Stochastic differential equations with oblique reflection on non-smooth time-dependent domains

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Motivation and approach

- The study of reflected stochastic differential equations (RSDE) can be motivated by many different applications.
  - Turbulence in confined spaces
  - Modelling of regulated financial markets
  - Queuing theory
- General results on existence and uniqueness of RSDE in non-smooth time-independent domains were derived by Dupuis & Ishii (1993).
- We derive a class of time-dependent domains for which similar results hold.
- Our approach is based on the Skorohod problem (SP).
- We define SP and RSDE, describe the connection between these two notions and prove existence and uniqueness of RSDE for a suitable class of domains.

What do we mean by a reflected SDE?

- The solution $X(t)$ to a reflected stochastic differential equation on $[0, \infty]$ should satisfy an SDE

$$X(t) = \underbrace{x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s)}_{:= Y(t)},$$

  whenever $X(t) > 0$ and continuously reflect in some sense into the positive half-line when $X(t)$ hits zero.
- In other words, when $X(t)$ is in the interior of the domain $X(t)$ follows the dynamics $Y(t)$. When $X(t)$ hits the boundary, then all of its intentions to proceed further down should be compensated. These compensations should immediately disappear when $X(t) > 0$.
- The problem of solving RSDE is to find, for a given dynamics $Y(t)$ and a domain Ω (in this case equal to $[0, \infty)$) a function $X(t)$ with these properties.

A formal definition of one-dimensional RSDE

Definition

A pair of continuous $\{\mathcal{F}_t\}$-adapted stochastic processes $(X(t), \Lambda(t))$ is said to be a (strong) solution to the RSDE confined to $[0, \infty)$ with coefficients b, σ and initial condition $x \geq 0$, if, $\mathbb{P}$-a.s. for $t \geq 0$,

(i) $X(t) = x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW + \Lambda(t)$,

(ii) $\Lambda(t)$ is non-decreasing and satisfies $\Lambda(0) = 0$,

(iii) $\int_0^t \mathbb{I}_{\{X(s)>0\}} d\Lambda(s) = 0$,

(iv) $X(t) \geq 0$.

- Criterion (iii) asserts that $\Lambda(t)$ only increases when $X(t) = 0$.
- We next state the SP and compare the two definitions.

The one-dimensional Skorohod problem (SP)

Definition

Let $\psi : [0,\infty) \to \mathbb{R}$ be a continuous function with $\psi(0) \geq 0$. A solution to the SP for ψ is given by a pair (ϕ, λ) of functions $\phi, \lambda : [0,\infty) \to [0,\infty)$ such that, for $t \geq 0$,

(1) $\phi = \psi + \lambda$,

(2) λ is non-decreasing, continuous and satisfies $\lambda(0) = 0$,

(3) λ increases only when $\phi(t) = 0$.

Theorem

There exists a unique solution to the SP and it is explicitly given by

$$\begin{aligned} \lambda(t) &= \sup_{0 \leq s \leq t} \{-\psi(s) \vee 0\}, \\ \phi(t) &= \psi(t) + \sup_{0 \leq s \leq t} \{-\psi(s) \vee 0\}. \end{aligned}$$

- We denote $\phi(\cdot) = (\Gamma\psi)(\cdot) := \psi(\cdot) + \sup_{0 \leq s \leq \cdot} \{-\psi(s) \vee 0\}$ as the Skorohod map. The Skorohod map is Lipschitz.

The relation between RSDE and SP

- Let ω be such that the deterministic functions $(X(t), \Lambda(t))$ satisfy conditions (i)-(iv) in the definition of solutions to RSDE and denote

$$Y(t) = x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW,$$

- Then $(X(t), \Lambda(t))$ satisfies criteria (1)-(3) in the definition of the SP for $Y(t)$. By the theorem, a unique such $X(t) = (\Gamma Y)(t)$ exists and $Y(t)$ is a solution to the Itō SDE

$$Y(t) = x + \int_0^t b(s, (\Gamma Y)(s))\, ds + \int_0^t \sigma(s, (\Gamma Y)(s))\, dW,$$

- Under standard conditions of Lipschitz continuity and linear growth of the coefficients, this Itō SDE has a unique solution.
- This proves existence and uniqueness for RSDE.

Generalizations of the results

- The SP can be used to prove existence and uniqueness of solutions to RSDE on multi-dimensional domains.
- On multi-dimensional domains, the direction of reflection at the boundary must be specified.
  - Normal reflection (specular reflection)
  - Oblique reflection
- Discontinuous (càdlàg) ψ can be considered. This arises when considering RSDE driven by Lévy processes.
- The domain and direction of reflection can be time-dependent.

Time-dependent domains and directions of reflection

Definition

Given $n \geq 1$, $T > 0$ and an open, connected set $\Omega' \subset \mathbb{R}^{n+1}$, let

$$\Omega = \Omega' \cap ([0, T] \times \mathbb{R}^n)$$

denote a time-dependent domain. For $t \in [0, T]$, let $\Omega_t = \{x : (t, x) \in \Omega\}$ be the time sections of Ω.

Definition

Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain. The direction of reflection at a point $x \in \partial\Omega_t$, $t \in [0, T]$, is given by a function γ defined on a neighbourhood of $\{\partial\Omega_t : t \in [0, T]\}$ taking values on the unit sphere in $\mathbb{R}^n$.

- We shall impose more restrictions on the domain and direction of reflection later on.

General definition of RSDE

- We state the definitions of RSDE and SP in this general setting.

Definition

Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain. A pair of continuous $\{\mathcal{F}_t\}$-adapted stochastic processes $(X(t), \Lambda(t))$ is said to be a (strong) solution to the RSDE confined to $\overline{\Omega}$ with coefficients b, σ and initial condition $x \in \overline{\Omega}_0$, if, $\mathbb{P}$-a.s. for $t \in [0, T]$,

(i) $X(t) = x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW + \Lambda(t)$,

(ii) $\Lambda(t) = \int_0^t \gamma(s, X(s))\, d|\Lambda|(s)$, $\quad d|\Lambda|$-a.s.,

(iii) $|\Lambda|(t) = \int_0^t \mathbb{I}_{\{X(s) \in \partial\Omega_s\}} d|\Lambda|(s) < \infty$

(iv) $X(t) \in \overline{\Omega}_t$.

General definition of SP

Definition

Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain and let $\psi \in [0, T] \to \mathbb{R}^n$ be a continuous function with $\psi(0) \in \overline{\Omega}_0$. A solution to the SP for (Ω, γ, ψ) is given by a pair (ϕ, λ) of functions $\phi, \lambda : [0, T] \to \mathbb{R}^n$ such that, for all $t \in [0, T]$,

(1) $\phi = \psi + \lambda$,

(2) $\lambda(t) = \int_0^t \gamma(s, \phi(s))\, d|\lambda|(s)$, $\quad d|\lambda|$-a.s.,

(3) $|\lambda|(t) = \int_0^t \mathbb{I}_{\{\phi(s) \in \partial\Omega_s\}} d|\lambda|(s) < \infty$,

(4) $\phi(t) \in \overline{\Omega}_t$.

- For oblique reflection, uniqueness of solutions to the SP can rarely be obtained. The strongest existence results for time-independent domains were derived by Costantini (1992). Nyström & Önskog (2010) obtained similar existence results in time-dependent domains.

Our assumptions

- Ω_t is non-empty, bounded and connected for all $t \in [0, T]$.
- Let $d(t, x) = d(x, \Omega_t)$, for all $t \in [0, T]$, $x \in \mathbb{R}^n$.

$$d(\cdot, x) \in \mathcal{W}^{1,p}([0, T]), \text{ for some } p \in (1, \infty)$$

$\Rightarrow d(t, x)$ is Hölder continuous with exponent $1 - 1/p$ in the time variable.

- The direction of reflection γ is a $\mathcal{C}^{1,2}$-function.
- The time sections satisfy a uniform exterior cone condition in the sense that there exists a constant $\rho \in (0, 1)$ such that

$$\bigcup_{0 \leq \zeta \leq \rho} B(x - \zeta\gamma(t, x), \zeta\rho) \subset \Omega_t^c,$$

for all $x \in \partial\Omega_t$, $t \in [0, T]$. By the spatial continuity of γ, they also satisfy a uniform interior cone condition.
$\Rightarrow$ The time sections are Lipschitz domains.

Main result and outline of the proof

Theorem

Under the assumptions on the previous slide, there exists a unique strong solution to RSDE.

The proof consists of the following four steps:

1. For smooth ψ, prove existence of solutions to the SP for (Ω, γ, ψ).
2. Prove relative compactness for a collection of solution to the SP for (Ω, γ, ψ).
3. Extend the existence result for the SP for (Ω, γ, ψ) to continuous ψ.
4. Prove a contraction estimate useful for Picard iteration and the proof of uniqueness.

Existence of solutions to SP for smooth paths

Theorem (Step 1)

Let $\psi \in \mathcal{C}^1([0,T])$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution $(\phi, \lambda) \in \mathcal{W}^{1,p}([0,T]) \times \mathcal{W}^{1,p}([0,T])$ to the SP for (Ω, γ, ψ).

- We use a penalty approach. Choose $\varepsilon > 0$ and consider an ordinary differential equation for $\phi_\varepsilon(t)$ (with unique solution)

$$\dot{\phi}_\varepsilon(t) = \dot{\psi}(t) + \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)) \gamma(t, \phi_\varepsilon(t)), \quad \phi_\varepsilon(0) = \psi(0).$$

- We obtain the upper bound

$$\frac{1}{\varepsilon^{p-1}} (d(t, \phi_\varepsilon(t)))^p + \frac{\kappa}{\varepsilon^p} \int_0^t (d(s, \phi_\varepsilon(s)))^p \, ds \leq K(T)$$

- $\lambda_\varepsilon(t) := \frac{1}{\varepsilon} \int_0^t d(s, \phi_\varepsilon(s)) \gamma(s, \phi_\varepsilon(s)) \, ds \xrightarrow{w} \lambda \in \mathcal{W}^{1,p}([0,T])$
- From $\frac{1}{\varepsilon^{p-1}} (d(t, \phi_\varepsilon(t)))^p \leq K(T)$, we deduce $\phi(t) \in \overline{\Omega}_t$.

Relative compactness of solutions to the SP

Theorem (Step 2)

Let A be a compact subset of $\mathcal{C}([0,T])$. Then

(i) *There exists a constant $L < \infty$ such that $|\lambda|(T) < L$, for all solutions $(\psi + \lambda, \lambda)$ to the SP for (Ω, γ, ψ) with $\psi \in A$.*

(ii) *The set $\{\phi : (\phi, \lambda)$ solves the SP for (Ω, γ, ψ) with $\psi \in A\}$ is relatively compact.*

- Fix $\psi \in A$ and let (ϕ, λ) be a solution to the SP for (Ω, γ, ψ).
- Fix $c > 0$. Define a sequence $\{T_m\}_{m=0,1,\ldots}$ by $T_0 = 0$ and

$$T_{m+1} = \min\{T, c, \inf\{t \in [T_m, T] : \phi(t) \notin B(\phi(T_m), c)\}\}.$$

- We show that $|\lambda|(T_{m+1}) - |\lambda|(T_m) \le M$.
- Using an appropriate test function and lengthy calculations, we show the a priori estimate

$$\|\lambda\|_{T_m,\tau} \le R\left(\|\psi\|_{T_m,\tau}^{1/2} + \|\psi\|_{T_m,\tau}^{3/2} + (\tau - T_m)^{1/2-1/2p}\right),$$

for $\tau \in [T_m, T_{m+1}]$ and some positive constant R. With this estimate, we prove that $\{T_m\}_{m=0,1,\ldots}$ do not accumulate.

Existence of solutions to SP for non-smooth paths

Theorem (Step 3)

Let $\psi \in \mathcal{C}([0,T])$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution (ϕ, λ) to the SP for (Ω, γ, ψ).

- Let $\psi_n \in \mathcal{C}^1([0,T])$ form a sequence of functions converging uniformly to ψ.
- By Step 1, there exists a solution (ϕ_n, λ_n) to the SP for (Ω, γ, ψ_n).
- By Step 2, we may assume

$$\begin{aligned} \sup_n |\lambda_n|(T) &\leq L < \infty. \\ \lim_{|s-t|\to 0} \sup_n |\lambda_n(s) - \lambda_n(t)| &= 0. \end{aligned}$$

- By the Arzela-Ascoli theorem there exists a function $\lambda \in \mathcal{C}([0,T])$ such that $\{\lambda_n\}$ converges uniformly to λ. Clearly $|\lambda|(T) \leq L$.
- We define ϕ as $\phi = \psi + \lambda$. The criteria in the definition of the SP are inherited from the properties of (ϕ_n, λ_n).

A contraction estimate for Picard iterations

Theorem (Step 4)

Assume that the triple (X, Y, Λ) *satisfies* $X(t), Y(t) \in \overline{\Omega}_t$ *and*

$$\begin{aligned} Y(t) &= x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s) + \Lambda(t), \\ \Lambda(t) &= \int_0^t \gamma(s, Y(s))\, d|\Lambda|(s), \quad d|\Lambda|\text{-a.s.} \\ |\Lambda|(t) &= \int_0^t \mathbb{I}_{\{Y(s) \in \partial\Omega_s\}} d|\Lambda|(s), \end{aligned}$$

where $x \in \overline{\Omega}_0$. *Let* (X', Y', Λ') *be another triple with initial value* $x' \in \overline{\Omega}_0$ *instead of* x. *Then there exists a positive constant* C *such that*

$$E\left[\sup_{0 \le s \le t} \left|Y(s) - Y'(s)\right|^4\right] \le C\left(\left|x - x'\right|^4 + \int_0^t E\left[\sup_{0 \le u \le s} \left|X(u) - X'(u)\right|^4\right] ds\right).$$

- The contraction estimate is derived using Itō calculus on $v(t, x, y) = e^{-\lambda(\alpha(t,x)+\alpha(t,y))} w_\varepsilon(t, x, y)$, where α and w_ε are two appropriate test functions in $\mathcal{C}^{1,2}\left(\overline{\Omega}\right)$.

Wrapping it all up...

- Existence in distribution of Y and Λ together with strong uniqueness implies existence of a unique strong solution.
- Picard iteration together with the contraction estimate proves strong uniqueness.
- Let ψ be a bounded variation path starting inside $\overline{\Omega}_0$. There can be at most one solution (ϕ, λ) to the SP for (Ω, γ, ψ). Let

$$S(t) = x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s),$$

  and let $\{S_n(\cdot)\}$ be a sequence of continuous bounded variation $\mathcal{F}_t$-adapted semimartingales which converges uniformly to $S(\cdot)$.
- Let (Y_n, Λ_n) be defined pathwise as solutions to the SP for S_n. By construction and the uniqueness of solutions to the SP for bounded variation paths, (Y_n, Λ_n) are $\mathcal{F}_t$-adapted.
- By the a priori estimate and uniform convergence of S_n to S, the joint distribution $(Y_n, S_n, \Lambda_n, |\Lambda|_n)$ is tight, and the limit $(Y, S, \Lambda, |\Lambda|)$ exists in distribution.
- This completes the proof!

References

1. Niklas Lundström & Thomas Önskog (2019). Stochastic and partial differential equations on non-smooth time-dependent domains. *Stoch. Proc. Appl.* **129**(4) 1097–1131.
2. Kaj Nyström & Thomas Önskog (2010). The Skorohod oblique reflection problem in time-dependent domains. *Ann. Prob.* **38**(6) 2170–2223.
3. Paul Dupuis & Hitoshi Ishii (1990). On oblique derivative problems for fully nonlinear second-order elliptic partial differential equations on nonsmooth domains. *Nonlinear Anal.* **15**(12) 1123–1138.
4. Paul Dupuis & Hitoshi Ishii (1993). SDEs with oblique reflection on nonsmooth domains. *Ann. Prob.* **21**(1) 554–580.

Thank you for your attention!