A generalized Monge-Ampère equation to compute freeform lens surfaces

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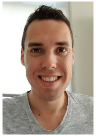
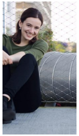
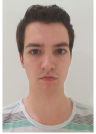

Maikel Bertens


Simon Kronberg


Lotte Romijn


Robert van Gestel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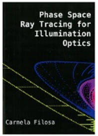
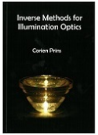
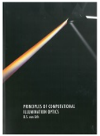
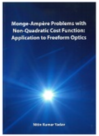


Carmela Filosa



Corien Prins


Bart van Lith



Nitin Yadav

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Outline

Application: Laser beam shaping

Expression for the optical map based on geometrical optics

Energy conservation

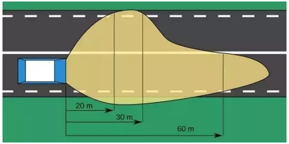
Numerical method and results

Concluding remarks

Illumination optics

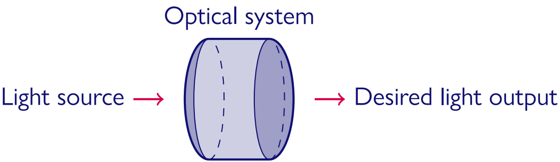
Design of optical systems for illumination purposes

- LED lighting
- Road lights
- Car headlights



Industry standard: Ray tracing with quasi Monte Carlo methods

- Easy to implement
- Slow convergence
- Design, ray trace, change design, ray trace, …



Goal: Inverse methods that directly compute the required optical system

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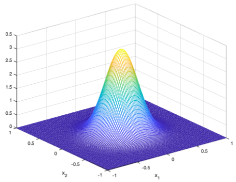
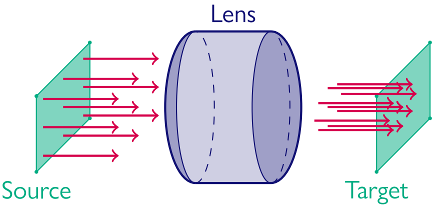
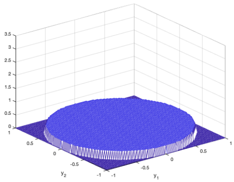
Laser beam shaping







Source distribution:
Gaussian profile

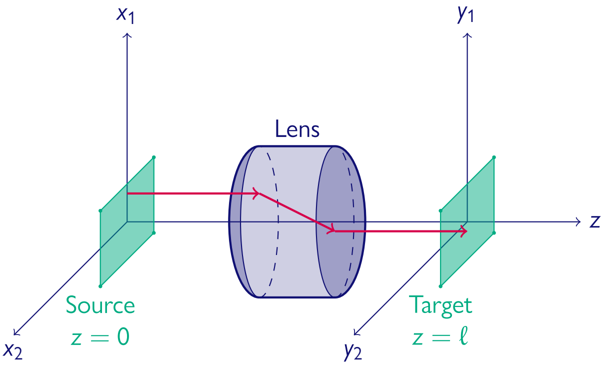
Source emits parallel light rays

Optical system:
One lens with two freeform surfaces

Desired target distribution:
Circular top hat profile

Output beam: Parallel light rays

Laser beam shaping



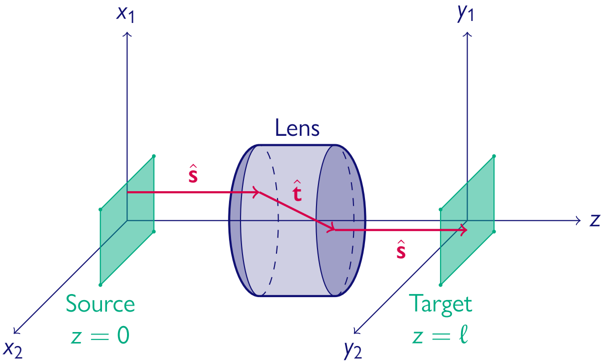
- Light rays travel from left to right
- Source plane: $z = 0$

  Cartesian coordinates $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Source S emits parallel light rays

  Exitance: $f(\mathbf{x})$, $\mathbf{x} \in S$
- First lens surface: $z = u_1(\mathbf{x})$, $\mathbf{x} \in S$
- Target plane: $z = \ell$

  Cartesian coordinates $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
- Second lens surface: $\ell - z = u_2(\mathbf{y})$, $\mathbf{y} \in T$
- Desired irradiance: $g(\mathbf{y})$, $\mathbf{y} \in T$

Let's follow a ray through the system

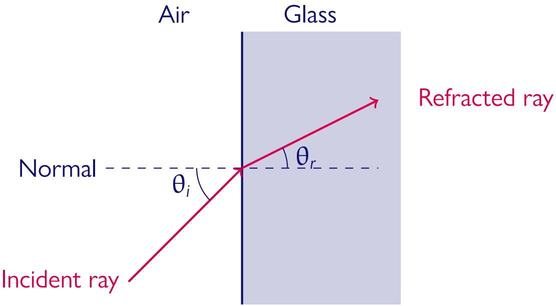


- Light rays travel from left to right
- Vectors with length 1 have hat, e.g. $\hat{\mathbf{s}}$
- Light ray leaves source in direction

$$\hat{\mathbf{s}} = \hat{\mathbf{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Refraction at first lens surface
  Refracted (or transmitted) ray has direction $\hat{\mathbf{t}}$
- Refraction at second lens surface
  Ray has direction $\hat{\mathbf{s}}$ again

Refraction at lens surface: Snell's law



- Snell's law:

$$n_{\text{air}} \sin\theta_i = n_{\text{glass}} \sin\theta_r$$

- Vector form:

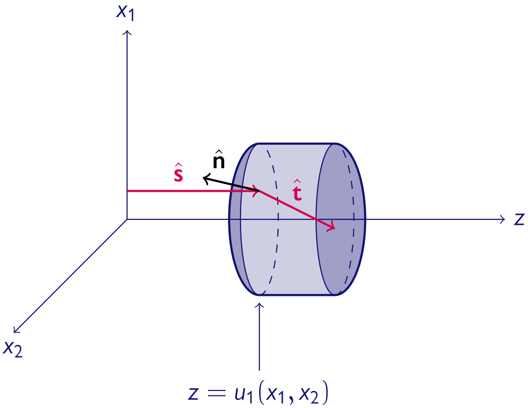
$$\hat{\mathbf{t}} = \alpha\hat{\mathbf{s}} + \beta\hat{\mathbf{n}}$$

with

$$\alpha = \frac{n_{\text{air}}}{n_{\text{glass}}} =: \eta$$

$$\beta = \eta c - \sqrt{1 - \eta^2(1 - c^2)}, \quad c := -\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$$

Snell's law at the first lens surface



- Snell's law:

$$\hat{\mathbf{t}} = \alpha\hat{\mathbf{s}} + \beta\hat{\mathbf{n}}$$

$$\alpha = \eta = \frac{n_{\text{air}}}{n_{\text{glass}}}$$

$$\beta = \eta c - \sqrt{1 - \eta^2(1 - c^2)}, \quad c := -\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$$

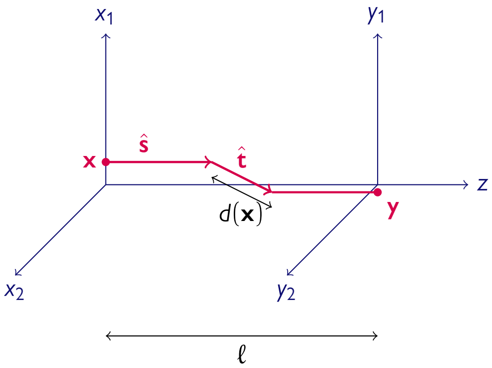
- First lens surface is $z = u_1(x_1, x_2)$, so

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}} \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \\ -1 \end{pmatrix}$$

- Together with $\hat{\mathbf{s}} = (0, 0, 1)^T$, we can write

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = (\eta - t_3) \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \end{pmatrix} = (\eta - t_3)\nabla u_1$$

Optical map



- We can now write the *optical map* **m** as:

$$\mathbf{y} = \mathbf{m}(\mathbf{x}) = \mathbf{x} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} d(\mathbf{x})$$

A point $\mathbf{x} \in S$ has image $\mathbf{y} = \mathbf{m}(\mathbf{x}) \in T$

- To find expression for $d(\mathbf{x})$, consider the *optical path length*:

$$L(\mathbf{x}) = n_{\text{air}} u_1(\mathbf{x}) + n_{\text{glass}} d(\mathbf{x}) + n_{\text{air}} u_2(\mathbf{y})$$

Theorem of Malus-Dupin: $L(\mathbf{x}) = L$

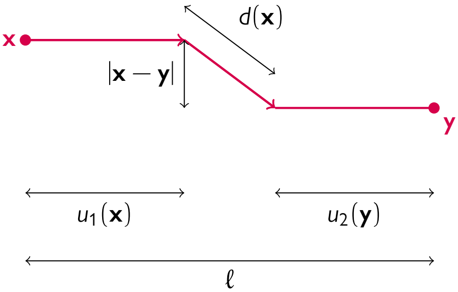
- Also:

$$\ell = u_1(\mathbf{x}) + \hat{\mathbf{s}} \cdot \hat{\mathbf{t}}\, d(\mathbf{x}) + u_2(\mathbf{y}) = u_1(\mathbf{x}) + t_3\, d(\mathbf{x}) + u_2(\mathbf{y})$$

- Conclusion:

$$\mathbf{m}(\mathbf{x}) = \mathbf{x} - \frac{L - n_{\text{air}}\ell}{n_{\text{glass}}} \frac{\nabla u_1(\mathbf{x})}{\sqrt{1 + (1 - \eta^2)|\nabla u_1(\mathbf{x})|^2}}$$

An equation for the freeform lens surfaces



- Pythagoras:

$$|\mathbf{x}-\mathbf{y}|^2 + (\ell - u_1(\mathbf{x}) - u_2(\mathbf{y}))^2 = (d(\mathbf{x}))^2$$

- Optical path length:

$$L = n_{\text{air}} u_1(\mathbf{x}) + n_{\text{glass}} d(\mathbf{x}) + n_{\text{air}} u_2(\mathbf{y})$$

- Rewrite as:

$$u_1(\mathbf{x}) + u_2(\mathbf{y}) = c(\mathbf{x},\mathbf{y})$$

$$c(\mathbf{x},\mathbf{y}) = \ell + \frac{n_{\text{air}}}{n_{\text{air}}^2 - n_{\text{glass}}^2} \cdot (L - n_{\text{air}}\ell) + \frac{n_{\text{glass}}}{n_{\text{air}}^2 - n_{\text{glass}}^2} \sqrt{(L - n_{\text{air}}\ell)^2 + (n_{\text{air}}^2 - n_{\text{glass}}^2)|\mathbf{x}-\mathbf{y}|^2}$$

- Optimal mass transport: c is cost function

A special choice for the lens surfaces

- We have found:

$$u_1(\mathbf{x}) + u_2(\mathbf{y}) = c(\mathbf{x}, \mathbf{y})$$

$$c(\mathbf{x}, \mathbf{y}) = \ell + \frac{n_{\text{air}}}{n_{\text{air}}^2 - n_{\text{glass}}^2} \cdot (L - n_{\text{air}}\ell) + \frac{n_{\text{glass}}}{n_{\text{air}}^2 - n_{\text{glass}}^2} \sqrt{(L - n_{\text{air}}\ell)^2 + (n_{\text{air}}^2 - n_{\text{glass}}^2)|\mathbf{x} - \mathbf{y}|^2}$$

- Many possible solutions. We choose a so-called *c-convex* pair:

$$u_1(\mathbf{x}) = \max_{\mathbf{y}\in T} \big(c(\mathbf{x}, \mathbf{y}) - u_2(\mathbf{y})\big), \qquad u_2(\mathbf{y}) = \max_{\mathbf{x}\in S} \big(c(\mathbf{x}, \mathbf{y}) - u_1(\mathbf{x})\big)$$

- It follows that

$$\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}) - \nabla u_1(\mathbf{x}) = \mathbf{0}$$

- Substitute $\mathbf{y} = \mathbf{m}(\mathbf{x})$ in previous equation and differentiate once more:

$$\mathrm{D}_{\mathbf{xx}} c(\mathbf{x}, \mathbf{m}(\mathbf{x})) + \mathrm{D}_{\mathbf{xy}} c(\mathbf{x}, \mathbf{m}(\mathbf{x}))\, \mathrm{D}\mathbf{m}(\mathbf{x}) - \mathrm{D}^2 u_1(\mathbf{x}) = \mathbf{0}$$

Matrix equation for the Jacobian of the optical map

- Previous slide

$$\mathrm{D}_{\mathbf{xx}}c(\mathbf{x},\mathbf{m}(\mathbf{x})) + \mathrm{D}_{\mathbf{xy}}c(\mathbf{x},\mathbf{m}(\mathbf{x}))\,\mathrm{D}\mathbf{m}(\mathbf{x}) - \mathrm{D}^2u_1(\mathbf{x}) = \mathbf{0}$$

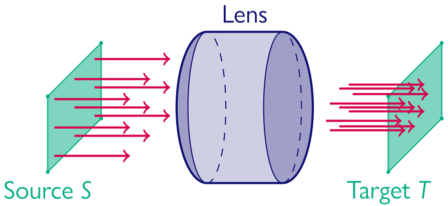
- Rewrite as

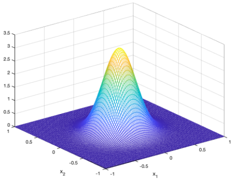
$$\underbrace{\mathrm{D}_{\mathbf{xy}}c(\mathbf{x},\mathbf{m}(\mathbf{x}))}_{\mathbf{C}}\,\mathrm{D}\mathbf{m}(\mathbf{x}) = \underbrace{\mathrm{D}^2u_1(\mathbf{x}) - \mathrm{D}_{\mathbf{xx}}c(\mathbf{x},\mathbf{m}(\mathbf{x}))}_{\mathbf{P}}$$

- Conclusion:

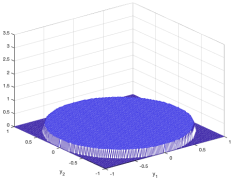
$$\mathbf{C}\,\mathrm{D}\mathbf{m}(\mathbf{x}) = \mathbf{P}$$

Energy conservation





Source distribution
$f(x_1, x_2)$
$(x_1, x_2) \in S$



Target distribution
$g(y_1, y_2)$
$(y_1, y_2) \in T$

- Global energy conservation:

$$\iint_S f(x_1, x_2)\, dx_1\, dx_2 = \iint_T g(y_1, y_2)\, dy_1\, dy_2$$

- Local energy conservation: For every $A \subseteq S$

$$\begin{aligned} &\iint_A f(x_1, x_2)\, dx_1\, dx_2 \\ &= \iint_{\mathbf{m}(A)} g(y_1, y_2)\, dy_1\, dy_2 \\ &= \iint_A g(\mathbf{m}(x_1, x_2)) \det(\mathrm{D}\mathbf{m}(x_1, x_2))\, dx_1\, dx_2 \end{aligned}$$

- Differential form:

$$f(\mathbf{x}) = g(\mathbf{m}(\mathbf{x})) \det(\mathrm{D}\mathbf{m}(\mathbf{x}))$$

Generalized Monge-Ampère equation

- Combine matrix equation

$$\mathbf{C}\,\mathrm{D}\mathbf{m}(\mathbf{x}) = \mathbf{P}$$

with local energy conservation

$$f(\mathbf{x}) = g(\mathbf{m}(\mathbf{x}))\,\det(\mathrm{D}\mathbf{m}(\mathbf{x}))$$

to find:

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \frac{\det(\mathbf{P})}{\det(\mathbf{C})} = \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

Generalized Monge-Ampère equation

- Transport boundary condition:

$$\mathbf{m}(\partial S) = \partial T$$

- For simpler optical systems (e.g. single reflector), optical map $\mathbf{m}$ has a potential, so $\mathbf{m} = \nabla\phi$
Then

$$\mathrm{D}\mathbf{m} = D^2\phi = \begin{pmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\,\partial x_2} \\ \frac{\partial^2\phi}{\partial x_2\,\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} \end{pmatrix}$$

- Differential equation:

$$\frac{\partial^2\phi}{\partial x_1^2}\frac{\partial^2\phi}{\partial x_2^2} - \left(\frac{\partial^2\phi}{\partial x_1\,\partial x_2}\right)^2 = \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

Monge-Ampère equation

Numerical method

1. Find the optical map $\mathbf{m} : S \longrightarrow T$ from

$$\det(\mathbf{C}\,\mathrm{D}\mathbf{m}) = \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

$$\mathbf{m}(\partial S) = \partial T$$

2. Find first lens surface $z = u_1(\mathbf{x})$ from

$$\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{m}(\mathbf{x})) - \nabla u_1(\mathbf{x}) = \mathbf{0}$$

3. Find second lens surface $\ell - z = u_2(\mathbf{y})$ from

$$u_2(\mathbf{m}(\mathbf{x})) = c(\mathbf{x}, \mathbf{m}(\mathbf{x})) - u_1(\mathbf{x})$$

Numerical method for finding the optical map

- Find mapping $\mathbf{m} : S \longrightarrow T$ such that

$$\det(\mathbf{C}\,\mathrm{D}\mathbf{m}) = \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$
$$\mathbf{m}(\partial S) = \partial T$$

- Break down in substeps
  We compute **P**, **b**, **m** such that

$$\mathbf{P} = \mathbf{C}\mathrm{D}\mathbf{m}$$
$$\det(\mathbf{P}) = \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}\det(\mathbf{C})$$
$$\mathbf{b}(\mathbf{x}) = \mathbf{m}(\mathbf{x}), \qquad \mathbf{x} \in \partial S$$

  b maps ∂S to ∂T

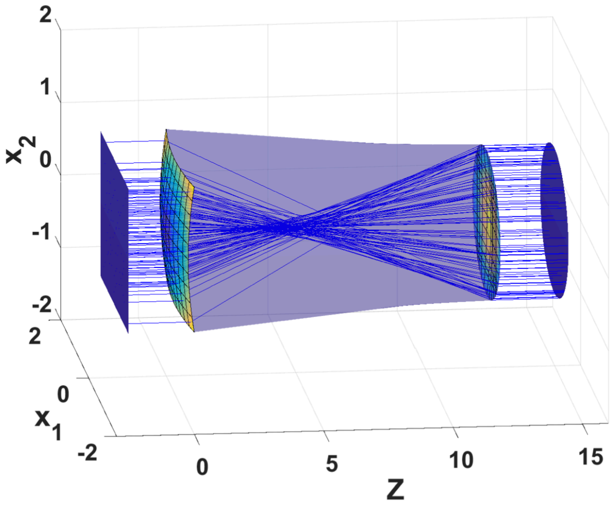
- Iterative procedure:
  1. Choose an initial guess $\mathbf{m}^0$
     Let n = 0
  2. Let: $J_I(\mathbf{m}, \mathbf{P}) = \frac{1}{2}\iint_S \|\mathbf{C}\mathrm{D}\mathbf{m} - \mathbf{P}\|^2 \, d\mathbf{x}$

     $\mathbf{P}^{n+1}$ minimizes $J_I(\mathbf{m}^n, \mathbf{P})$
     Constrained minimization problem
  3. Let: $J_B(\mathbf{m}, \mathbf{b}) = \frac{1}{2}\int_{\partial S} \|\mathbf{m} - \mathbf{b}\|^2 \, ds$

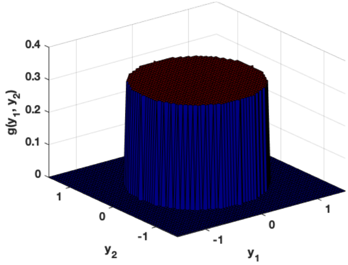
     $\mathbf{b}^{n+1}$ minimizes $J_B(\mathbf{m}^n, \mathbf{b})$
     Projection on boundary ∂T
  4. Let: $J(\mathbf{m}, \mathbf{P}, \mathbf{b}) = \alpha J_I(\mathbf{m}, \mathbf{P}) + (1 - \alpha) J_B(\mathbf{m}, \mathbf{b})$

     $\mathbf{m}^{n+1}$ minimizes $J(\mathbf{m}, \mathbf{P}^{n+1}, \mathbf{b}^{n+1})$
     Elliptic PDEs for m_1 and m_2 — FVM

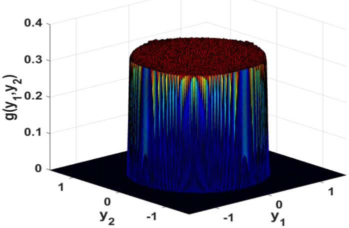
Numerical results for the laser beam shaping problem



Desired target distribution



Achieved



Alternative optical systems

c-convex pair:

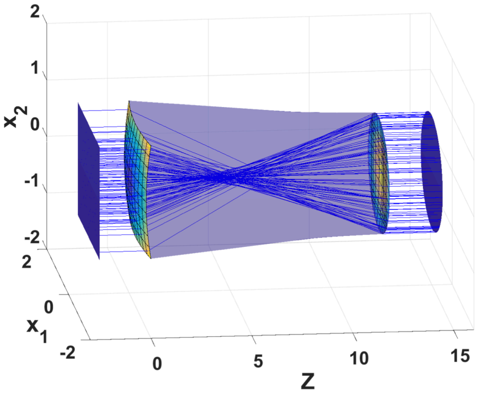
$$u_1(\mathbf{x}) = \max_{\mathbf{y}\in T}\left(c(\mathbf{x},\mathbf{y}) - u_2(\mathbf{y})\right)$$

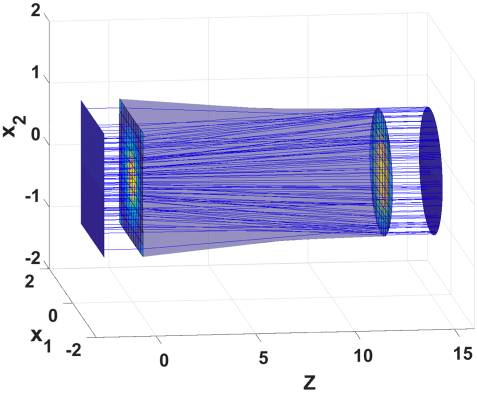
$$u_2(\mathbf{y}) = \max_{\mathbf{x}\in S}\left(c(\mathbf{x},\mathbf{y}) - u_1(\mathbf{x})\right)$$

c-concave pair

$$u_1(\mathbf{x}) = \min_{\mathbf{y}\in T}\left(c(\mathbf{x},\mathbf{y}) - u_2(\mathbf{y})\right)$$

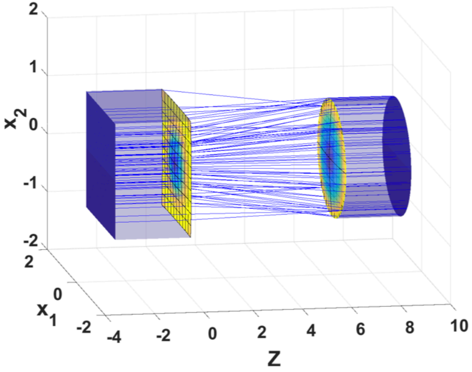
$$u_2(\mathbf{y}) = \min_{\mathbf{x}\in S}\left(c(\mathbf{x},\mathbf{y}) - u_1(\mathbf{x})\right)$$



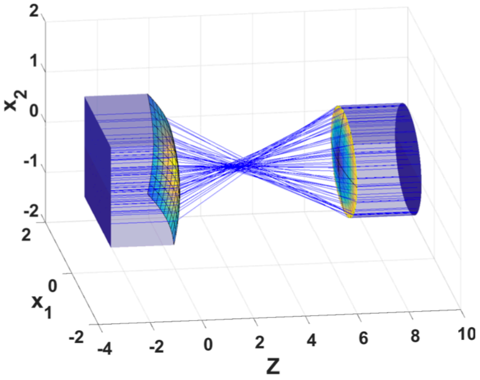


Alternative optical systems (cont)

Two lenses, c-convex

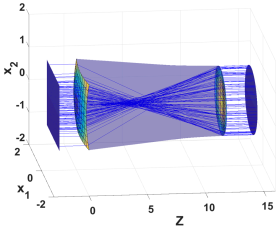


Two lenses, c-concave



Concluding remarks

- Modeled challenging optical design problem
- Based on principles of geometrical optics and energy conservation
- Generalized Monge-Ampère equation with transport boundary condition
- Least-squares solver



More information:

- Yadav, N. K., ten Thije Boonkkamp, J. H. M., IJzerman, W. L. (2019).
  Computation of double freeform optical surfaces using a Monge–Ampère solver: Application to beam shaping.
  Optics Communications, 439, 251-259. https://doi.org/10.1016/j.optcom.2019.01.069
- Yadav, N. K. (2018).
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  PhD thesis. Eindhoven University of Technology.

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