On modeling the behavior of pedestrians near walls and the mean-field approach to crowd dynamics

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KAAS – Karlstad Applied Analysis Seminar

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Content

Joint work with **Boualem Djehiche** (KTH).
Behavior near walls in the mean-field approach to crowd dynamics.
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1. The mean-field approach to multi-agent systems
2. Pedestrian crowd dynamics
3. Sticky boundary model

Large human crowds as dynamical systems

Human stampedes: a frequently reoccurring event.

Recorded stampedes with deadly outcome in 2020:

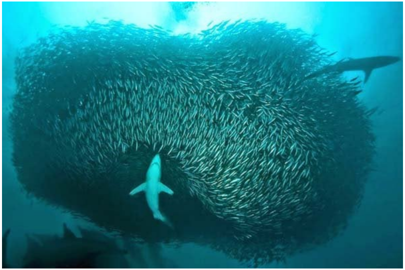
- January 7: Funeral of general Qasem Soleimani (Iran)
  >56 dead, >200 injured
- February 1: Prayer meeting (Tanzania)
  20 dead, 16 injured
- February 4: School (Kenya)
  14 dead.
- February 17: Market (Niger)
  >20 dead
- May 21: Religious festivity (Sri Lanka)
  3 dead.

Sources: see Wikipedia article "List of human stampedes".

Overview: MFG & MFTC

MEAN FIELD GAMES (MFG): **dynamic stochastic decision making** where a **very large** number of agents

- are non-cooperative (they seek a Nash equilibrium)
- are (to some extent) symmetric
- interact through aggregate effects



OPTIMAL CONTROL OF MCKEAN-VLASOV EQUATIONS (MFTC): cooperative agents can "shape the aggregates".

Overview: MFG & MFTC in the finite horizon stoch. diff. case

- MFG: equilibrium (fixed point) $\hat{u}$

$$\begin{cases} \inf_{u\in\mathcal{U}} E\Big[\int_0^T f(t, X_t^u, \mu(t), u_t)dt + g(X_T^u, \mu(T))\Big] \\ \text{s.t. } dX_t^u = b(t, X_t^u, \mu(t), u_t)dt + \sigma dW_t,\ X_0^u = x \end{cases} \quad \Bigg| \quad \hat{\mu} = \mathbb{P} \circ (X^{\hat{u}})^{-1}$$

- MFTC: optimization
- Mean-field type game (MFTG): a game between McKean-Vlasov equations.

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Overview: MFG & MFTC

Relaxations of exchangeability include:
agent classes, major/minor agents, etc.

MFG and MFTC, are they meaningful approximations?

- Lasry, Lions (2006, 2007) and Huang, Malhamé, Caines (2006).
- Noncontrolled diffusion processes Oelschläger (1984), El Karoui, Du Huu, Jeanblanc-Picqué (1987). Extended e.g. by Lacker (2017).
- Carmona, Delarue (2018)

Overview: MFG & MFTC

Two questions raised in the theory

- Existence of a minimizer/equilibrium.
- Explicit computation of such points:
  - The Bellman principle, which yields the Hamilton-Jacobi-Bellman equation (HJB) for the value function.
  - Pontryagin's maximum principle which yields the Hamiltonian system for *"the derivative"* of the value function.

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The two main techniques for solving mean-field games/control problems

DP

- Dynamical programming yields to coupled system of equations: HJB backward in time and the evolution of the state forward in time.
- Possible time-inconsistency issues.
- DP for MFTC: Bensoussan, Frehse, Yam (2013), Laurière, Pironneau (2014)

PMP

- The stochastic maximum principle yields a forward-backward SDE for the state dynamics and the adjoint state dynamics.
- PMP for MFTC: Andersson, Djehiche (2011).
- Existence and uniqueness results for MF-FBSDEs: Carmona, Delarue (2018), Djehiche, Hamadène (2019).

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Example: MFG and MFTC

Linear dynamics

$$dX_t = u_t dt + \sigma dW_t, \; X_0 = x_0 \in \mathbb{R},$$

and quadratic mean-field cost

$$J(u) = \frac{1}{2}\mathbb{E}\left[X_T^2 - \mathbb{E}[X_T]^2 + \int_0^T u_t^2 dt\right].$$

Pontryagin's maximum principle: $\hat{u}_t = p_t$ (a.e.-t, $\mathbb{P}$-a.s.)

Pontryagin's maximum principle: $(X_\cdot^{\hat{u}}, p_\cdot)$ solves the FBSDE

MFG:

$$\begin{cases} dX_t^{\hat{u}} = p_t dt + \sigma dW_t, \; X_0^{\hat{u}} = x_0, \\ dp_t = q_t dW_t, \; p_T = -X_T^{\hat{u}}. \end{cases}$$

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Pedestrian crowd dynamics: Overview

Figure: Lab experiment

Figure: Kumbh Mela Experiment

Empirical studies

The empirical study of human crowds has been conducted since Hankin & Wright (1958) (flows in London's undergound).

Empirical studies confirm that pedestrians have a will to reach specific targets and act as if there is a repulsion from other individuals.

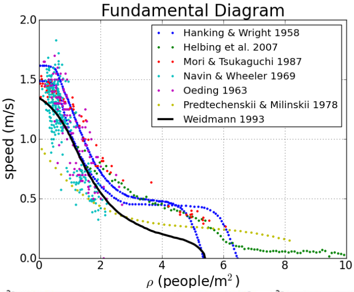
Solid particles

- Interaction only through collisions
- Dynamics ruled by inertia
- All directions equally influential

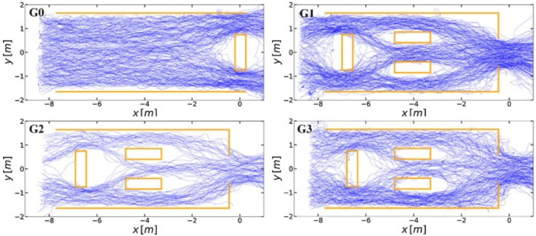
Humans

- Avoidance of collisions and obstacles
- Dynamics ruled (partially) by decision
- Some directions more influential than others

Empirical data: fundamental diagrams and route choice



Fundamental diagram



Route choice

Mathematical approaches

- Microscopic models
  - Social force: Helbing & Molnár (1995)
  - Cellular automaton: Schadschneider et al (2001)
  - Optimal control: Hoodendororn & Bovy (2003)
- Mesoscopic models
- Macroscopic models

The mean-field approach to crowd dynamics bridges the micro and macro scale.

Mathematical approaches

- Microscopic models
- Mesoscopic models
  - Kinetic equations: Albi et al (2016)
  - Swarming: Fornasier et al (2010)
- Macroscopic models

The mean-field approach to crowd dynamics bridges the micro and macro scale.

Mathematical approaches

- Microscopic models
- Mesoscopic models
- Macroscopic models
  - Fluid-like models: Henderson (1971); Hudges (2003)
  - Optimal transport: Santambrogio et al (2010)
  - Mean-field games: Dogbé (2010); Lachapelle & Wolfram (2011); Djehiche, Tcheukam & Tembine (2017)
  - Mean-field type control: Burger et al (2013, 2014); Djehiche, Tcheukam & Tembine (2017)
  - Mean-field type games: Aurell & Djehiche (2018, 2019)

The mean-field approach to crowd dynamics bridges the micro and macro scale.

Pedestrian crowds in confined domains

Boundary conditions in the mean-field approach

Treatment of walls in pedestrian crowd models

Model class	**Wall modeling**
Social force	Repulsive forces, disutility
Cellular automata (CA)	Forbidden cells
Continuum limit of CA	Neumann/no-flux b.c.
Hughes flow model	Neumann/no-flux b.c., oblique reflection
Mean-field games/control/type games	Neumann/no-flux b.c., disutility

Neumann/no-flux boundary conditions on the pedestrian density correspond to *instantaneous reflection*.

Sticky reflected Brownian motion

Consider the SDE system

$$\begin{cases} dX_t = \frac{1}{2} d\ell_t^0(X_\cdot) + 1_{\{X_t>0\}} dB_t, \quad X_0 = x_0, \\ 1_{\{X_t=0\}} dt = \frac{1}{2\gamma} d\ell_t^0(X_\cdot), \end{cases} \tag{1}$$

where $x_0 \in \mathbb{R}_+$, $\gamma \in (0,\infty)$ is a given stickyness level of the boundary $\{0\}$, $\ell_0(X_\cdot)$ is the local time of $X_\cdot$ at 0, B is a standard Brownian motion.

Skorokhods conjecture: System (1) has no strong solution. The system has a unique weak solution, the *reflected Brownian motion X in $\mathbb{R}_+$ sticky at 0.*

Feller (1952), Wentzell (1959), Itô-Mckean (1963), Graham (1988), Chitashvili (1989), Warren (1997), Engelbert & Peskir (2014), ...

Sticky reflected Brownian motion with boundary diffusion

Grothaus and Vosshall (2017) extentend the result to a *bounded* domain $\mathcal{D} \subset \mathbb{R}^d$ with sticky C^2*-smooth boundary* $\partial\mathcal{D}$.

- Let $n(x)$ be the outward normal of $\partial\mathcal{D}$ at x,
- $\pi(x) := E - n(x)(n(x))^*$, the orthogonal projection on the tangent space of $\partial\mathcal{D}$ at x,
- $\kappa(x) := (\pi(x)\nabla) \cdot n(x)$, the mean curvature of $\partial\mathcal{D}$ at x.

These quantities are uniformly bounded over $\partial\mathcal{D}$.

Sticky reflected Brownian motion with boundary diffusion

Let $\Omega := C([0,T];\mathbb{R}^d)$ be path space, $\mathcal{F}$ the Borel σ-field over Ω, and $X_t(\omega) = \omega(t)$ be the coordinate process.

Theorem (Grohaus and Vosshall (2017))

There exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ under which

$$\begin{cases} dX_t = 1_{\mathcal{D}}(X_t)dB_t + 1_{\partial\mathcal{D}}(X_t)\big(dB_t^{\partial\mathcal{D}} - \frac{1}{2\gamma}n(X_t)dt\big), \\ dB_t^{\partial\mathcal{D}} = \pi(X_t) \circ dB_t = -\frac{1}{2}\kappa(X_t)n(X_t)dt + \pi(X_t)dB_t, \\ B \textit{ is a standard } \mathbb{P}\textit{-BM in } \mathbb{R}^d,\ X_0 = x_0 \in \bar{\mathcal{D}},\ \gamma > 0, \end{cases}$$

and X is $C([0,T];\bar{\mathcal{D}})$-valued $\mathbb{P}$-a.s. (in particular, X is $\mathbb{P}$-a.s. uniformly bounded).

Sticky reflected Brownian motion with boundary diffusion

$$dX_t = \left(1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t)\right) dB_t - 1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t)dt$$

The sticky reflected SDE with boundary diffusion is composed of

- interior diffusion $1_{\mathcal{D}}(X_t)dB_t$,
- boundary diffusion $1_{\partial\mathcal{D}}(X_t)dB_t^{\partial\mathcal{D}}$
- normal sticky reflection $-1_{\partial\mathcal{D}}(X_t)\frac{1}{2\gamma}n(X_t)dt$

From now on, we abbreviate $dX_t =: \sigma(X_t)dB_t + a(X_t)dt$ where

$$\sigma(X_t) := 1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t),\ a(X_t) := -1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t).$$

The stickiness level γ

Let

- λ be the Lebesgue measure on $\mathbb{R}^d$
- s be the surface measure on $\partial\mathcal{D}$
- $\rho := 1_{\mathcal{D}}\alpha\lambda + 1_{\partial\mathcal{D}}\alpha' s, \quad \alpha, \alpha' \in \mathbb{R}$

ρ becomes a probability measure on $\mathbb{R}^d$ with full support on $\bar{\mathcal{D}}$ if we choose

$$\alpha = \bar{\alpha}/\lambda(\mathcal{D}), \quad \alpha' = (1-\bar{\alpha})/s(\partial\mathcal{D}), \quad \bar{\alpha} \in [0,1],$$

The probability measure ρ is then the invariant distribution of X_t whenever

$$\frac{1}{\gamma} = \frac{\bar{\alpha}}{1-\bar{\alpha}} \frac{s(\partial\mathcal{D})}{\lambda(\mathcal{D})}$$

As $\gamma \to 0$, $\bar{\alpha} \to 1$ and the invariant distribution concentrates on $\mathcal{D}$.
As $\gamma \to \infty$, $\bar{\alpha} \to 0$ and the invariant distribution concentrates on $\partial\mathcal{D}$

The sticky reflected SDE of mean-field type with boundary diffusion

For β of linear growth in (t, x, μ) and Lipschitz in μ w.r.t. TV.; $\mathcal{U}$ prog.meas. processes; $\mathcal{D}$ as before.

Theorem (A., Djehiche (2020))

Given $u \in \mathcal{U}$, there exists a unique weak solution $(\mathbb{P}^u)$ to the sticky reflected SDE of mean-field type with boundary diffusion

$$dX_t = \sigma(X_t)dB_t^u + \Big(a(X_t) + \sigma(X_t)\beta(t, X_t, \mathbb{P}^u(t), u_t)\Big)dt$$

Under $\mathbb{P}^u$ the t-marginal distribution of $X.$ is $\mathbb{P}^u(t)$ for $t \in [0, T]$ and $X.$ is almost surely $C([0, T]; \bar{\mathcal{D}})$-valued. Furthermore, $\mathbb{P}^u \in \mathcal{P}_p(\Omega)$.

Proof: Start from base case $\mathbb{P}$. The **Girsanov transform** given by $L_T = \mathcal{E}_T(\int_0^{\cdot} \beta(t, X_t, Q(t), u_t)dB_t)$ transforms $\mathbb{P}$ to $\mathbb{P}^{Q,u}$ and is well defined for any $Q \in \mathcal{P}(\Omega)$. The map $Q \mapsto \mathbb{P}^Q$ is a contraction in the complete space $(\mathcal{P}(\Omega), D_{TV})$, and has fixed point $\mathbb{P}^u$.

CONTROL of sticky reflected SDE of mean-field type with boundary diffusion

Consider the following finite time-horizon problem:

$$\inf_{u\in\mathcal{U}} \mathbb{E}^u \left[\int_0^T f(t, X_\cdot, \mathbb{P}^u(t), u_t)dt + g(X_T, \mathbb{P}^u(T)) \right]$$

A *weak form* mean-field type control problem.
Controlled object: $\mathbb{P}^u$ ($X_\cdot$ is the coordinate process).

Grisanov transformation relates $\mathbb{P}^u$ to $\mathbb{P}$, which is indep. of u!
Reformulation: control via the likelihood process L^u

$$\begin{cases} \inf_{u\in\mathcal{U}} E \left[\int_0^T L_t^u f(t, X, \mathbb{P}^u(t), u_t)dt + L_T^u g(X_T, \mathbb{P}^u(T)) \right] \\ \text{s.t. } dL_t^u = L_t^u \beta(t, X_\cdot, \mathbb{P}^u(t), u_t)dB_t, \quad L_0^u = 1, \\ \qquad X_\cdot \text{ coordinate process under } \mathbb{P}, \quad d\mathbb{P}^u(t) = L_t^u d\mathbb{P} \end{cases}$$

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CONTROL of sticky reflected SDE of mean-field type with boundary diffusion

A mean-field control problem where the likelihood process *acts* as the state process.

We have a PMP under some additional assumptions:

- For $\phi = \beta, f, g$,

  $$\phi(t, X_\cdot, \mathbb{P}^{u}(t), u_t) = \phi(t, X_\cdot, \mathbb{E}[L_t^{u} r_\phi(X_t)], u_t),$$

  where $r_\beta, r_f, r_g : \mathbb{R}^d \to \mathbb{R}^d$.
- For every $u \in \mathcal{U}$, the process $(f(t, X, \mathbb{E}[L_t^u r_f(X_t)], u_t))_t$ is progressively measurable with respect to $\mathbb{F}$ and $(x, y) \mapsto g(x, y)$ is Borel measurable.
- The functions $(t, x, y, u) \mapsto (f, \beta)(t, x, y, u)$ and $(x, y) \mapsto g(x, y)$ are twice continuously differentiable with respect to y. Moreover, β, f and g and all their derivatives up to second order with respect to y are continuous in (y, u), and bounded.

CONTROL of sticky reflected SDE of mean-field type with boundary diffusion

Using Buchdahn *et al.* (2011) and Honsker (2012):

Theorem (A., Djehiche (2020))

Assume $(\hat{u}, L^{\hat{u}})$ *is optimal for the control problem. Then for all* $v \in U$,

$$\begin{aligned} &\mathcal{H}(L_t^{\hat{u}}, v, p_t, q_t) - \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) \\ &\qquad + \frac{1}{2}[\delta(L\beta)_t]^T P_t [\delta(L\beta)_t] \leq 0, \quad a.e.\text{-}t \in [0,T],\ \mathbb{P}\text{-}a.s. \end{aligned} \tag{2}$$

$\mathcal{H}(\ell, u, p, q) := \ell(\beta(\dots)q - f(\cdot))$, $\delta(L\beta)_t := L_t^{\hat{u}}(\beta(\dots, v) - \beta(\dots, \hat{u}_t))$, *and the adjoint processes* (p, q), (P, Q) *solve the first and second order adjoint equation (BSDEs), respectively.*

The optimality condition above leads to
an FBSDE for $(L^{\hat{u}}, (p,q), (P,Q))$.

Whenever U is convex, (P,Q) is not needed.
Condition (2) simplifies to $(v - \hat{u}_t)^T \nabla_u \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) \leq 0$.

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INTERACTING sticky reflected SDEs with boundary diffusion

Consider $N \in \mathbb{N}$ (non-transformed, independent) sticky reflected SDEs with boundary diffusion

$$dX_t^i = a(X_t^i)dt + \sigma(X_t^i)dB_t^i, \quad X_0^i = x_i. \tag{3}$$

Theorem (Grothaus, Vosshall (2017))

There exists a unique probability measure $\mathbb{P}^N$ on $(\mathbf{\Omega}, \mathbf{F})$, where $\mathbf{\Omega} := C([0,T];\mathbb{R}^{Nd})$ and $\mathbf{F}$ is the corresponding filtration. Under $\mathbb{P}^N$, $(X^1, \dots, X^N)$ satisfies (27) and is $C([0,T];\bar{\mathcal{D}}^N)$-valued $\mathbb{P}^N$-a.s.

Interaction and control can be introduced
with a Girsanov transformation.

INTERACTING sticky reflected SDEs with boundary diffusion

Introducing controls – setup for cooperative minimization

Given $\mathbf{u} := (u^1, \dots, u^N) \in \mathcal{U}^N$, let $\mu_t^N := \frac{1}{N}\sum_i \delta_{X_t^i}$ and

$$dL_{i,t}^{\mathbf{u}} = L_{i,t}^{\mathbf{u}}\beta(t, X_\cdot^i, \mu_t^N, u_t^i)dB_t^i, \quad L_{i,0}^{\mathbf{u}} = 1.$$

$L_t^{N,\mathbf{u}} := \prod_i L_{i,t}^{\mathbf{u}}$ defines a Girsanov transform of $\mathbb{P}^N$ to $\mathbb{P}^{N,\mathbf{u}}$.

Under $\mathbb{P}^{N,\mathbf{u}}$ the coordinate process is $C([0,T];\bar{\mathcal{D}})$-valued a.s. and satisfies

$$dX_t^i = (\sigma(X_t^i)\beta(t, X_t^i, \mu_t^N, u_t^i) + a(X_t^i))dt + \sigma(X_t^i)dB_t^{i,\mathbf{u}}, \; X_0^i = x_i,$$

where $B^{i,\mathbf{u}}$ is a $\mathbb{P}^{N,\mathbf{u}}$-BM. Also, $\mathbb{P}^{N,\mathbf{u}} \in \mathcal{P}_p((C([0,T];\mathcal{D})^N)$.

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An optimization problem

Social cost for the particle system:

$$J_N(\mathbf{u}) := \frac{1}{N}\sum_i \mathbb{E}^{N,\mathbf{u}}\left[\int_0^T f(t, X^i_\cdot, \mu^N_t, u^i_t)dt + g(X^i_T, \mu^N_T)\right]$$

Minimization of J_N is a cooperative scenario.

Two common questions asked in the MFG/MFTC literature:

- Approximation: $J_N(\hat{u}, \ldots, \hat{u}) = \inf_{\mathbf{u}} J_N(\mathbf{u}) + \varepsilon_N$
- Convergence: $J_N(\hat{\mathbf{u}}(N)) \to J(\hat{u})$, where $\hat{\mathbf{u}}(N) \in \arg\inf_{\mathbf{u}} J_N(\mathbf{u})$.

Insipred by Lacker (2018), we provide an approximation result.

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Theorem (A., Djehiche (2020))

Let $u \in \mathcal{U}$ be a closed-loop control, i.e., $u_t(\omega) = \varphi(\omega_{\cdot\wedge t})$ for some measurable function $\varphi : (\Omega, \mathcal{F}) \to (U, \mathcal{B}(U))$.

Given u and ξ, r.v. with nonatomic law λ with support only on $\bar{\mathcal{D}}$, let

$$dX_t = (a(X_t) + \sigma(X_t)\beta(t, X_\cdot, \mathbb{P}^u(t), \varphi(X_{\cdot\wedge t})))\,dt + \sigma(X_t)dB_t, \;\; X_0 = \xi$$

and $(X^{1,N}, \ldots, X^{N,N})$ solve the interacting, non-MF, sticky equations, all using the control u, with $X^{i,N} = \xi^i \sim \lambda$ i.i.d.
Then, for any $k \leq N$,

$$\lim_{N\to\infty} D_T\left(\mathbb{P}^{N,\mathbf{u}} \circ (X_\cdot^{1,N}, \ldots, X_\cdot^{N,k})^{-1}, (\mathbb{P}^u \circ X_\cdot^{-1})^{\otimes k}\right) = 0$$

$$\lim_{N\to\infty} J_N(u, \ldots, u) = J(u)$$

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EXAMPLE: Unidirectional pedestrian flow

Experimental results show that average pedestrian speed in a cross-section of a corridor can be higher in the center than near the walls (Daamen *et al.* (2007)), but also higher near the walls (Zanlungo *et al.* (2012)), depending on the circumstances (congestion, geometry, etc).

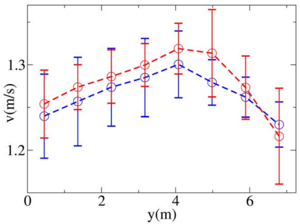

Figure 2. Velocity distributions as measured in the environment E_1 ($\bar{v}^+$ in red, $\bar{v}^-$ in blue). Error bars are obtained as standard deviations of values of $\bar{v}$ averaged over time windows of length 1200 s.
doi:10.1371/journal.pone.0050720.g002

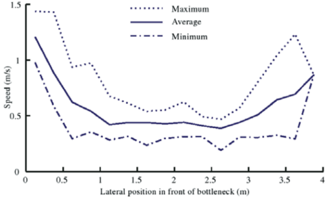

Fig. 5. Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

EXAMPLE: Unidirectional pedestrian flow

Let $\mathcal{D}$ be a long narrow corridor with exit x_T and entrance x_0 in opposite ends.

$$\begin{cases} \min\limits_{u.\in\mathcal{U}} \dfrac{1}{2}E\left[\displaystyle\int_0^1 L_t^u f(t, X., E[L_t^u r_f(X_t)], u_t)dt + L_T^u|X_T - x_T|^2\right], \\ \text{s.t. } dL_t^u = L_t^u u_t dB_t,\ L_0^u = 1. \end{cases}$$

f is a congestion-type running cost:

$$f(t, X., E[L_t^u r_f(X_t)], u_t) = \mathcal{C}(X_t)\left\{1 + h\left(t, X., E^u[r_f(X_t)]\right)\right\}|u_t|^2,$$

where

- $|u|^2$ is the cost of moving in free space;
- $h|u|^2$ is the additional cost to move in congested areas;
- $\mathcal{C}(X_t) := \xi 1_\Gamma(X_t) + 1_\mathcal{D}(X_t)$, $\xi > 0$, monitors f on the boundary $\partial\mathcal{D}$.

Lower ξ yields lower overall cost of moving on $\partial\mathcal{D}$ and vice versa.

EXAMPLE: Unidirectional pedestrian flow

Assuming U is convex, an optimal control satisfies

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\,(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

By using $\hat{u}$, the agents implements the following strategy:

- Move towards the exit x_T,
- Scale the speed according to local congestion.

EXAMPLE: Unidirectional pedestrian flow

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\left(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])\right)}.$$

We will compare two congestion-type costs

- friendly

$$h = h_1 := |X_2(t) - E^{\hat{u}}[X_2(t)]|$$

- averse

$$h = h_2 := \frac{1}{|X_2(t) - E^{\hat{u}}[X_2(t)]|}$$

In both cases,

- $r_f((x_1, x_2)) = x_2$
- $X_2(t)$ is the y-component of X_t (perpendicular to the corridor walls).

EXAMPLE: Unidirectional pedestrian flow

Estimated cross-section mean speed profiles

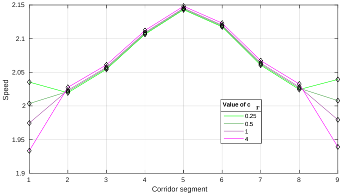

Figure: Congestion friendly ($h = h_1$).

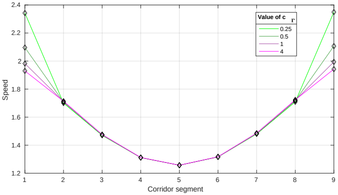

Figure: Congestion averse ($h = h_2$).

CONCLUSIONS

- Control of scenarios through Girsanov transformation: a MFTC problem.
- The stickiness level is not controlled.
- Convergence of non-controlled particle system to sticky reflected SDE of MFT with boundary diffusion.
- Sticky boundaries allows pedestrians to spend time, move, and interact at the boundary.

Thank you!