Stochastic homogenization of multicontinuum models

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Outline

- Introduction to the models
- Literature to averaging
- Introduction to homogenization
- Main results
- Some proofs
- Some remarks

General model

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \nabla \cdot [A(\frac{x}{\varepsilon}, v^\varepsilon)\nabla u^\varepsilon(t,x)] + \text{additional terms}$$
$$dv^\varepsilon(t,x) = -\frac{1}{\varepsilon}(v^\varepsilon(t,x) - u^\varepsilon(t,x))dt + \sqrt{\frac{Q}{\varepsilon}}dW(t,x), \quad (1)$$

$A(\cdot,\cdot)$ is the permeability of the medium, Q is a bounded linear operator of trace class defined on $L^2(D)$ and W a standard $L^2(D)$-valued Brownian motion.

Scales

- Here we have time scale and spatial scale: $(t, \frac{t}{\varepsilon})$ and $(x, \frac{x}{\varepsilon})$.
- v^ε is the fast variable while u^ε is the slow variable.
- Then the goal is to pass to the limit in ε.
- To find the homogenized (effective) equation.

Diffusion Reaction model

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(t,x)\right) + \alpha\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right)u^\varepsilon(t,x) + f(t,x)$$

$$dv^\varepsilon(t,x) = -\frac{1}{\varepsilon}(v^\varepsilon(t,x) - u^\varepsilon(t,x))dt + \sqrt{\frac{Q}{\varepsilon}}dW(t,x),$$

Diffusion Convection model

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(t,x)\right) + \alpha\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right)\nabla u^\varepsilon(t,x) + f(t,x)$$

$$dv^\varepsilon(t,x) = -\frac{1}{\varepsilon}(v^\varepsilon(t,x) - u^\varepsilon(t,x))dt + \sqrt{\frac{Q}{\varepsilon}}dW(t,x),$$

Multicontinuum model

$$\frac{\partial u_1^\varepsilon}{\partial t} = \mathrm{div}\left(A_1^\epsilon \nabla u_1^\varepsilon\right) + \alpha^\epsilon\left(v_1^\varepsilon, v_2^\varepsilon\right)\left(u_2^\varepsilon - u_1^\varepsilon\right) + f_1,$$

$$\frac{\partial u_2^\varepsilon}{\partial t} = \mathrm{div}\left(A_2^\epsilon \nabla u_2^\varepsilon\right) + \alpha^\epsilon\left(v_1^\varepsilon, v_2^\varepsilon\right)\left(u_1^\varepsilon - u_2^\varepsilon\right) + f_2,$$

$$dv_1^\varepsilon = -\frac{1}{\varepsilon}(v_1^\varepsilon - g_1(u_1^\varepsilon, u_2^\varepsilon)dt + \sqrt{\frac{Q_1}{\varepsilon}}dW_1(t,x),$$

$$dv_2^\varepsilon = -\frac{1}{\varepsilon}(v_2^\varepsilon - g_2(u_1^\varepsilon, u_2^\varepsilon)dt + \sqrt{\frac{Q_2}{\varepsilon}}dW_2(t,x)$$

$$+ \text{Boundary Conditions, Initial Conditions,}$$

where $g_1,\ g_2$ are Lipschitz.

Literature on averaging

1. For stochastic differential equations:
   - R.Z. Khasminskii, *on the principle of averaging the Itô stochastic differential equation*, Kybernetika, (1968).
   - A.Yu. Veretennikov, *On the averaging principle for systems of stochastic differential equations*, Mat. USSR Sb., (1991).
   - M. Freidlin, A. Wentzell, *Averaging principle for stochastic perturbations of multifrequency systems*, Stochastics and Dynamics, (2003).
2. For stochastic partial differential equations:
   - S. Cerrai, *A Khasminskii type averaging principle for stochastic reaction-diffusion equations*, Ann. Appl. Probab., (2009)
   - S. Cerrai, M. Freidlin, *Averaging principle for a class of stochastic reaction-diffusion equations*, Probab. Theory Related Fields, (2009).

Introduction to the homogenization

Assume to have a sequence of partial differential operators L_ε (with oscillating coefficients) and a sequence of solutions u_ε which for a given domain D and source f

$$L_\varepsilon u_\varepsilon = f \quad in\ D \tag{2}$$

complemented by appropriate boundary conditions. If we assume that u_ε converges in some sense to some u, we look for a so-called homogenized operator $\overline{L}$ such that

$$\overline{L}u = f \quad in\ D \tag{3}$$

Passing from (2) to (3) is the homogenization process.

Typically

$$L_\varepsilon u^\varepsilon = -\nabla \cdot (a(x, \frac{x}{\varepsilon})\nabla u^\varepsilon).$$

Formally, in order to find the form of $\overline{L}$, one writes the expansion

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots \tag{4}$$

where each $u_i(x, y)$ is periodic in y. Inserting (4) into (2) leads to a cascade of equations for u_i and averaging wrt to y the equation for u_0 gives (3).

Typically

$$\overline{L}u_0(x) = -\nabla \cdot (\bar{a}(x)\nabla u_0(x)),$$

where

$$\bar{a}(x) = \int_Y \langle a(x,y)(I+\nabla_y N), (I+\nabla_y N)\rangle dy,$$

such that N is solution of the cell problem:

$$-\operatorname{div}(a(I+\nabla N)) = 0 \quad inY$$

and $y \to N(x,y)$- is Y periodic.
Now, other arguments are needed to prove the convergence of the sequence u_ε to u_0.

Homogenization methods

1. The energy method
2. The two-scale convergence

Literature on homogenization

1. Homogenization of the Stokes problem in perforated domains
   - Sánchez-Palencia (1980) (asymptotic expansion method)
   - L. Tartar (the energy method)
   - G. Allaire (1992) (two scale convergence method)
2. Homogenization of PDEs with random coefficients or stochastic forcing in non perforated domains
   - Bourgeat, A. Mikelić and Wright in (1994)
   - P. A. Razafimandimby, M. Sango, and J. L. Woukeng (2012)
3. Homogenization of SPDEs in perforated domains (at its infancy)
   - W. Wang and J. Duan (2007)
   - H. Bessaih, Y. Efendiev and F. Maris (2015, 2016)

Assumptions

The α satisfies the following conditions:

$$\alpha(\cdot, x) \in L^2_{per}(Y), \tag{5}$$

for every $x \in \mathbb{R}^3$.

$$\alpha(y, \cdot) \in Lip_b(\mathbb{R}), \tag{6}$$

for every $y \in Y$, with

$$\|\alpha(y, \cdot)\|_{Lip_b(\mathbb{R})} \leq C, \tag{7}$$

The matrix $A = (a_{ij})_{1 \leq i,j \leq 3} \in L^\infty(Y; \mathbb{R}^{3\times 3})$ is strictly positive and bounded uniformly in $y \in Y$, i. e. there exist $0 < m < M$ such that

$$m\xi^2 \leq A(y)\xi\xi \leq M\xi^2, \tag{8}$$

for almost every $y \in Y$ and $\xi \in \mathbb{R}^3$.

For any $\varepsilon > 0$, we denote by

$$A^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \quad A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \tag{9}$$

and

$$\alpha^\varepsilon : L^2(D) \to L^\infty(D), \quad \alpha^\varepsilon(\eta)(x) = \alpha\left(\frac{x}{\varepsilon}, \eta(x)\right). \tag{10}$$

We assume that W is Bm on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Q is a bounded operator on $L^2(D)$ of trace class, and that $f \in L^2(0, T; L^2(D))$ and $u_0^\varepsilon, v_0^\varepsilon \in L^2(D)$

Compare with the existing literature

E. Pardoux, A. L. Piatniski (2003), *Homogenization of a nonlinear random parabolic partial differential equation*

$$\frac{\partial u^\varepsilon}{\partial t} = \nabla \cdot (A(\frac{x}{\varepsilon}, v(\frac{t}{\varepsilon^2}))\nabla u^\varepsilon) + \frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, v(\frac{t}{\varepsilon^2}), u^\varepsilon)$$

$$dv = b(v)dt + \sigma(v)dW,$$

In Cerrai-Friedlin (2009), they consider

$$du = [A_1 u + B_1(u, v)]dt + G_1(u, v)dW$$
$$dv = \frac{1}{\varepsilon}[A_2 v + B_2(u, v)]dt + \frac{1}{\sqrt{\varepsilon}}G_2(u, v)dW.$$

Where B_1 and B_2 are Lipschitz-Continuous.
In particular, our term $\alpha(\cdot, v^\varepsilon)u^\varepsilon$ or $\alpha(\cdot, v^\varepsilon)\nabla u^\varepsilon$ do not satisfy these assumptions.

If $u_0^\varepsilon \in L^2(D)$ and $v_0^\varepsilon \in L^2(\Omega; L^2(D))$ then there exists a unique global solution $u^\varepsilon \in L^\infty(\Omega; C([0,T]; L^2(D) \cap L^2(0,T; H_0^1(D))))$ and $v^\varepsilon \in L^2(\Omega; C([0,T]; L^2(D))$: $\mathbb{P}$ a.s.

$$\int_D u^\varepsilon(t)\phi - \int_D u_0^\varepsilon \phi + \int_0^t \int_D A^\varepsilon \nabla u^\varepsilon(s) \nabla \phi + \int_0^t \int_D \alpha^\varepsilon(v^\varepsilon) u^\varepsilon \phi = \int_0^t \int_D f(s)\phi,$$

for every $t \in [0,T]$ and every $\phi \in H_0^1(D)$, and

$$v^\varepsilon(t) = v_0^\varepsilon e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t u^\varepsilon(s) e^{-(t-s)/\varepsilon} ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{-(t-s)/\varepsilon} dW(s).$$

Uniform estimates

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega;L^2(0,T;H_0^1(D)))} \leq C_T, \tag{11}$$

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega;C([0,T];L^2(D)))} \leq C_T, \tag{12}$$

and

$$\sup_{\varepsilon>0} \left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^\infty(\Omega;L^2(0,T;H^{-1}(D)))} \leq C_T. \tag{13}$$

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \|v^\varepsilon(t)\|^2_{L^2(D)} \leq C_T. \tag{14}$$

The fast motion equation

For fixed $\xi \in L^2(D)$:

$$\begin{cases} dv^\xi & = -(v^\xi - \xi)dt + \sqrt{Q}dW, \\ v(0) & = \eta. \end{cases} \tag{15}$$

This equation admits a unique mild solution $v^\xi(t) \in L^2(\Omega; C(0, T; L^2(D)))$ given by:

$$v^\xi(t) = \eta e^{-t} + \xi(1 - e^{-t}) + \int_0^t e^{-(t-s)}\sqrt{Q}dW. \tag{16}$$

Let us define the transition semigroup P_t^ξ associated to the equation (15)

$$P_t^\xi \Phi(\eta) = \mathbb{E}\Phi(v^{\xi,\eta}(t)), \tag{17}$$

for every $\Phi \in B_b(L^2(D))$ and every $\eta \in L^2(D)$.
Let μ^ξ be the associated invariant measure on $L^2(D)$. We recall that it is invariant for the semigroup P_t^ξ if

$$\int_{L^2(D)} P_t^\xi \Phi(z) d\mu^\xi(z) = \int_{L^2(D)} \Phi(z) d\mu^\xi(z),$$

for every $\Phi \in B_b(L^2(D))$.

The equation (15) admits a unique ergodic invariant measure μ^{ξ} that is strongly mixing and gaussian with mean ξ and operator Q. We also have:

$$\left| P_t^{\xi}\Phi(\eta) - \int_{L^2(D)} \Phi(z)d\mu^{\xi}(z) \right| \leq c[\Phi]e^{-t}(1 + \|\eta\|_{L^2(D)} + \|\xi\|_{L^2(D)}),$$

for any Lipschitz function Φ defined on $L^2(D)$, where $[\Phi]$ is the Lipschitz constant of Φ.

We need more refined results for the fast motion equation.

So for any $\xi, \eta \in L^2(\Omega, \mathcal{F}_{t_0}, L^2(D))$, and a.e. $\omega \in \Omega$ we have:

$$\mathbb{E}\left(\|v^{\xi,\eta}(t)\|^2_{L^2(D)} | \mathcal{F}_{t_0}\right) \leq 2\left(\|\eta\|^2_{L^2(D)} e^{-2(t-t_0)} + \|\xi\|^2_{L^2(D)} + TrQ\right),$$

and

$$\mathbb{E}\left(\left|P_t^{\xi(\omega)}\Phi(\eta(\omega)) - \int_{L^2(D)} \Phi(z) d\mu^{\xi(\omega)}(z)\right| \middle| \mathcal{F}_{t_0}\right) \leq c[\Phi] e^{-(t-t_0)}(1 + \|\eta(\omega)\|_{L^2(D)} + \|\xi(\omega)\|_{L^2(D)}),$$

Lemma

Let $\Phi \in C^u([0,T]; L^\infty(\Omega; Lip(L^2(D))))$ be an $\mathcal{F}_t$ - measurable process on $Lip(L^2(D))$, and let $0 \leq t_0 < t_0 + \delta \leq T$. For $\xi, \eta \in L^2(\Omega, \mathcal{F}_{t_0}, L^2(D))$, let $v^{\xi,\eta}$ be the previous solution. We have:

$$\mathbb{E}\left(\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\left(\Phi(s, v^{\xi,\eta}(s)) - \int_{L^2(D)} \Phi(s,z)d\mu^\xi(z)\right) ds\right| \Big| \mathcal{F}_{t_0}\right) \leq c\left(1 + \|\eta\|_{L^2(D)} + \|\xi\|_{L^2(D)}\right)\left(\frac{\|\Phi\|}{\sqrt{\delta}} + \sqrt{\|\Phi\|[\Phi](\delta)}\right), \tag{18}$$

where $[\Phi]$ is the modulus of uniform continuity of Φ.

This lemma is crucial because, we need to apply the semigroup P_t^{ξ} to a function of the form

$$\Phi^{\varepsilon}(s,\eta) = \int_D \alpha^{\varepsilon}(\eta) u^{\varepsilon}(\varepsilon s)\, dx$$

The cell problem

We introduce $\chi : Y \to \mathbb{R}$ the solution of the cell problem

$$\begin{cases} \operatorname{div}\left(A(y)\left(I+\nabla\chi(y)\right)\right) & = 0 & & \text{in } Y, \\ \chi & -Y & \text{periodic,} & \end{cases}$$

We introduce the following averaged operators:

$$\overline{\alpha^\varepsilon} : L^2(D) \to L^\infty(D), \quad \overline{\alpha^\varepsilon}(\xi) = \int_{L^2(D)} \alpha^\varepsilon(\eta) d\mu^\xi(\eta)$$

$$\overline{\alpha}(\xi) = \int_{L^2(D)} \left(\int_Y \alpha(y,z) dy \right) d\mu^\xi(z).$$

$$\overline{A} = \int_Y A(y) \left(I + \nabla \chi(y) \right) dy.$$

Our main result of the diffusion reaction equation

Theorem (Bessaih-Efendiev-Maris, 2019)

Assume the sequence u_0^ε is uniformly bounded in $H_0^1(D))$ and strongly convergent in $L^2(D)$ to some function u_0, and v_0^ε is uniformly bounded in $L^2(\Omega; L^2(D))$. Then, there exists $\overline{u} \in L^2(0, T; H_0^1(D)))$ such that u^ε converges in probability to $\overline{u}$ in $L^2(0, T; H_0^1(D)))$ and $\overline{u}$ is the solution of the following deterministic equation:

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} &= \operatorname{div}\left(\overline{A}\nabla\overline{u}\right) + \overline{\alpha}(\overline{u})\overline{u} + f \quad \text{in } D, \\ \overline{u} &= 0 \quad \text{on } \partial D, \\ \overline{u}(0) &= u_0 \quad \text{in } D. \end{cases} \tag{19}$$

Our main result of the convection diffusion equation

Theorem (Bessaih-Efendiev-Maris, 2020)

Assume the sequence u_0^ε *is uniformly bounded in* $H_0^1(D))$ *and strongly convergent in* $L^2(D)$ *to some function* u_0*, and* v_0^ε *is uniformly bounded in* $L^2(\Omega; L^2(D))$*. Then, there exists* $\overline{u} \in L^2(0,T; H_0^1(D)))$ *such that* u^ε *converges in probability to* $\overline{u}$ *in* $L^2(0,T; H_0^s(D)))$ *where* $0 \leq s < 1$ *and* $\overline{u}$ *is the solution of the following deterministic equation:*

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} = \operatorname{div}\left(\overline{A}\nabla\overline{u}\right) + \overline{\alpha_K}(\overline{u})\nabla\overline{u} + f & \text{in } D, \\ \overline{u} = 0 & \text{on } \partial D, \\ \overline{u}(0) = u_0 & \text{in } D. \end{cases} \tag{20}$$

$$\alpha_K(y,\eta) = \alpha(y,\eta)\left(I + \nabla\chi(y)\right)$$

Our main result of the multicontinuum equation

Theorem (Bessaih-Efendiev-Maris, 2020)

Assume similar properties for initial conditions. Then, there exist $\overline{u}_1, \overline{u}_2 \in L^2(0, T; H_0^1(D)) \cap C([0, T]; L^2(D))$ *such that* $u_1^\varepsilon, u_2^\varepsilon$ *converge in probability to* $\overline{u}_1, \overline{u}_2$*:*

$$\begin{cases} \dfrac{\partial \overline{u}_1}{\partial t} = \text{div}\left(\overline{A}_1 \nabla \overline{u}_1\right) + \overline{\alpha}(g_1(\overline{u}_1, \overline{u}_2), g_2(\overline{u}_1, \overline{u}_2))(\overline{u}_2 - \overline{u}_1) + f_1 \text{ in } D, \\ \dfrac{\partial \overline{u}_2}{\partial t} = \text{div}\left(\overline{A}_2 \nabla \overline{u}_2\right) + \overline{\alpha}(g_1(\overline{u}_1, \overline{u}_2), g_2(\overline{u}_1, \overline{u}_2))(\overline{u}_1 - \overline{u}_2) + f_2 \text{ in } D, \\ + \quad \text{initial conditions, boundary conditions,} \end{cases} \tag{21}$$

We need to pass to the limit in ε on the variational formulation

$$\int_D u^\varepsilon(t)\phi - \int_D u_0^\varepsilon \phi + \int_0^t \int_D A^\epsilon \nabla u^\varepsilon(s) \nabla \phi + \int_0^t \int_D \alpha^\varepsilon(v^\varepsilon) u^\varepsilon \phi = \int_0^t \int_D f(s)\phi,$$

Here, we use tightness arguments and pass to the limit in distribution only. After changing the space of probability, the sequence u^ε given by Skorokhod theorem converges a.s. to $\overline{u}$ strongly in $L^2(0, T; H_0^1(D))$

$$\int_0^t \int_D \alpha^\varepsilon(v^\varepsilon) u^\varepsilon \phi = S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon$$

$$S_1^\varepsilon = \int_0^T \int_D (\alpha^\varepsilon(v^\varepsilon(t)) - \overline{\alpha^\varepsilon}(u^\varepsilon(t))) u^\varepsilon(t) \phi dxdt,$$

$$S_2^\varepsilon = \int_0^T \int_D (\overline{\alpha^\varepsilon}(u^\varepsilon(t)) u^\varepsilon(t) - \overline{\alpha^\varepsilon}(\overline{u}(t))\overline{u}(t)) \phi dxdt,$$

$$S_3^\varepsilon = \int_0^T \int_D (\overline{\alpha^\varepsilon}(\overline{u}(t))\overline{u}(t) - \overline{\alpha}(\overline{u}(t))\overline{u}(t)) \phi dxdt.$$

Fix n^ε a positive integer and let $\delta^\varepsilon = \dfrac{T}{n^\varepsilon}$. We define $\widetilde{u}^\varepsilon$ as the piecewise constant function:

$$\widetilde{u}^\varepsilon(t) = u^\varepsilon(k\delta^\varepsilon) \ \text{ for } t \in [k\delta^\varepsilon, (k+1)\delta^\varepsilon). \tag{22}$$

We define also the sequence $\widetilde{v}^\varepsilon$ as the solution of:

$$\begin{cases} d\widetilde{v}^\varepsilon(t,x) & = -\dfrac{1}{\varepsilon}(\widetilde{v}^\varepsilon(t,x) - \widetilde{u}^\varepsilon(t,x))dt + \sqrt{\dfrac{Q}{\varepsilon}}dW(t,x), \\ \widetilde{v}^\varepsilon(0,x) & = v_0^\varepsilon(x). \end{cases}$$

A simple calculation shows that

$$\lim_{\delta^{\varepsilon}\to 0} \|\widetilde{u}^{\varepsilon} - u^{\varepsilon}\|_{L^{\infty}(0,T;L^2(D))} = 0, \tag{23}$$

so we also have that

$$\lim_{\delta^{\varepsilon}\to 0} \|\widetilde{v}^{\varepsilon} - v^{\varepsilon}\|_{L^{\infty}(0,T;L^2(D))} = 0, \tag{24}$$

Now

$$
\begin{aligned}
&\int_0^T \int_D \left(\alpha^\varepsilon(v^\varepsilon(t)) - \overline{\alpha}^\varepsilon(u^\varepsilon(t))\right)\phi^\varepsilon(t)dxdt \\
-&\int_0^T \int_D \left(\alpha^\varepsilon(\widetilde{v}^\varepsilon(t)) - \overline{\alpha}^\varepsilon(\widetilde{u}^\varepsilon(t))\right)\phi^\varepsilon(t)dxdt = \\
&\int_0^T \int_D \phi^\varepsilon(t)\left(\alpha^\varepsilon(v^\varepsilon(t)) - \alpha^\varepsilon(\widetilde{v}^\varepsilon(t))\right)dxdt \\
+&\int_0^T \int_D \phi^\varepsilon(t)\left(\overline{\alpha}^\varepsilon(\widetilde{u}^\varepsilon(t)) - \overline{\alpha}^\varepsilon(u^\varepsilon(t))\right)dxdt, \\
&\to 0
\end{aligned}
$$

$$\int_0^T \int_D \left(\alpha^\varepsilon(\widetilde{v}^\varepsilon(t)) - \overline{\alpha}^\varepsilon(\widetilde{u}^\varepsilon(t))\right) \phi^\varepsilon(t) dx dt$$

$$= \sum_{k=0}^{n^\varepsilon - 1} \int_{k\delta^\varepsilon}^{(k+1)\delta^\varepsilon} \int_D \left(\alpha^\varepsilon(\widetilde{v}^\varepsilon(t)) - \overline{\alpha}^\varepsilon(\widetilde{u}^\varepsilon(t))\right) \phi^\varepsilon(t) dx dt.$$

So, given the estimates uniform estimates for u^ε, and choosing the δ^ε appropriately we can get that

$$\lim_{\varepsilon \to 0} \mathbb{E} \left| S_1^\varepsilon \right| = 0.$$

For the convergence of S_3^ε, we used the homogenization results

- G. Allaire (1991), *Homogenization of the Navier-Stokes Equations with a Slip Boundary Condition.*

For any $t \in [0, T]$, $F_t^\varepsilon : L^2(D) \to L^2(D)$,

$$F_t^\varepsilon(z)(x) = \left(\alpha\left(\frac{x}{\varepsilon}, z(x)\right) - \int_Y \alpha\left(y, z(x)\right)\right) u(t, x).$$

By a density argument, we show that: for any $z \in L^2(D)$, for every $t \in [0, T]$ and a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} F_t^\varepsilon(z) = 0 \quad \text{in} \quad L^2(D), \ \textit{weakly}.$$

The sequence being also uniformly bounded by $\|\overline{u}\|_{L^\infty(\Omega;C([0,T];L^2(D)))}$, Vitali's convergence theorem implies

$$\lim_{\varepsilon\to 0}\int_{L^2(D)} F_t^\varepsilon(z)d\mu^{\overline{u}(t)}(z) = 0 \text{ in } L^2(D),$$

which can be rewritten as

$$\lim_{\varepsilon\to 0}\overline{\alpha^\varepsilon}(\overline{u}(t))\overline{u}(t) - \overline{\alpha}(\overline{u}(t))\overline{u}(t) = 0 \text{ in } L^2(D).$$

This implies that $\mathbb{P}$ a.s. and for every $t \in [0, T]$

$$\lim_{\varepsilon\to 0}\int_D \left(\overline{\alpha^\varepsilon}(\overline{u}(t))\overline{u}(t) - \overline{\alpha}(\overline{u}(t))\overline{u}(t)\right)\phi\psi'(t)dx = 0.$$

The sequence being also uniformly bounded by $c\|\overline{u}\|_{L^\infty(\Omega, C([0,T];L^2(D)))}\|\phi\|_{L^\infty(D)}\|\psi'\|_{L^\infty[0,T]}$. We apply the bounded convergence theorem and integrate over $\Omega \times [0, T]$ and get that

$$\lim_{\varepsilon \to 0} \mathbb{E}\,|S_3^\varepsilon| = 0.$$

The convergence of S_2^ε is simpler:

$$\mathbb{E}\,|S_2^\varepsilon| \leq c\|\phi\|_{L^\infty(D)}\|\psi\|_{L^\infty[0,T]}\mathbb{E}\,\|u^\varepsilon - \overline{u}\|_{L^1(0,T;H_0^1(D)))}\,.$$

implies that

$$\lim_{\varepsilon \to 0} \mathbb{E}\,|S_2^\varepsilon| = 0.$$

Combining the convergences of S_1^ε, S_2^ε and S_3^ε we get that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left|\int_0^T \int_D \left(\alpha^\varepsilon(v^\varepsilon(t))u^\varepsilon(t) - \overline{\alpha}(\overline{u}(t))\overline{u}(t)\right)\phi dxdt\right| = 0.$$

Some remarks

- Tackle the full diffusion problem
- Tackle the case of coefficient dependent on time, the non-autonomous case
- Generalize to the case of SPDEs for the particle equations
- Find some rate of convergence. This is related to better convergence, like convergence in mean.

References

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THANK YOU