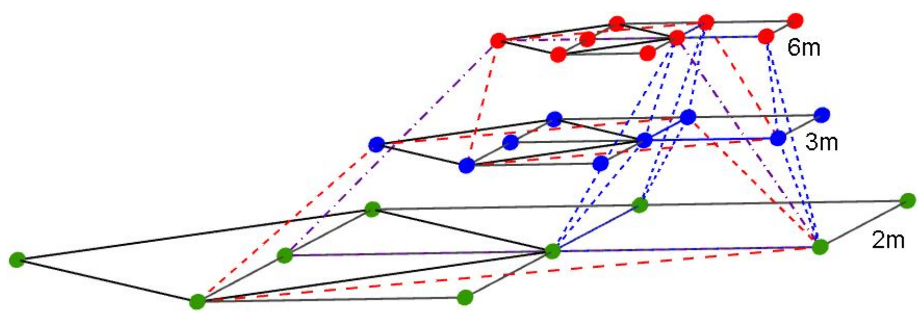

FIGURE 1. 27-velocity model for a mixture of three species with mass ratios 2, 3/2, and 3

3.3. **Polyatomic molecules.** We now assume that we have s different internal energies $E^1,...,E^s$. Fixing a set of velocity vectors $V_i = \left\{\xi_1^i,...,\xi_{n_i}^i\right\} \subset \mathbb{R}^d$ for each internal energy E^i we obtain a set of $N = n_1 + ... + n_s$ pairs (each pair being composed of a velocity vector and an internal energy), cf. [20],

$$\begin{aligned}\mathcal{P} &= \left\{\left(\xi_1^1, E^1\right),...,\left(\xi_{n_1}^1, E^1\right),...,\left(\xi_1^s, E^s\right),...,\left(\xi_{n_s}^s, E^s\right)\right\} \\ &= \{(\mathbf{p}_1, E_1),...,(\mathbf{p}_N, E_N)\}, \text{ with } N = n_1 + ... + n_s.\end{aligned}$$

Obviously, the same velocity can be repeated many times, but only for different internal energies. We might need to scale the distribution functions (see below, cf. [26, 36])

$$f_r' = \frac{f_r}{g_i} \text{ if } E_r = E^i, r = 1,...,N \text{ for some numbers } g_1,...,g_s. \quad (34)$$

Then we consider the system (9) − (10) ($\varepsilon = 0$ for the discrete Boltzmann equation) with the collision coefficients

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0 \tag{35}$$

with equality unless

$$\begin{aligned} \mathbf{p}_i + \mathbf{p}_j &= \mathbf{p}_k + \mathbf{p}_l \text{ and} \\ \frac{m\left|\mathbf{p}_i\right|^2}{2} + \frac{m\left|\mathbf{p}_j\right|^2}{2} + E_i + E_j &= \frac{m\left|\mathbf{p}_k\right|^2}{2} + \frac{m\left|\mathbf{p}_l\right|^2}{2} + E_k + E_l. \end{aligned} \tag{36}$$

We assume that we can obtain the symmetry relations (35), possibly after a scaling (34). Actually assuming a convenient reciprocity relation [26], this will be the case.

The collision invariants include, and for normal models are restricted to

$$\phi = \phi(\mathbf{p}) = a + \mathbf{b} \cdot \mathbf{p} + c(m\left|\mathbf{p}\right|^2 + 2E(\mathbf{p})) \tag{37}$$

for some constant $a, c \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^d$. For normal models we will have $d + 2$ linearly independent collision invariants. How to construct such normal models is considered in [12].

The Maxwellians are

$$M = e^{\phi}, \tag{38}$$

where (for normal models) ϕ is given by Eq.(37).

Definition 3.5. [12] A DVM $\left\{\left\{\mathrm{V}_1, E^1\right\}, ..., \left\{\mathrm{V}_s, E^s\right\}\right\}$, with internal energies $\left\{E^1, ..., E^s\right\}$, is called supernormal if the restriction to each collection

$$\left\{\left\{\mathrm{V}_{r_1}, E^{r_1}\right\}, ..., \left\{\mathrm{V}_{r_i}, E^{r_i}\right\}\right\} \subseteq \left\{\left\{\mathrm{V}_1, E^1\right\}, ..., \left\{\mathrm{V}_s, E_s\right\}\right\}, \, 1 \leq i \leq s,$$

is a normal DVM.

Theorem 3.6. [12] *A DVM $\left\{\left\{\mathrm{V}_1, E^1\right\}, ..., \left\{\mathrm{V}_s, E^s\right\}\right\}$, with internal energies $\left\{E_1, ..., E_s\right\}$, is supernormal if and only if the restriction to each pair $\left\{\left\{\mathrm{V}_i, E^i\right\}, \left\{\mathrm{V}_j, E^j\right\}\right\}$, $1 \leq i < j \leq s$, of velocity sets is a supernormal DVM.*

Theorem 3.7. [12] *Let $d = 2$ or $d = 3$. For any given set of internal energies $\left\{r_1 E, ..., r_s E\right\}$, where $r_1, ..., r_s$ are positive integers, there is a supernormal DVM $\left\{\left\{\mathrm{V}_{r_1}, r_1 E\right\}, ..., \left\{\mathrm{V}_{r_s}, r_s E\right\}\right\}$.*

Assume that $d = 2$, $s = 3$, and $r_i = i$, for $i = 1, 2, 3$, and let

$$\mathrm{V}_i = \frac{\sqrt{E}}{2\sqrt{m}}\mathrm{V}, \, i = 1, 2, 3,$$

where m denotes the mass, and

$$\mathrm{V} = \left\{(\pm 1, \pm 1), (3, \pm 1), (1, 3), (3, 3)\right\},$$

which is a normal 8-velocity DVM, then we obtain a supernormal 24-velocity DVM (see figure 2).

3.4. **Bimolecular chemical reactions.** We can combine the two different approaches in the two preceding sections in an obvious way to obtain models for mixtures with internal energies, see [11]. It is then also possible to add bimolecular reactive collisions [15] and by that extend to models for bimolecular chemical reactions, cf. [11]. We will below consider an example (cf. [39, 31]), but our method is not limited to this case in any way.

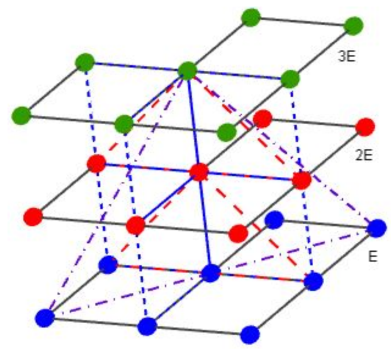

FIGURE 2. 24-velocity supernormal DVM for internal energies E, 2E, and 3E

We assume that we have four different species $A^1, ..., A^4$, with masses $m_1, ..., m_4$ and internal energies $E_1, ..., E_4$, respectively. We assume that we have all elastic collisions (as a mixture), but also a reaction

$$A^1 + A^2 \leftrightharpoons A^3 + A^4$$

if

$$\begin{aligned} m_1\xi^1 + m_2\xi^2 &= m_3\xi^3 + m_4\xi^4 \text{ and} \\ m_1 \left|\xi^1\right|^2 + m_2 \left|\xi^2\right|^2 + 2E_1 + 2E_2 &= m_3 \left|\xi^3\right|^2 + m_4 \left|\xi^4\right|^2 + 2E_3 + 2E_4, \end{aligned}$$

which implies one less collision invariant. The collision invariants for normal models are then

$$\phi = (\phi^{\alpha_1}, \phi^{\alpha_2}, \phi^{\alpha_3}, \phi^{\alpha_4}), \text{ with } \phi^{\alpha_i} = \phi^{\alpha_i}(\xi) = a_{\alpha_i} + m_{\alpha_i}\mathbf{b} \cdot \xi + cm_{\alpha_i} |\xi|^2$$
$$\text{and } a_{\alpha_4} = a_{\alpha_1} + a_{\alpha_2} - a_{\alpha_3},$$

for some constant $a_{\alpha_1}, a_{\alpha_2}, a_{\alpha_3}, c \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^d$. Furthermore, we still assume (after some scaling of the distribution functions) that the collision coefficients fulfill

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0. \tag{39}$$

Assumption (39) is needed to be able to obtain the same structure as for single species, which we, in turn, need to be able to apply our results for boundary layers presented in Section 4 below. However, assuming a convenient reciprocity relation [26], this assumption (39) is fulfilled after a suitable scaling of the distribution function.

3.5. **Bose condensate with excitations.** A general discrete model for excitations in a Bose gas interacting with a Bose condensate, under the assumption that the density of the Bose condensate is constant (cf. [2]), first presented in [10], reads

$$\frac{\partial F_i}{\partial t} + \mathbf{p}_i \cdot \nabla_{\mathbf{x}} F_i = C_{12i}(F) + \Gamma C_{22i}(F), \, i = 1, ..., N, \tag{40}$$

where $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set, $F = (F_1, ..., F_N)$, $F_i = F_i(x) = F(x, \mathbf{p}_i)$, is the distribution function of the excitations, and $\Gamma \in \mathbb{R}_+$ is constant. For generality, we allow $\mathbf{p} = (p^1, ..., p^d)$ to be of dimension d, rather than of dimension 3.

The collision operators $C_{12i}(F)$ are given by

$$C_{12i}(F) = \sum_{j,k,l=1}^{N} (\delta_{il} - \delta_{ij} - \delta_{ik}) \Gamma_{jk}^{l} \left((1 + F_l) F_j F_k - F_l (1 + F_j)(1 + F_k) \right),$$

where $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$, with $\Gamma_{jk}^{i} = 1$ if

$$\mathbf{p}_i = \mathbf{p}_j + \mathbf{p}_k \text{ and } |\mathbf{p}_i|^2 = |\mathbf{p}_j|^2 + |\mathbf{p}_k|^2 + n, \tag{41}$$

and $\Gamma_{jk}^{i} = 0$ otherwise. Furthermore, the collision operators $C_{22i}(F)$ are given by $Q_i^1(F)$ in Eq.(10) with $\Gamma_{ij}^{kl} = 1$ if Eq.(12) is satisfied, and $\Gamma_{ij}^{kl} = 0$ otherwise. The system (2) can also be written as

$$\frac{\partial F}{\partial t} + \mathbf{p} \cdot \nabla_{\mathbf{x}} F = C_{12}(F) + \Gamma C_{22}(F). \tag{42}$$

The collision operator $C_{12}(F)$ in (42) is also given by the expression

$$C_{12}(F) = n\widetilde{L}F + n\widetilde{Q}(F, F), \tag{43}$$

where

$$\begin{aligned} \left(\widetilde{L}F\right)_i &= \sum_{j,k=1}^{N} 2\Gamma_{ij}^{k} F_k - \Gamma_{jk}^{i} F_i \text{ and} \\ \widetilde{Q}_i(F, G) &= \sum_{j,k=1}^{N} \Gamma_{jk}^{i} Q_{jk}^{i}(F, G) - 2\Gamma_{ij}^{k} Q_{ij}^{k}(F, G), \text{ with} \\ Q_{jk}^{i}(F, G) &= \frac{1}{2} \left(F_j G_k + G_j F_k - F_i (G_j + G_k) - G_i (F_j + F_k) \right). \end{aligned}$$

A function $\phi = \phi(\mathbf{p})$ is a collision invariant, if and only if,

$$\phi_i = \phi_j + \phi_k,$$

for all indices such that $\Gamma_{jk}^{i} \neq 0$, if $\Gamma = 0$, with the additional condition (14) for all indices such that $\Gamma_{ij}^{kl} \neq 0$, if $\Gamma \neq 0$. The collision invariants include, and for normal models (without spurious or non-physical collision invariants) are limited to

$$\phi = \phi(\mathbf{p}) = -\alpha \left(|\mathbf{p}|^2 + n \right) - \beta \cdot \mathbf{p}, \tag{44}$$

for some constant $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$. Then the equation

$$\langle C_{12}(F) + \Gamma C_{22}(F), \phi \rangle = 0 \tag{45}$$

has the general solution (44). Also, see [10],

$$\left\langle \log \frac{F}{1 + F}, C_{12}(F) + \Gamma C_{22}(F) \right\rangle \leq 0.$$

The Maxwellians are (for normal models)

$$M = (M_1, ..., M_n), \text{ with } M_i = e^{\phi_i} = e^{-\alpha\left(|\mathbf{p}_i|^2 + n\right) - \beta \cdot \mathbf{p}_i},$$

and the Planckians are (again for normal models)

$$P = (P_1, ..., P_n), \text{ with } P_i = \frac{M_i}{1 - M_i} = \frac{1}{e^{\alpha\left(|\mathbf{p}_i|^2 + n\right) + \beta \cdot \mathbf{p}_i} - 1} = \frac{1}{e^{\alpha\left(|\mathbf{p_i} - \mathbf{p}_0|^2 + n_0\right)} - 1}, \tag{46}$$

with $\alpha > 0$, $\beta \in \mathbb{R}^d$, $\mathbf{p}_0 = \dfrac{\beta}{2}$ and $n_0 = n - |\mathbf{p}_0|^2$.

We define

$$\mathcal{H}[F] = \mathcal{H}[F](x) = \sum_{i=1}^{n} p_i^1 \mu(F_i(x)),$$

where

$$\mu(y) = \begin{cases} y \log y - (1+y)\log(1+y) & \text{if } y > 0 \\ 0 \text{ if } y = 0 \end{cases}.$$

For the planar stationary system

$$B\frac{dF}{dx} = C_{12}(F) + \Gamma C_{22}(F), \text{ with } B = \operatorname{diag}(p_1^1, ..., p_N^1), \tag{47}$$

we obtain

$$\frac{d}{dx}\mathcal{H}[F] = \sum_{i=1}^{n} p_i^1 \frac{dF_i}{dx} \log \frac{F_i}{1+F_i} = \left\langle \log \frac{F}{1+F}, C_{12}(F) + \Gamma C_{22}(F) \right\rangle \leq 0,$$

with equality if, and only if, F is a Planckian. Denote by

$$\begin{aligned} j_i &= \left\langle Bp^i, F \right\rangle, \ i = 1, ..., d, \\ j_{d+1} &= \left\langle B\left(|\mathbf{p}|^2 + n\right), F \right\rangle. \end{aligned} \tag{48}$$

By Eqs.(47),(45) the numbers $j_1, ..., j_{d+1}$ are independent with respect to x in the planar stationary case. For some fixed numbers $j_1, ..., j_{d+1}$, we denote by $\mathbb{P}$ the manifold of all Planckians $F = P$ (46), such that Eq.(48) is fulfilled. Then we can prove the following analogue to Theorem 3.1 by similar arguments (used for the discrete Boltzmann equation in [24] and also [13]).

Theorem 3.8. *If $F = F(x)$ is a bounded nonnegative solution to Eq.(47), then*

$$\lim_{x \to \infty} dist(F(x), \mathbb{P}) = 0,$$

where $\mathbb{P}$ is the Planckian manifold associated with the invariants (48) of F. If there are only finitely many Planckians in $\mathbb{P}$, then then there is a Planckian P in $\mathbb{P}$, such that $\lim\limits_{x \to \infty} F(x) = P$.

Given a Planckian (46) we denote

$$F = P + R^{1/2} f, \text{ with } R = P(1+P),$$

in Eq.(42), and obtain

$$\frac{\partial F}{\partial t} + \mathbf{p} \cdot \nabla_{\mathbf{x}} F + Lf = S(f),$$

where $L = L_{12} + \Gamma L_{22}$, with

$$L_{12} f = -2nR^{-1/2}\widetilde{Q}(P, R^{1/2} f) - n\widetilde{L}R^{1/2} f \tag{49}$$

and L_{22} given by Eqs.(26),(28), is the linearized collision operator ($N \times N$ matrix), and $S(f) = S_{12}(f,f) + S_{22}(f,f,f)$, with

$$S_{12}(f,g) = nR^{-1/2}\widetilde{Q}(R^{1/2} f, R^{1/2} g) \tag{50}$$

and $S_{22}(f,f,f)$ given by Eqs.(27),(29), is the nonlinear part. In more explicit forms, the operators (49) and (50) read

$$\begin{aligned}(L_{12}f)_i &= n\sum_{j,k=1}^{N}\frac{\Gamma_{jk}^{i}L_{jk}^{i}f-2\Gamma_{ij}^{k}L_{ij}^{k}f}{R_i^{1/2}},\ i=1,...,n,\text{ with}\\ L_{jk}^{i}f &= (1+P_j+P_k)\,R_i^{1/2}f_i-(P_k-P_i)\,R_j^{1/2}f_j-(P_j-P_i)\,R_k^{1/2}f_k, \quad (51)\end{aligned}$$

and

$$\begin{aligned}S_{12i}(f,g) &= n\sum_{j,k=1}^{N}\frac{\Gamma_{jk}^{i}S_{jk}^{i}(f,g)-2\Gamma_{ij}^{k}S_{ij}^{k}(f,g)}{R_i^{1/2}},\ i=1,...,N,\text{ with}\\ S_{jk}^{i}(f,g) &= \frac{1}{2}\Big(R_j^{1/2}R_k^{1/2}\left(f_jg_k+g_jf_k\right)-R_i^{1/2}R_j^{1/2}\left(f_ig_j+g_if_j\right)-\\ &\quad R_i^{1/2}R_k^{1/2}\left(f_ig_k+g_if_k\right)\Big).\end{aligned}$$

The linearized collision operator L is symmetric and positive semi-definite with, for normal models, the null-space

$$\begin{aligned}N(L) &= \operatorname{span}\left(R^{1/2},R^{1/2}p^1,...,R^{1/2}p^d,R^{1/2}\left|\mathbf{p}\right|^2\right)\\ &= \operatorname{span}\left\{\sqrt{P(1+\varepsilon P)},\ \sqrt{P(1+\varepsilon P)}\mathbf{p},\ \sqrt{P(1+\varepsilon P)}\left|\mathbf{p}\right|^2\right\}.\end{aligned}$$

Then also

$$\left\langle S\left(f\right),R^{1/2}\phi\right\rangle=\left\langle C_{12}\left(F\right)+\Gamma C_{22}\left(F\right),\phi\right\rangle+\left\langle F,LR^{1/2}\phi\right\rangle=0$$

for all collision invariants ϕ, and for some constant $\widetilde{K}$

$$\begin{aligned}&\left|S_{12}\left(f,f\right)-S_{12}\left(h,h\right)\right|=\\ &\qquad\left|nR^{-1/2}\left(\widetilde{Q}(R^{1/2}\left(f-h\right),R^{1/2}\left(f+h\right))\right)\right|\leq\widetilde{K}(\left|f\right|+\left|h\right|)\left|f-h\right|.\end{aligned}$$

Hence, by the inequality (31) there is some constant $\widetilde{K}$ (possibly different from the one above) such that

$$\left|S\left(f\right)-S\left(h\right)\right|\leq\widetilde{K}(\left|f\right|+\left|h\right|)(1+\left|f\right|+\left|h\right|)\left|f-h\right|.$$

We can also, before prescribing the set of velocities, make the change of variables

$$\mathbf{p}\rightarrow\mathbf{p}+\mathbf{p}_0 \quad (52)$$

(cf. Eq.(46)). We then, instead of relations (41), obtain the relations

$$\mathbf{p}_i=\mathbf{p}_j+\mathbf{p}_k+\mathbf{p}_0\text{ and }\left|\mathbf{p}_i\right|^2=\left|\mathbf{p}_j\right|^2+\left|\mathbf{p}_k\right|^2+n_0,$$

where $n_0=n-\left|\mathbf{p}_0\right|^2$, and the collision invariants

$$\phi=\mathbf{a}\cdot\left(\mathbf{p}+\mathbf{p}_0\right)+b\left(\left|\mathbf{p}\right|^2+n_0\right).$$

Moreover

$$N(L)=\operatorname{span}\left(R^{1/2}\left(p^1+p_0^1\right),...,R^{1/2}\left(p^d+p_0^d\right),R^{1/2}\left(\left|\mathbf{p}\right|^2+n_0\right)\right),$$

and if $p_0^1\neq 0$, then the matrix B have to be replaced with $B+p_0^1I$.

The planar stationary system reads as before

$$B\frac{df}{dx}+Lf=S(f),\text{ with }B=\operatorname{diag}(p_1^1,...,p_N^1),$$

where
$$\mathbf{x} = (x = x^1, x^2, ..., x^d) \text{ and } \mathbf{p} = (p^1, ..., p^d).$$

We assume that the sets $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\}$ are chosen in such a way that $p_i^1 \neq 0$ for $i = 1, ..., N$.

4. **Boundary layers.**

We can (without loss of generality) assume that
$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \tag{53}$$
where
$$B_+ = \operatorname{diag}(b_1, ..., b_{n^+}) \text{ and } B_- = -\operatorname{diag}(b_{n^++1}, ..., b_N), \text{ with}$$
$$b_1, ..., b_{n^+} > 0 \text{ and } b_{n^++1}, ..., b_N < 0. \tag{54}$$
We also define the projections $R_+ : \mathbb{R}^N \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^N \to \mathbb{R}^{n^-}$, $n^- = N - n^+$, by
$$R_+ s = s^+ = (s_1, ..., s_{n^+}) \text{ and } R_- s = s^- = (s_{n^++1}, ..., s_N)$$
for $s = (s_1, ..., s_N)$, and consider the non-linear system
$$\begin{cases} B\frac{df}{dx} + Lf = S(f) \\ f^+(0) = Cf^-(0) + h_0 \\ f(x) \to 0 \text{ as } x \to \infty \end{cases}, \tag{55}$$
where C is a given $n^+ \times n^-$ matrix, $h_0 \in \mathbb{R}^{n^+}$, and the non-linear part fulfills
$$S(f) \in N(L)^\perp$$
and
$$|S(g) - S(h)| \leq \widetilde{K} G(|g|, |h|) |g - h|$$
for some positive constant $\widetilde{K} > 0$ and differentiable function $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with positive partial derivatives and $G(0, 0) = 0$.

The boundary condition $f(x) \to 0$ as $x \to \infty$ corresponds to the case when we have made the transformation (3) for a stationary point $P = P_\infty$, such that $F \to P_\infty$ as $x \to \infty$.

We introduce the operator $\mathcal{C} : \mathbb{R}^N \to \mathbb{R}^{n^+}$, given by
$$\mathcal{C} = R_+ - CR_-,$$
and assume that
$$\dim \mathcal{C}X_+ = n^+, \text{ with } X_+ = \operatorname{span}(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, w_1, ..., w_l). \tag{56}$$

We have the following result [7].

Lemma 4.1. *Let B_+ and B_- be the matrices defined by Eq.(54). Then condition (56) is fulfilled, if*
$$C^T B_+ C < B_- \text{ on } R_- X_+. \tag{57}$$

Proof. Let $u \in X_+$ and $C^T B_+ C < B_-$ on $R_- X_+$. Then
$$\langle u, u \rangle_B \geq 0.$$
Furthermore, if $u \neq 0$ and $\mathcal{C}u = 0$, then
$$\langle u, u \rangle_B = \left\langle Cu^-, Cu^- \right\rangle_{B_+} - \left\langle u^-, u^- \right\rangle_{B_-} = \left\langle (C^T B_+ C - B_-)u^-, u^- \right\rangle < 0.$$
Hence, if $\mathcal{C}u = 0$, then $u = 0$. That is, $\dim \mathcal{C}X_+ = \dim X_+ = n^+$, and the lemma is proved. □

Remark 3. Condition (57) can, if $l = 0$, be weakened to

$$C^T B_+ C \leq B_- \text{ on } R_- X_+.$$

If $C = 0$, then condition (56) is fulfilled. In particular,

$$\left\{u_1^+, ..., u_{m^+}^+, y_1^+, ..., y_{k^+}^+, w_1^+, ..., w_l^+\right\}$$

is a basis of $\mathbb{R}^{n^+}$. When the Maxwell-type boundary conditions, in the case of the discrete Boltzmann equation, fulfill condition (56) is also studied in [7].

Our main result on boundary layers gives the number of conditions that must be posed on the given data h_0 to obtain a well-posed problem.

Theorem 4.2. *Let condition (56) be fulfilled and suppose that $\langle h_0, h_0 \rangle_{B_+}$ is sufficiently small. Then with $k^+ + l$ conditions on h_0, the system (55) has a locally unique solution.*

Theorem 4.2 is proved below in Section 6.

For the discrete Boltzmann equation Theorem 4.2 improves the results in [7] for the degenerate case $l > 0$ by getting rid of some restrictive assumptions on the nonlinear part. An interesting thing and one of the main results of this paper is that the generalizations made, make it possible to apply the results also for mixtures, polyatomic gases with a discrete number of internal energies, and bimolecular reactive flows [15, 31, 39], but also for some discrete quantum kinetic equations, Nordheim-Boltzmann equation [1, 37] (maybe more known as the Uehling-Uhlenbeck equation [42]) and an equation for excitations in a Bose gas interacting with a Bose condensate at low temperatures [2, 34, 46] (see Section 3 above).

Remark 4. Our results can be extended in a natural way, to yield also for singular matrices B, cf. [7], if

$$N(L) \cap N(B) = \{0\}.$$

5. Critical numbers for axially symmetric discrete models.

In this section we study, instead of Eq.(5), the equation

$$(B + uI)\frac{df}{dx} + Lf = S(f), \tag{58}$$

and consider such symmetric sets $\mathcal{P}$, such that

$$\text{if } \mathbf{p}_i = (p_i^1, p_i^2, ..., p_i^d) \in \mathcal{P}, \text{ then } (\pm p_i^1, \pm p_i^2, ..., \pm p_i^d) \in \mathcal{P}. \tag{59}$$

We also assume that (i) we have a symmetric set (59); (ii) our DKM is normal; and (iii)

$$B = \text{diag}(p_1^1, ..., p_{\widetilde{N}}^1, -p_1^1, ..., -p_{\widetilde{N}}^1), \text{ with } p_1^1, ..., p_{\widetilde{N}}^1 > 0.$$

Below we will omit the tildes, and just write N instead of $\widetilde{N}$.

In this case a possible reduction is as follows: the equation (5) (or (58)) admit a class of solutions satisfying

$$F_i = F_{i'} \text{ if } p_i^1 = p_{i'}^1 \text{ and } |\mathbf{p}_i|^2 = |\mathbf{p}_{i'}|^2. \tag{60}$$

This reduces the number N of equations (2) to the number $2\widetilde{N} \leq N$ of different combinations $(p_i^1, |\mathbf{p}_i|^2)$. The structure of the collision terms (43) (including extensions) and (13) (in slightly different notations) remains unchanged. However, to be able to keep the structure, we might need to add equal equations (instead of just taking them away). Hence, the elements in the diagonal matrix (53) might change, but will still be multiples (with positive multipliers $r_i > 0$) of the previous ones.

Below we will omit the tildes, and just write N instead of $\widetilde{N}$. We can, without loss of generality, assume that

$$(p^1_{i+N}, |\mathbf{p}_{i+N}|^2) = (-p^1_i, |\mathbf{p}_i|^2) \text{ and } p^1_i > 0$$

for $i = 1, ..., N$, and obtain

$$B = \operatorname{diag}(r_1 p^1_1, ..., r_N p^1_N, -r_1 p^1_1, ..., -r_N p^1_N), \text{ with } p^1_1, ..., p^1_N > 0.$$

5.1. **Nordheim-Boltzmann equation.** We assume that the Maxwellian (21) in the Planckian (22), which the transformation (25) is made around, is non-drifting, i.e. with $\mathbf{b} = \mathbf{0}$ in Eq.(21). The linearized collision operator L has the null-space

$$N(L) = \operatorname{span}(\phi_1, ..., \phi_{d+2}),$$

where, with $R = P(1+P)$,

$$\begin{array}{l} \phi_1 = R^{1/2} = R^{1/2} \cdot (1, ..., 1) \\ \phi_2 = R^{1/2} p^1 = R^{1/2} \cdot (p^1_1, ..., p^1_N, -p^1_1, ..., -p^1_N) \\ \phi_3 = R^{1/2} |\mathbf{p}|^2 = R^{1/2} \cdot (|\mathbf{p}_1|^2, ..., |\mathbf{p}_N|^2, |\mathbf{p}_1|^2, ..., |\mathbf{p}_N|^2) \\ \phi_{i+2} = R^{1/2} p^i = R^{1/2} \cdot (p^i_1, ..., p^i_N, -p^i_1, ..., -p^i_N),\ i = 2, ..., d. \end{array} \tag{61}$$

Then the degenerate values of u, i.e. the values of u for which $l \geq 1$, are

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5 - \chi_3^2)}}, \tag{62}$$

where $\chi_1 = \langle \phi_1, \phi_1 \rangle$, $\chi_2 = \langle \phi_2, \phi_2 \rangle$, $\chi_3 = \langle \phi_1, \phi_3 \rangle$, $\chi_4 = \langle \phi_2, \phi_3 \rangle_B$, $\chi_5 = \langle \phi_3, \phi_3 \rangle$, cf. [6, 7, 13]. Moreover, we can obtain the following table for the values of k^+, k^- and l ([6, 7, 13]):

	$u < u_-$	$u = u_-$	$u_- < u < 0$	$u = 0$	$0 < u < u_+$	$u = u_+$	$u_+ < u$
k^+	0	0	1	1	$d+1$	$d+1$	$d+2$
k^-	$d+2$	$d+1$	$d+1$	1	1	0	0
l	0	1	0	d	0	1	0

. (63)

In the continuous case $\langle f, g \rangle = \int fg \, d\mathbf{p}$ and $[f, g] = \int ufg \, d\mathbf{p}$ corresponds to $\langle f, g \rangle_B$.

For the continuous Boltzmann equation ($\varepsilon = 0$), with $d = 3$, the numbers $\chi_1, ..., \chi_5$ are given by

$$\chi_1 = \rho,\ \chi_2 = \rho T,\ \chi_3 = 3\rho T,\ \chi_4 = 5\rho T^2 \text{ and } \chi_5 = 15\rho T^2,$$

(where ρ and T denote the density and the temperature respectively), if we have made the expansion (3) around a non-drifting Maxwellian

$$M = \frac{\rho}{(2\pi T)^{3/2}} e^{-|\xi|^2/2T}.$$

Therefore, for the Boltzmann equation (with $d = 3$) the degenerate values (62) are (cf. [25])

$$u_0 = 0 \text{ and } u_\pm = u^0_\pm = \pm\sqrt{\frac{5T}{3}}.$$

On the other hand, in the continuous case, assuming for bosons ($\varepsilon = 1$) and fermions ($\varepsilon = -1$), with $d = 3$, that

$$P = P_\pm = \frac{1}{e^{\frac{|\mathbf{p}|^2}{2T}} \mp 1},$$

respectively, we have

$$R = R_{\pm} = P_{\pm}(1 \pm P_{\pm}) = \frac{e^{\frac{|\mathbf{p}|^2}{2T}}}{\left(e^{\frac{|\mathbf{p}|^2}{2T}} \mp 1\right)^2}.$$

By a change to spherical coordinates, we obtain

$$\begin{aligned}
\chi_1 &= \int R\, d\mathbf{p} = 8\pi T\, I_2^{\pm}, \chi_2 = \int R \left(p^1\right)^2\, d\mathbf{p} = \frac{16\pi}{3} T^2 I_4^{\pm}, \\
\chi_3 &= \int R \left|\mathbf{p}\right|^2\, d\mathbf{p} = 16\pi T^2 I_4^{\pm},\ \chi_4 = \int R \left(p^1\right)^2 \left|\mathbf{p}\right|^2\, d\mathbf{p} = \frac{32\pi}{3} T^3 I_6^{\pm}, \text{ and} \\
\chi_5 &= \int R \left|\mathbf{p}\right|^4\, d\mathbf{p} = 32\pi T^3 I_6^{\pm}, \text{ where} \\
I_n^{+} &= \int\limits_{\lambda}^{\infty} r^n \frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\, dr \text{ and } I_n^{-} = \int\limits_{0}^{\infty} r^n \frac{e^{r^2}}{\left(e^{r^2}+1\right)^2}\, dr.
\end{aligned}$$

Here we have for bosons considered the restriction $|\mathbf{p}| \geq \lambda\sqrt{2T}$, for some $\lambda > 0$, cf. [2, 10]. Then the degenerate values are

$$u_0 = 0,\ u_{+}^{\pm 1} = \sqrt{\frac{I_6^{\pm}}{I_4^{\pm}}}\sqrt{\frac{2T}{3}}, \text{and } u_{-}^{\pm 1} = -\sqrt{\frac{I_6^{\pm}}{I_4^{\pm}}}\sqrt{\frac{2T}{3}},$$

with $I_n^{+} = \int\limits_{\lambda}^{\infty} r^n \frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\, dr$ and $I_n^{-} = \int\limits_{0}^{\infty} r^n \frac{e^{r^2}}{\left(e^{r^2}+1\right)^2}\, dr$.

Considering fermions,

$$I_{2n}^{-} = \sqrt{\pi}\frac{(2n-1)!!}{2^{n+1}}\eta\left(n - \frac{1}{2}\right),$$

where η is the Dirichlet eta-function or alternating zeta-function, and hence

$$u_{\pm}^{-1} = \pm\sqrt{\frac{\eta\left(5/2\right)}{\eta\left(3/2\right)}}\sqrt{\frac{5T}{3}}.$$

On the other hand, considering bosons,

$$I_{2n}^{+} \to \sqrt{\pi}\frac{(2n-1)!!}{2^{n+1}}\zeta\left(n - \frac{1}{2}\right) \text{ as } \lambda \to 0,$$

where ζ is the zeta-function, and hence

$$u_{\pm}^{+1} \to \pm\sqrt{\frac{\zeta\left(5/2\right)}{\zeta\left(3/2\right)}}\sqrt{\frac{5T}{3}} \text{ as } \lambda \to 0.$$

However, remind that $\zeta\left(\frac{1}{2}\right)$ is infinite.

The values of k^+, k^- and l for the (continuous) Nordheim-Boltzmann equation, with $d = 3$, are given by the table (cf. [25, 44] etc. for the Boltzmann equation)

		$u = u_-^{\varepsilon}$		$u = 0$		$u = u_+^{\varepsilon}$	
k^+	0	0	1	1	4	4	5
k^-	5	4	4	1	1	0	0
l	0	1	0	3	0	1	0

. (64)

5.2. **Multicomponent mixtures.** We assume that the symmetric set $\mathcal{P}$ consists of s symmetric (in the sense of Eq.(59)) sets of $2N_{\alpha_i}$, $i = 1, ..., s$, velocities respectively, which constitute normal models considered by themselves, but also a normal model all together (cf. semi-supernormal DVMs in [14]), and that we have made the transformation (25) around a non-drifting Maxwellian M (i.e. with $\mathbf{b} = \mathbf{0}$ in Eqs.(32),(33)). Let

$$B = \operatorname{diag}\left(B_{\alpha_1}, ..., B_{\alpha_s}\right), \text{ with } B_\alpha = \operatorname{diag}(\xi_1^{\alpha,1}, ..., \xi_{N_\alpha}^{\alpha,1}, -\xi_1^{\alpha,1}, ..., -\xi_{N_\alpha}^{\alpha,1}). \tag{65}$$

The linearized collision operator L has the null-space

$$N(L) = \operatorname{span}\left(\phi_1^{\alpha_1}, ..., \phi_1^{\alpha_s}, \phi_2, ..., \phi_{d+2}\right),$$

where

$$\phi_1^{\alpha_i} = M^{1/2} \cdot (\underbrace{0, ..., 0}_{2\sum_{j=1}^{i-1} N_{\alpha_j}}, \underbrace{1, ..., 1}_{2N_{\alpha_i}}, \underbrace{0, ..., 0}_{2\sum_{j=i+1}^{s} N_{\alpha_j}}),\ i = 1, ..., s,$$
$$\phi_2 = M^{1/2} \cdot (m_{\alpha_1}\phi_2^{\alpha_1}, ..., m_{\alpha_s}\phi_2^{\alpha_s}), \text{ with } \phi_2^\alpha = (\xi_1^{\alpha,1}, ..., \xi_{N_\alpha}^{\alpha,1}, -\xi_1^{\alpha,1}, ..., -\xi_{N_\alpha}^{\alpha,1})$$
$$\phi_3 = M^{1/2} \cdot (m_{\alpha_1}\phi_3^{\alpha_1}, ..., m_{\alpha_s}\phi_3^{\alpha_s}), \text{ with } \phi_3^\alpha = (\left|\xi_1^\alpha\right|^2, ..., \left|\xi_{N_\alpha}^\alpha\right|^2, \left|\xi_1^\alpha\right|^2, ..., \left|\xi_{N_\alpha}^\alpha\right|^2)$$
$$\phi_{2+i} = M^{1/2} \cdot (m_{\alpha_1}\phi_{2+i}^{\alpha_1}, ..., m_{\alpha_s}\phi_{2+i}^{\alpha_s}), \text{with } \phi_{2+i}^\alpha = (\xi_1^{\alpha,i}, ..., \xi_{2N_\alpha}^{\alpha,i}), i = 2, ..., d.$$

The degenerate values of u are

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\mathcal{X}}{\chi_2\left(\sum_{i=1}^{s}\frac{(\chi_3^{\alpha_i})^2}{\chi_1^{\alpha_i}} - \chi_5\right)}}, \text{ with}$$
$$\mathcal{X} = \chi_4^2 + \chi_5\sum_{i=1}^{s}\frac{(\chi_2^{\alpha_i})^2}{\chi_1^{\alpha_i}} - 2\chi_4\sum_{i=1}^{s}\frac{\chi_2^{\alpha_i}\chi_3^{\alpha_i}}{\chi_1^{\alpha_i}} - \sum_{i=1}^{s}\frac{(\chi_2^{\alpha_i}\chi_3^{\alpha_j} - \chi_2^{\alpha_j}\chi_3^{\alpha_i})^2}{\chi_1^{\alpha_i}\chi_1^{\alpha_j}}.$$

where $\chi_1^{\alpha_i} = \langle\phi_1^{\alpha_i}, \phi_1^{\alpha_i}\rangle$, $\chi_2^{\alpha_i} = \langle\phi_1^{\alpha_i}, \phi_2\rangle_{B_{\alpha_i}} = m_{\alpha_i}\langle\phi_2^{\alpha_i}, \phi_2^{\alpha_i}\rangle$, $\chi_2 = \langle\phi_2, \phi_2\rangle$, $\chi_3^{\alpha_i} = \langle\phi_1^{\alpha_i}, \phi_3\rangle = m_{\alpha_i}\langle\phi_1^{\alpha_i}, \phi_3^{\alpha_i}\rangle$, $\chi_4 = \langle\phi_2, \phi_3\rangle_B$, and $\chi_5 = \langle\phi_3, \phi_3\rangle$. Moreover, we can obtain the following table for the values of k^+, k^- and l (cf. [8] for $s = 2$):

		$u = u_-$		$u = 0$		$u = u_+$	
k^+	0	0	1	1	$s+d$	$s+d$	$s+d+1$
k^-	$s+d+1$	$s+d$	$s+d$	1	1	0	0
l	0	1	0	$s+d-1$	0	1	0

For the continuous Boltzmann equation, with $d = 3$,

$$\chi_1^{\alpha_i} = n_{\alpha_i},\ \chi_2^{\alpha_i} = n_{\alpha_i}T,\ \chi_2 = \sum_{i=1}^{s} m_{\alpha_i}n_{\alpha_i}T,\ \chi_3^{\alpha_i} = 3n_{\alpha_i}T,\ \chi_4 = 5\sum_{i=1}^{s} n_{\alpha_i}T^2,$$
$$\chi_5 = 15\sum_{i=1}^{s} n_{\alpha_i}T^2,$$

(where $n_{\alpha_1}, ..., n_{\alpha_s}$, and T denote the number densities of the species $\alpha_1, ..., \alpha_s$ and the temperature respectively), if we have made the expansion (3) around a non-drifting Maxwellian

$$M = \left(M_{\alpha_1}, ..., M_{\alpha_s}\right), \text{ with } M_{\alpha_i} = \frac{n_{\alpha_i}}{(2\pi T)^{3/2}}e^{-m_{\alpha_i}|\xi|^2/2T}.$$

Therefore, for the Boltzmann equation, with $d=3$, for a mixture of s species the degenerate values (62) are

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\sum_{i=1}^s n_{\alpha_i}}{\sum_{i=1}^s m_{\alpha_i} n_{\alpha_i}}}\sqrt{\frac{5T}{3}}.$$

The values of k^+, k^- and l for the Boltzmann equation, with $d=3$, for a mixture of s species are given by the table

		$u=u_-$		$u=0$		$u=u_+$	
k^+	0	0	1	1	$s+3$	$s+3$	$s+4$
k^-	$s+4$	$s+3$	$s+3$	1	1	0	0
l	0	1	0	$s+2$	0	1	0

.

5.3. **Polyatomic molecules.** We assume that the symmetric set $\mathcal{P}$ consists of s copies of the same symmetric (in the sense of Eq.(59)) set of $2N$ velocities, which constitutes a normal model, and that we have made the change of variables (34) and the transformation (25) around a non-drifting Maxwellian M (i.e. with $\mathbf{b}=\mathbf{0}$ in Eqs.(38),(37)). Let

$$B = \operatorname{diag}(B_1, ..., B_s)\,,\ \text{with } B_i = g_i \operatorname{diag}(\xi_1, ..., \xi_N, -\xi_1, ..., -\xi_N),$$

and replace uI in Eq.(59) with

$$\widetilde{u} = \operatorname{diag}(u_1, ..., u_s)\,,\ \text{with } u_i = g_i \operatorname{diag}(u, ..., u),$$

The linearized collision operator L has the null-space

$$N(L) = \operatorname{span}(\phi_1, \phi_2, ..., \phi_{d+2})\,,$$

where

$$\begin{aligned}
&\phi_1 = M^{1/2}\cdot(1, ..., 1)\\
&\phi_2 = M^{1/2}\cdot(\widetilde{\phi}_2, ..., \widetilde{\phi}_2),\ \text{with } \widetilde{\phi}_2 = (\xi_1^1, ..., \xi_N^1, -\xi_1^1, ..., -\xi_N^1)\\
&\phi_3 = M^{1/2}\cdot(\phi_3^1, ..., \phi_3^s)\\
&\phi_{2+i} = M^{1/2}\cdot(\widetilde{\phi}_{2+i}, ..., \widetilde{\phi}_{2+i}), \text{with } \widetilde{\phi}_{2+i} = (\xi_1^i, ..., \xi_{2N}^i), i = 2, ..., d,\\
&\text{with } \phi_3^r = (|\xi_1|^2 + 2E^r, ..., |\xi_N|^2 + 2E^r, |\xi_1|^2 + 2E^r, ..., |\xi_N|^2 + 2E^r)
\end{aligned}$$

The degenerate values of u are

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5 - \chi_3^2)}}.$$

where $\chi_1 = \langle\phi_1, \phi_1\rangle$, $\chi_2 = \langle\phi_2, \phi_2\rangle$, $\chi_3 = \langle\phi_1, \phi_3\rangle$, $\chi_4 = \langle\phi_2, \phi_3\rangle_B$, and $\chi_5 = \langle\phi_3, \phi_3\rangle$. Moreover, the values of k^+, k^- and l are given by the table (63).

For the continuous Boltzmann equation, with $d=3$, the numbers $\chi_1, ..., \chi_5$ are given by (see also [11])

$$\begin{aligned}
\chi_1 &= n,\ \chi_2 = nT,\ \chi_3 = 3nT + \frac{2n}{Q}\sum_{i=1}^s g_i E^i e^{-E^i/T},\\
\chi_4 &= \frac{n}{Q}\sum_{i=1}^s \left(5T^2 + 2TE^i\right) g_i e^{-E^i/T}, \text{ and}\\
\chi_5 &= 15nT^2 + \frac{4n}{Q}\sum_{i=1}^s \left(3TE^i + \left(E^i\right)^2\right) g_i e^{-E^i/T},
\end{aligned}$$

(where $E^1, ..., E^s$, and T denote the different internal energies and the temperature respectively), if we have made the expansion (3) around a non-drifting Maxwellian

$$M = (M_1, ..., M_s)\,, \text{ with } M_i = \frac{n}{(2\pi T)^{3/2} Q} e^{-\left(|\xi|^2/2+E^i\right)/T},$$

where

$$Q = \sum_{i=1}^{s} g_i e^{-E^i/T}.$$

5.4. **Bose condensate with excitations.** We assume that we have made the transformation (52) and that the reduction induced by Eq.(60) is made. The linearized collision operator L has the null-space

$$N(L) = \text{span}\,(\phi_1, \phi_2)\,,$$

where

$$\begin{aligned}
\phi_1 &= R^{1/2}\left(p^1 + p_0^1\right) = R^{1/2} \cdot (p_1^1 + p_0^1, ..., p_N^1 + p_0^1, -p_1^1 + p_0^1, ..., -p_N^1 + p_0^1) \\
\phi_2 &= R^{1/2}\left(|\mathbf{p}|^2 + \mathbf{n}_0\right) \\
&= R^{1/2} \cdot (|\mathbf{p}_1|^2 + \mathbf{n}_0, ..., |\mathbf{p}_N|^2 + \mathbf{n}_0, |\mathbf{p}_1|^2 + \mathbf{n}_0, ..., |\mathbf{p}_N|^2 + \mathbf{n}_0),
\end{aligned}$$

with $R = P(1 + P)$. See also [10], where we, without stating it, for simplicity assumed the approximation $\mathbf{n}_0 = \mathbf{n} - |\mathbf{p}_0|^2 = 0$.

The degenerate values of p_0^1, i.e. the values of p_0^1 for which $l \geq 1$, are

$$p_{0\pm}^1 = \pm\sqrt{\frac{3\chi_2\chi_5 - 2\chi_3\chi_4 + \sqrt{(3\chi_2\chi_5 - 2\chi_3\chi_4)^2 + 4\left(\chi_1\chi_5 - \chi_3^2\right)\chi_4^2}}{2\left(\chi_1\chi_5 - \chi_3^2\right)}},$$

where

$$\begin{aligned}
\chi_1 &= \left\langle R^{1/2}, R^{1/2} \right\rangle, \chi_2 = \left\langle R^{1/2}p^1, R^{1/2}p^1 \right\rangle = \left\langle R^{1/2}, R^{1/2}p^1 \right\rangle_B, \\
\chi_3 &= \left\langle R^{1/2}, R^{1/2}\left(|\mathbf{p}|^2 + \mathbf{n}_0\right) \right\rangle, \chi_4 = \left\langle R^{1/2}p^1, R^{1/2}\left(|\mathbf{p}|^2 + \mathbf{n}_0\right) \right\rangle_B, \text{ and} \\
\chi_5 &= \left\langle R^{1/2}\left(|\mathbf{p}|^2 + \mathbf{n}_0\right), R^{1/2}\left(|\mathbf{p}|^2 + \mathbf{n}_0\right) \right\rangle.
\end{aligned}$$

We obtain the following table for the values of k^+, k^- and l

	$p_0^1 < p_{0-}^1$	$p_0^1 = p_{0-}^1$	$p_{0-}^1 < p_0^1 < p_{0+}^1$	$p_0^1 = p_{0+}^1$	$p_{0+}^1 < p_0^1$
k^+	0	0	1	1	2
k^-	2	1	1	0	0
l	0	1	0	1	0

.

If we assume that the reduction induced by Eq.(60) isn't made and consider symmetric sets $\mathcal{P}$, such that

$$\text{if } \mathbf{p}_i = (p_i^1, p_i^2, ..., p_i^d) \in \mathcal{P}, \text{ then also } (\pm p_i^1, \pm p_i^2, ..., \pm p_i^d) \in \mathcal{P},$$

then 0 is added to the degenerate values and the values of k^+, k^- and l are

		$p_0^1 = -\widetilde{p}_{0+}^1$		$p_0^1 = 0$		$p_0^1 = \widetilde{p}_{0+}^1$	
k^+	0	0	1	1	d	d	$d+1$
k^-	$d+1$	d	d	1	1	0	0
l	0	1	0	$d-1$	0	1	0

,

for some number $\widetilde{p}_{0+}^1$, where $\widetilde{p}_{0+}^1 = p_{0+}^1$ if $p_{0+}^2 = ... = p_{0+}^d = 0$.

Assuming that

$$P=\frac{1}{e^{\frac{|\mathbf{p}|^2}{2T}}-1},$$

in the continuous case, we have

$$R=P(1+P)=\frac{e^{\frac{|\mathbf{p}|^2}{2T}}}{\left(e^{\frac{|\mathbf{p}|^2}{2T}}-1\right)^2},$$

and, hence [10] (if we assume the approximation $\mathbf{n}_0=\mathbf{n}-\mathbf{n}_0=0$ for simplicity; motivated by that we are "close to diffusive thermal equilibrium" in [2])

$$\begin{aligned}\chi_0 &= 8\pi T\,I_2,\ \chi_1=\frac{16\pi}{3}T^2I_4,\ \chi_2=\frac{32\pi}{3}T^3I_6,\ \chi_3=32\pi T^3I_6,\text{ and}\\ \chi_4 &= 16\pi T^2I_4,\text{ with } I_n=\int_\lambda^\infty r^n\frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\,dr.\end{aligned}$$

Here we have considered the restriction $|\mathbf{p}|\geq\lambda\sqrt{2T}$, for some $\lambda>0$ (cf. [2, 10]).

The degenerate values are (assuming that $p_{0+}^2=p_{0+}^3=0$)

$$p_{0\pm}^1=\pm\sqrt{\frac{T}{3}}\sqrt{\frac{I_4I_6+I_6\sqrt{4I_2I_6-3I_4^2}}{I_2I_6-I_4^2}},\text{ with } I_n=\int_\lambda^\infty r^n\frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\,dr.$$

Here $I_2I_6\geq I_4^2$ by the Cauchy-Schwarz inequality. Furthermore, the values of k^+, k^- and l are given by the table

		$p_0^1=-p_{0+}^1$		$p_0^1=0$		$p_0^1=p_{0+}^1$	
k^+	0	0	1	1	3	3	4
k^-	4	3	3	1	1	0	0
l	0	1	0	2	0	1	0

.

Note that for $\lambda=0$

$$I_4=\frac{3}{8}\sqrt{\pi}\zeta\left(\frac{3}{2}\right)\text{ and } I_6=\frac{15}{16}\sqrt{\pi}\zeta\left(\frac{5}{2}\right),$$

where $\zeta=\zeta(z)$ is the zeta-function, while

$$I_2\to\infty\text{ as }\lambda\to0,$$

and, hence

$$p_{0\pm}^1\to0\text{ as }\lambda\to0.$$

6. **Proof of Theorem 4.2.** We add (cf. [7] and [44]) a damping term $-\gamma BP_0^+f$ to the right-hand side of the system (55) and obtain the damped system

$$\left\{\begin{array}{l}B\frac{df}{dx}+Lf=S(f)-\gamma BP_0^+f\\ f^+(0)=Cf^-(0)+h_0\\ f(x)\to0\text{ as }x\to\infty\end{array}\right.,\tag{66}$$

where $\gamma>0$ and, with the notations in Eqs.(6)-(8),

$$P_0^+f=\sum_{i=1}^{k^+}\frac{\langle f(x),y_i\rangle_B}{\langle y_i,y_i\rangle_B}y_i+\sum_{j=1}^{l}\langle f(x),z_j\rangle_B\,(w_j-xz_j).\tag{67}$$

The projection (67) coincides with the ones in [7] and [44] in the non-degenerate cases as $l=0$. However, in the degenerate cases the projection (67) is different from the ones in [7] and [44].

We follow the ideas in [7]. First we consider the corresponding linearized inhomogeneous system

$$\left\{ \begin{array}{l} B\frac{df}{dx}+Lf=g-\gamma BP_{0}^{+}f \\ f^{+}(0)=Cf^{-}(0)+h_{0} \\ f(x)\rightarrow 0 \text{ as } x\rightarrow \infty \end{array} \right. , \tag{68}$$

where $g=g(x):\mathbb{R}_{+}\rightarrow \mathbb{R}^{N}$ is a given function such that

$$g\left(x\right)\in N(L)^{\perp} \text{ for all } x\in \mathbb{R}_{+}. \tag{69}$$

The system (68) has (under the assumption that all necessary integrals exist) the solution, using the notations in Eqs.(6)-(8),

$$f(x)=\sum_{i=1}^{k^{+}}\mu_{i}\left(x\right)y_{i}+\sum_{j=1}^{l}\alpha_{j}\left(x\right)\left(w_{j}-xz_{j}\right)+\sum_{r=1}^{q}\beta_{r}\left(x\right)u_{r}, \tag{70}$$

where

$$\begin{array}{l} \mu_{i}\left(x\right)=\mu_{i}\left(0\right)e^{-\gamma x},\; i=1,...,k^{+}, \\ \alpha_{j}\left(x\right)=\alpha_{j}\left(0\right)e^{-\gamma x},\; j=1,...,l, \\ \beta_{r}\left(x\right)=\beta_{r}\left(0\right)e^{-\lambda_{r}x}+\int\limits_{0}^{x}e^{(\tau-x)\lambda_{r}}\widetilde{\beta}_{r}\left(\tau\right)\,d\tau,\; r=1,...,m^{+}, \\ \beta_{r}\left(x\right)=-\int\limits_{x}^{\infty}e^{(\tau-x)\lambda_{r}}\widetilde{\beta}_{r}\left(\tau\right)\,d\tau,\; r=m^{+}+1,...,q, \end{array} \tag{71}$$

with

$$\widetilde{\beta}_{r}\left(x\right)=\frac{\left\langle g\left(x\right),u_{r}\right\rangle}{\lambda_{r}}. \tag{72}$$

and $\beta_{1}\left(0\right),...,\beta_{m^{+}}\left(0\right),\mu_{1}\left(0\right),...,\mu_{k^{+}}\left(0\right),\alpha_{1}\left(0\right),...,\alpha_{l}\left(0\right)$ are given by the system

$$\begin{aligned} &\sum_{r=1}^{m^{+}}\beta_{r}\left(0\right)\mathcal{C}u_{r}+\sum_{i=1}^{k^{+}}\mu_{i}\left(0\right)\mathcal{C}y_{i}+\sum_{j=1}^{l}\alpha_{j}\left(0\right)\mathcal{C}w_{j} \\ = \;\; & h_{0}+\sum_{r=m^{+}+1}^{q}\int\limits_{0}^{\infty}e^{\tau\lambda_{r}}\widetilde{\beta}_{r}\left(\tau\right)\,d\tau\,\mathcal{C}u_{r}, \text{ with } \mathcal{C}=R_{+}-CR_{-}. \end{aligned} \tag{73}$$

For $h_{0}=0$ in (66), we have the trivial solution $f(x)\equiv 0$. Therefore, we consider only non-zero h_{0}, $h_{0}\neq 0$, below. The system (73) has (under the assumption that all necessary integrals exist) a unique solution if we assume that the condition (56) is fulfilled.

Theorem 6.1. *Assume that conditions (56) and (69) are fulfilled and that all necessary integrals exist. Then the system (68) has a unique solution given by Eqs.(70)-(73).*

We now fix a number σ, such that

$$0<2\sigma\leq \min\left\{\left|\lambda\right|\neq 0;\; \det(\lambda B-L)=0\right\} \text{ and } 2\sigma\leq\gamma$$

and introduce the norm (cf. [7] and [33])

$$\left|h\right|_{\sigma}=\sup_{x\geq 0}(e^{\sigma x}\left|h\left(x\right)\right|),$$

the Banach space

$$\mathcal{X}=\left\{h\in\mathcal{B}^{0}[0,\infty)\mid\left|h\right|_{\sigma}<\infty\right\},$$

and its closed convex subset

$$\mathcal{S}_{R}=\left\{h\in\mathcal{B}^{0}[0,\infty)\mid\left|h\right|_{\sigma}\leq R\left|h_{0}\right|\right\},$$

where R is a, so far, undetermined positive constant.

We assume that condition (56) is fulfilled and introduce the operator $\Theta(f)$ on $\mathcal{X}$, defined by

$$\Theta(f)=\sum_{i=1}^{k^{+}}\mu_{i}\left(f(x)\right)y_{i}+\sum_{j=1}^{l}\alpha_{j}\left(f(x)\right)\left(w_{j}-xz_{j}\right)+\sum_{r=1}^{q}\beta_{r}\left(f(x)\right)u_{r},$$

where

$$\begin{aligned}
&\mu_{i}\left(f(x)\right)=\mu_{i}\left(f(0)\right)e^{-\gamma x},\ i=1,...,k^{+},\\
&\alpha_{j}\left(f(x)\right)=\alpha_{j}\left(f(0)\right)e^{-\gamma x},\ j=1,...,l,\\
&\beta_{r}\left(f(x)\right)=\beta_{r}\left(f(0)\right)e^{-\lambda_{r}x}+\int_{0}^{x}e^{(\tau-x)\lambda_{r}}\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)\,d\tau,\ r=1,...,m^{+},\\
&\beta_{r}\left(f(x)\right)=-\int_{x}^{\infty}e^{(\tau-x)\lambda_{r}}\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)\,d\tau,\ \ r=m^{+}+1,...,q,
\end{aligned}$$

with $\beta_{1}\left(f(0)\right),...,\beta_{m^{+}}\left(f(0)\right),\mu_{1}\left(f(0)\right),...,\mu_{k^{+}}\left(f(0)\right),$ and $\alpha_{1}\left(f(0)\right),...,\alpha_{l}\left(f(0)\right)$ given by the system

$$\begin{aligned}
&\sum_{r=1}^{m^{+}}\beta_{r}\left(f(0)\right)\mathcal{C}u_{r}+\sum_{i=1}^{k^{+}}\mu_{i}\left(f(0)\right)\mathcal{C}y_{i}+\sum_{j=1}^{l}\alpha_{j}\left(f(0)\right)\mathcal{C}w_{j}\\
=\ &h_{0}+\sum_{r=m^{+}+1}^{q}\int_{0}^{\infty}e^{\tau\lambda_{r}}\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)\,d\tau\,\mathcal{C}u_{r},
\end{aligned}$$

where

$$\mathcal{C}=R_{+}-CR_{-}\text{ and }\widetilde{\beta}_{r}\left(f\right)=\frac{\left\langle S\left(f\right),u_{r}\right\rangle}{\lambda_{r}}.$$

Lemma 6.2. *Let $f,h\in\mathcal{X}$ and assume that condition (56) is fulfilled. Then there is a positive constant K (independent of f and h), such that*

$$\left|\Theta(0)\right|_{\sigma}\leq K\left|h_{0}\right|, \tag{74}$$

$$\left|\Theta(f)-\Theta(h)\right|_{\sigma}\leq G(\left|f\right|_{\sigma},\left|h\right|_{\sigma})\left|f-h\right|_{\sigma}, \tag{75}$$

where $G:\mathbb{R}_{+}\times\mathbb{R}_{+}\rightarrow\mathbb{R}_{+}$ is a differentiable function with positive partial derivatives and $G(0,0)=0$.

Proof. By condition (56) the linear map $\mathcal{C}=R_{+}-CR_{-}$ is invertible on $X_{+}=\operatorname{span}\left(u_{1},...,u_{m^{+}},y_{1},...,y_{k^{+}},w_{1},....,w_{l}\right)$. The inverse map $\mathcal{C}^{-1}$ is bounded. We denote by P the matrix with $u_{1},...,u_{q},y_{1},...,y_{k},z_{1},...,z_{l},w_{1},...,w_{l}$ as columns (in that order). Then the inverse matrix of P has the following expression

$$P^{-1}=D^{-1}\widetilde{P}^{t}B,\text{ where }D=\operatorname{diag}(\lambda_{1},...,\lambda_{q},\gamma_{1},...,\gamma_{k},1,...,1),$$

and $\widetilde{P}$ is the matrix with $u_1, ..., u_q, y_1, ..., y_k, w_1, ..., w_l, z_1, ..., z_l,$ as columns (note the interchanged order of the columns). We obtain that

$$\begin{aligned}
|\Theta(0)|_\sigma &= \left|PP^{-1}\Theta(0)\right|_\sigma \leq |P| \left|P^{-1}\Theta(0)\right|_\sigma \\
&= |P| \left| \sum_{r=1}^{m^+} \beta_r(0) e^{-\lambda_r x} P^{-1} u_r \right. \\
&\left. + e^{-\gamma x} P^{-1} \left(\sum_{i=1}^{k^+} \mu_i(0) y_i + \sum_{j=1}^{l} \alpha_j(0)(w_j - x z_j) \right) \right|_\sigma \\
&\leq |P| \left| P^{-1} \left(\sum_{r=1}^{m^+} \beta_r(0) u_r + \sum_{i=1}^{k^+} \mu_i(0) y_i + \sum_{j=1}^{l} \alpha_j(0) w_j \right) \right| \\
&+ |P| \sup_{x \geq 0} (e^{-\frac{\gamma}{2}x} x) \left| P^{-1} \sum_{j=1}^{l} \alpha_j(0) z_j \right| \\
&= |P| \left(\left| P^{-1} \mathcal{C}^{-1} h_0 \right| + \frac{2}{\gamma e} \left| P^{-1} \sum_{j=1}^{l} \alpha_j(0) w_j \right| \right) \leq K_0 |h_0|,
\end{aligned}$$

with $K_0 = |P| \left|P^{-1}\mathcal{C}^{-1}\right| (1 + \dfrac{2}{\gamma e})$.

Here we used that

$$e^{-\frac{\gamma}{2}x} x \leq \frac{2}{\gamma e} \text{ for all } x \in \mathbb{R},$$

with equality if and only if $x = \dfrac{2}{\gamma}$.

Clearly,

$$|f|_\sigma < \infty \Rightarrow |S(f)|_\sigma < \infty,$$

$$\begin{aligned}
& \left| P^{-1} \left(\sum_{r=1}^{m^+} (\beta_r(f(0)) - \beta_r(h(0))) u_r + \sum_{j=1}^{l} (\alpha_j(f(0)) - \alpha_j(h(0))) w_j \right. \right. \\
& \left. \left. + \sum_{i=1}^{k^+} (\mu_i(f(0)) - \mu_i(h(0))) y_i \right) \right| \\
= & \left| P^{-1}\mathcal{C}^{-1}\mathcal{C} \sum_{r=m^++1}^{q} \int_0^\infty e^{\tau\lambda_r} \left(\widetilde{\beta}_r(f(\tau)) - \widetilde{\beta}_r(h(\tau)) \right) d\tau \, u_r \right| \\
\leq & \left| P^{-1}\mathcal{C}^{-1}\mathcal{C}P \right| \left| \int_0^\infty e^{-2\tau\sigma} \sum_{r=m^++1}^{q} \left(\widetilde{\beta}_r(f(\tau)) - \widetilde{\beta}_r(h(\tau)) \right) P^{-1} u_r \, d\tau \right| \\
\leq & \left| P^{-1}\mathcal{C}^{-1}\mathcal{C}P \right| \int_0^\infty e^{-3\tau\sigma} \, d\tau \left| P^{-1} B^{-1} (S(f) - S(h)) \right|_\sigma,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{x\geq 0}(e^{-\frac{\gamma}{2}x}x)\left|P^{-1}\sum_{j=1}^{l}\left(\alpha_j\left(f\left(0\right)\right)-\alpha_j\left(h\left(0\right)\right)\right)z_j\right| \\
= \;&\sup_{x\geq 0}(e^{-\frac{\gamma}{2}x}x)\left|P^{-1}\sum_{j=1}^{l}\left(\alpha_j\left(f\left(0\right)\right)-\alpha_j\left(h\left(0\right)\right)\right)w_j\right| \\
\leq \;&\frac{2}{\gamma e}\left|P^{-1}\left(\sum_{r=1}^{m^+}(\beta_r\left(f(0)\right)-\beta_r\left(h(0)\right))u_r+\sum_{j=1}^{l}(\alpha_j\left(f\left(0\right)\right)-\alpha_j\left(h\left(0\right)\right))w_j\right.\right. \\
&\left.\left.+\sum_{i=1}^{k^+}(\mu_i\left(f\left(0\right)\right)-\mu_i\left(h\left(0\right)\right))y_i)\right)\right|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left|\Theta(f)-\Theta(h)\right|_\sigma \\
&=\left|PP^{-1}\left(\Theta(f)-\Theta(h)\right)\right|_\sigma\leq\left|P\right|\left|P^{-1}\left(\Theta(f)-\Theta(h)\right)\right|_\sigma \\
&\leq\left|P\right|\sup_{x\geq 0}\left(\int_x^\infty e^{(3x-2\tau)\sigma}\left|P^{-1}\sum_{r=m^++1}^{q}(\widetilde{\beta}_r\left(f\left(\tau\right)\right)-\widetilde{\beta}_r\left(h\left(\tau\right)\right))u_r\right|d\tau\right. \\
&+\int_0^x e^{(2\tau-x)\sigma}\left|P^{-1}\sum_{r=1}^{m^+}(\widetilde{\beta}_r\left(f\left(\tau\right)\right)-\widetilde{\beta}_r\left(h\left(\tau\right)\right))u_r\right|d\tau \\
&+(e^{-\frac{\gamma}{2}x}x)\left|P^{-1}\sum_{j=1}^{l}\left(\alpha_j\left(f\left(0\right)\right)-\alpha_j\left(h\left(0\right)\right)\right)z_j\right| \\
&+\left|P^{-1}\left(\sum_{r=1}^{m^+}(\beta_r\left(f(0)\right)-\beta_r\left(h(0)\right))u_r+\sum_{j=1}^{l}(\alpha_j\left(f\left(0\right)\right)-\alpha_j\left(h\left(0\right)\right))w_j\right.\right. \\
&\left.\left.\left.+\sum_{i=1}^{k^+}(\mu_i\left(f\left(0\right)\right)-\mu_i\left(h\left(0\right)\right))y_i)\right)\right|\right) \\
&\leq\left|P\right|(\sup_{x\geq 0}\left(\int_x^\infty e^{3(x-\tau)\sigma}\,d\tau+\int_0^\infty e^{(\tau-x)\sigma}\,d\tau\right)+(1+\frac{2}{\gamma e})\left|P^{-1}\mathcal{C}^{-1}\mathcal{C}P\right|\int_0^\infty e^{-3\tau\sigma}\,d\tau) \\
&*\left|P^{-1}B^{-1}(S(f)-S(h))\right|_\sigma \\
&\leq K_1\left|S(f)-S(h)\right|_\sigma,\text{ with }K_1=\frac{1}{3\sigma}\left|P\right|\left|D^{-1}\widetilde{P}^t\right|\left(4+(1+\frac{2}{\gamma e})\left|P^{-1}\mathcal{C}^{-1}\mathcal{C}P\right|\right).
\end{aligned}$$

By the assumption

$$\left|S(f)-S(h)\right|\leq K_2G(\left|f\right|,\left|h\right|)\left|f-h\right|$$

Therefore,

$$\left|S(f)-S(h)\right|_\sigma\leq K_2G(\left|f\right|,\left|h\right|)\left|f-h\right|_\sigma\leq K_2G(\left|f\right|_\sigma,\left|h\right|_\sigma)\left|f-h\right|_\sigma.$$

Let $K=\max(K_0,K_1K_2)$. □

Theorem 6.3. *Let condition (56) be fulfilled. Then there is a positive number δ_0, such that if*

$$|h_0| \leq \delta_0,$$

then the system (66) has a unique solution $f = f(x)$ in $\mathcal{S}_R$ for a suitable chosen R.

Proof. By estimates (74) and (75), there is a positive number K such that

$$|\Theta(f)|_\sigma = |\Theta(f) - \Theta(0) + \Theta(0)|_\sigma \leq K(|h_0| + G(|f|_\sigma, 0)\,|f|_\sigma) \tag{76}$$

if $f \in \mathcal{X}$.

Let $R = K + 1$ and let δ_0 be a positive number, such that $G(R\delta_0, R\delta_0) \leq \dfrac{1}{R}$. By estimates (75) and (76)

$$|\Theta(f)|_\sigma \leq (\frac{K}{R} + G(R\,|h_0|\,, R\,|h_0|))R\,|h_0| \leq R\,|h_0|$$

and

$$|\Theta(f) - \Theta(h)|_\sigma \leq KG(R\,|h_0|\,, R\,|h_0|)\,|f-h|_\sigma \leq \frac{K}{K+1}\,|f-h|_\sigma\,,$$

if $f, h \in \mathcal{S}_R$ and $|h_0| \leq \delta_0$. The theorem follows by the contraction mapping theorem. □

Theorem 6.4. *The solution $f = f(x)$ of Theorem 6.3 is a solution of the system (55) if and only if $P_0^+ f(0) = 0$.*

Proof. The solution $f = f(x)$ of Theorem 6.3 is a solution of the system (55) if and only if $P_0^+ f(x) = 0$. The theorem follows by the relations

$$\begin{aligned} \mu_i(f(x)) &= \mu_i(f(0))e^{-\gamma x},\ i = 1, ..., k^+, \\ \alpha_j(f(x)) &= \alpha_j(f(0))e^{-\gamma x},\ j = 1, ..., l, \end{aligned}$$

that are fulfilled for any solution $f = f(x)$ of Theorem 6.3. □

We denote by $\mathbb{I}^\gamma$ the linear solution operator

$$\mathbb{I}^\gamma(h_0) = f(0),$$

where $f(x)$ is given by

$$\left\{ \begin{array}{l} B\dfrac{df}{dx} + Lf + \gamma BP_0^+ f = 0 \\ \mathcal{C}f(0) = h_0 \\ f \to 0 \text{ as } x \to \infty \end{array} \right. .$$

Similarly, we denote by $\mathcal{I}^\gamma$ the nonlinear solution operator

$$\mathcal{I}^\gamma(h_0) = f(0),$$

where $f(x)$ is given by

$$\left\{ \begin{array}{l} B\dfrac{df}{dx} + Lf = S(f, f) - \gamma BP_0^+ f \\ \mathcal{C}f(0) = h_0 \\ f \to 0 \text{ as } x \to \infty \end{array} \right. .$$

By Theorem 6.4, the solution of Theorem 6.3 is a solution of the problem (55) if and only if $P_0^+ \mathcal{I}^\gamma(h_0) \equiv 0$.

We now proceed with an orthonormalization process. Let

$$r_i = \frac{r_i'}{\sqrt{\langle r_i', r_i' \rangle_{B_+}}}, \text{ with}$$

$$r_i' = \mathcal{C}y_i - \sum_{r=1}^{m^+} \frac{\langle \mathcal{C}y_i, \mathcal{C}u_r \rangle_{B_+}}{\langle \mathcal{C}u_r, \mathcal{C}u_r \rangle_{B_+}} \mathcal{C}u_r - \sum_{j=1}^{i-1} \langle \mathcal{C}y_i, r_j \rangle_{B_+} r_j \neq 0, \, i = 1, ..., k^+,$$

and

$$r_{k^++i} = \frac{r_{k^++i}'}{\sqrt{\left\langle r_{k^++i}', r_{k^++i}' \right\rangle_{B_+}}}, \text{ with}$$

$$r_{k^++i}' = \mathcal{C}w_i - \sum_{r=1}^{m^+} \frac{\langle \mathcal{C}w_i, \mathcal{C}u_r \rangle_{B_+}}{\langle \mathcal{C}u_r, \mathcal{C}u_r \rangle_{B_+}} \mathcal{C}u_r - \sum_{j=1}^{k^++i-1} \langle \mathcal{C}w_i, r_j \rangle_{B_+} r_j \neq 0, \, i = 1, ..., l.$$

Then

$$P_0^+ \mathbb{I}^\gamma \equiv 0 \Leftrightarrow h_0 \in \mathcal{R}^{\perp_{B_+}}, \text{ where}$$
$$\mathcal{R}^{\perp_{B_+}} = \left\{ u \in \mathbb{R}^{n^+} \;\middle|\; \langle u, r_i \rangle_{B_+} = 0 \text{ for } i = 1, ..., k^+ + l \right\}$$

and

$$\begin{aligned} \mathcal{I}^\gamma(h_0) &\equiv \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1), \; h_0 = \sum_{i=1}^{k^++l} a_i r_i + h_1, \text{ with} \\ h_1 &\in \mathcal{R}^{\perp_{B_+}} \text{ and } a_i = \langle h_0, r_i \rangle_{B_+}. \end{aligned}$$

Lemma 6.5. *Suppose that $P_0^+ \mathcal{I}^\gamma(h_0) \equiv 0$. Then h_0 is a function of h_1 if $\langle h_0, h_0 \rangle_{B_+}$ is sufficiently small.*

Proof. It is obvious that $\mathcal{I}^\gamma(0) = 0$ and that we for the Fréchet derivative of $\mathcal{I}^\gamma(\epsilon h_0)$ have

$$\left. \frac{d}{d\epsilon} \mathcal{I}^\gamma(\epsilon h_0) \right|_{\epsilon=0} = \mathbb{I}^\gamma(h_0).$$

Then

$$\left. \frac{\partial}{\partial a_i} \left\langle \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1), u \right\rangle_B \right|_{(0,...,0)} = \left. \frac{d}{d\epsilon} \langle \mathcal{I}^\gamma(\epsilon r_i), u \rangle_B \right|_{\epsilon=0} = \langle \mathbb{I}^\gamma(r_i), u \rangle_B \neq 0,$$

where $u = y_i$ if $i = 1, ..., k^+$ and $u = z_{i-k^+}$ if $i = k^+ + 1, ..., k^+ + l$. By the implicit function theorem, $\left\langle \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1), y_1 \right\rangle_B = 0$ defines $a_1 = a_1(a_2, ..., a_{k^++l}, h_1)$. By induction

$$a_1 = a_1(h_1), ..., a_{k^++l} = a_{k^++l}(h_1).$$

□

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Received xxxx 20xx; revised xxxx 20xx.

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