The initial and boundary value problem representing stretching motion of elastic materials

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This research is a joint work with Prof. Aiki of Japan Women’s University.
figure 1: Domain

\( u(t, x) \in \mathbb{R}^2 \) is the place of \( x \in [0, 1] \) at time \( t \).

The function \( u \) is defined on a domain \( Q(T) := (0, T) \times (0, 1), \ T > 0 \).

\( u \) satisfies the following initial and boundary value problem \( P \).
Initial and boundary value problem P

\[ \rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left( f(\varepsilon) \frac{\partial u}{\partial x} \right) = 0, \quad \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \text{ on } Q(T), \]

\[ \frac{\partial^i}{\partial x^i} u(t, 0) = \frac{\partial^i}{\partial x^i} u(t, 1) \text{ for } 0 \leq t \leq T \text{ and } i = 0, 1, 2, 3, \]

\[ u(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = v_0(x) \text{ for } 0 \leq x \leq 1, \]

where \( \rho \) is the density of the elastic material, \( \gamma \) is a positive constant, \( \varepsilon \) is the strain of the elastic material, \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, \( u_0 \) is the initial position and \( v_0 \) is the initial velocity.

The feature of problem P

By the definition of \( \varepsilon \), it is difficult to prove the uniqueness of solutions. We can not expect existence of a strong solution.
How to define the strain $\varepsilon$

Let us divide the elastic material onto $N$ parts (like fig.). $l_{N*}$ is the natural length of the each part. Put $l_{N*} = \frac{l_*}{N}$ where $l_*$ is the total natural length.

Put $u = (u_1, u_2)$ and $l_N = |u(t, l_{N*} + x) - u(t, x)|$; length of the each part.
How to define the strain $\varepsilon$

\[
\varepsilon_N = \frac{l_N - l_{N^*}}{l_{N^*}}
\]

\[
= \frac{|u(t, l_{N^*} + x) - u(t, x)|}{l_{N^*}} - 1
\]

\[
= \sqrt{\left(\frac{u_1(t, x + l_{N^*}) - u_1(t, x)}{l_{N^*}}\right)^2 + \left(\frac{u_2(t, x + l_{N^*}) - u_2(t, x)}{l_{N^*}}\right)^2} - 1
\]

\[
\rightarrow \sqrt{\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2} - 1 =: \left|\frac{\partial u}{\partial x}\right| - 1 \quad (N \to \infty).
\]

Thus, strain $\varepsilon$ is given by $\varepsilon = \left|\frac{\partial u}{\partial x}\right| - 1$. 
1. Main result for problem P (PDE)
2. Motivation, ODE model
3. Results for ODE model
4. Proof of main theorem (PDE)
5. Future work
Main result for problem P (PDE)

Notation \( H = L^2(0, 1)^2, \quad V = \{ z \in W^{2,2}(0, 1)^2 | z(0) = z(1), z_x(0) = z_x(1) \} \).

**Definition of weak solutions of problem P**

If a function \( u \) satisfies the following condition, we call \( u \) a weak solution of problem P.

\[
\begin{align*}
 u & \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), u(0) = u_0, \quad \text{and} \\
-\rho \int_{Q(T)} u_t \cdot \eta_t \, dx \, dt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} \, dx \, dt + \int_{Q(T)} f(\varepsilon)u_x \cdot \eta_x \, dx \, dt = \int_0^1 v_0 \cdot \eta(0) \, dx,
\end{align*}
\]

for any \( \eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V) \) with \( \eta(T) = 0 \).

**Theorem 1.**

If \( f: \mathbb{R} \to \mathbb{R} \); Lipschitz continuous, monotone increasing and \( f(0) = 0, u_0 \in V \) and \( v_0 \in H \), then problem P has a weak solution.
Our research aim is analysis of rotational motion of shape memory alloy ring. However, when we try to construct a model for this motion, we have a lot of difficulties.

### Difficulties of the modeling
- To consider gravity, force by the pulleys.
- To consider shape memory effect.
- To consider heat from the hot water.

As a first step, we establish the following model by ordinary differential equations.
Model 1 (When the elastic material is divided into $N$ parts)

$$m \frac{d^2 X_i}{dt^2} = f(\varepsilon_i) \cdot \frac{X_{i+1} - X_i}{l_i} - f(\varepsilon_{i-1}) \cdot \frac{X_i - X_{i-1}}{l_{i-1}},$$

$$f(\varepsilon_i) = \frac{\kappa}{2} \left( \varepsilon_i + \frac{1}{2} - \frac{1}{2(\varepsilon_i + 1)^2} \right), \quad \kappa > 0,$$

$$X_i(0) = X_{0i}, \quad \frac{dX_i(0)}{dt} = V_{0i}, \quad i = 0, 1, \ldots, N - 1.$$
The feature of Model 1 is the definition of the stress function $f$.

$$f(\varepsilon_i) = \frac{\kappa}{2} \left( \varepsilon_i + \frac{1}{2} - \frac{1}{2(\varepsilon_i + 1)^2} \right).$$

When we use Hook’s law $f(\varepsilon_i) = \kappa \varepsilon_i$ and calculate by the Euler method, the solution behavior is bad.

$\Rightarrow$ It means the preserved energy was broken.

So, we modify the model such that the structure preserving numerical method (SPNM) is available and obtain Model 1.
Numerical result by the Euler method \( (f(\varepsilon_i) = \kappa \varepsilon_i) \)
In order to get stable numerical results, we defined the stress as follows.

\[ f(\varepsilon_i) = \frac{\kappa}{2} \left( \varepsilon_i + \frac{1}{2} - \frac{1}{2(\varepsilon_i + 1)^2} \right). \]

Then Model 1 has the preserved energy as follows.

\[ \sum_{i=0}^{N-1} \left\{ \frac{m}{2} (V_i)^2 + l_{N*} \hat{f}(\varepsilon_i) \right\}, \quad \left( \hat{f}(\varepsilon_i) = \frac{\kappa}{2} \left( \frac{\varepsilon_i^2}{2} + \frac{\varepsilon_i}{2} + \frac{1}{2(1 + \varepsilon_i)} \right) \right\). \]

⇒ So, we applied the SPNM, since, in generally, this method is said to be able to get stable results.

On SPNM (the structure preserving numerical method),

- We can get the numerical scheme for Model 1 which does not break the preserved energy.
- By Banach’s fixed point theorem, we can prove the existence and uniqueness of numerical solutions and solutions of Model 1.
- We can also prove the convergence of numerical solutions.
We construct the following numerical scheme for Model 1 ($\Delta t > 0$).

**Numerical scheme (NS) for Model 1**

\[
\begin{align*}
\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} &= -\frac{\kappa}{4} \left\{ \varepsilon_i^{(n+1)} + \varepsilon_i^{(n)} + 1 - \frac{1}{(1 + \varepsilon_i^{(n+1)})(1 + \varepsilon_i^{(n)})} \right\} \frac{X_i^{(n+1)} - X_i^{(n+1)} + X_i^{(n)} - X_i^{(n)}}{|X_i^{(n+1)} - X_i^{(n+1)}| + |X_i^{(n)} - X_i^{(n)}|} \\
&\quad + \frac{\kappa}{4} \left\{ \varepsilon_i^{(n+1)} + \varepsilon_i^{(n)} + 1 - \frac{1}{(1 + \varepsilon_i^{(n+1)})(1 + \varepsilon_i^{(n)})} \right\} \frac{X_i^{(n+1)} - X_i^{(n+1)} + X_i^{(n)} - X_i^{(n)}}{|X_i^{(n+1)} - X_i^{(n+1)}| + |X_i^{(n)} - X_i^{(n)}|}, \\
\frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t} &= \frac{V_i^{(n+1)} + V_i^{(n)}}{2}.
\end{align*}
\]

In many cases, the numerical scheme is not unique when we use the SPNM. In fact, there exists the numerical scheme, not (NS).
Let $K$ be a positive integer, $\Delta t = \frac{T}{K}$.

**Theorem of Model 1 and (NS)**

1. If $X_{0i} \neq X_{0j} (i \neq j)$, $V_{0i} \in \mathbb{R}^2$, then Model 1 has the unique solution $X \in C^2([0, T]; \mathbb{R}^{2N})$.

2. For a large enough $K$, in other word, for a small enough $\Delta t$, (NS) has the unique solution $(X_{i(n+1)}, V_{i(n+1)}) \in \mathbb{R}^4$ for all $i = 0, 1, \cdots, N - 1$.

3. Let $X_K : [0, T] \rightarrow \mathbb{R}^{2N}$ be a function obtained by joining the solutions of (NS) $(X_{i(n+1)}, V_{i(n+1)}) \in \mathbb{R}^4$ with lines. For some positive constant $C$,

$$|X(t) - X_K(t)| \leq C|\Delta t|$$

for $0 \leq t \leq T$ and large $K$.

We show you the numerical result by using (NS).
The results of numerical calculation by using (NS)
Let us consider the case that Model 1 has periodic solutions in time. In these results, the solution is periodic in time, mathematically. ($R(t)$: distance between the origin and $X_i(t)$)

**figure 3**: Euler method

**figure 4**: SPNM

Now, we are trying to find a better scheme.
## Summary for Model 1

### Result about Model 1
- Construction of Model 1.
- The numerical scheme for Model 1 by using SPNM.
- The existence and uniqueness of solutions of Model 1 and (NS).
- The convergence of the numerical solution (convergence rate: $\Delta t$).

### Why the numerical result is not stable?
- Model 1 is not suitable for SPNM.

⇒ Future work
- Development of the proper scheme.
- To improve the accuracy rate of numerical calculations.
Modeling of the partial differential equation model

Model 1 (When the elastic body is divided into N parts)

\[
m \frac{d^2 X_i}{dt^2} = f(\varepsilon_i) \cdot \frac{X_{i+1} - X_i}{l_i} - f(\varepsilon_{i-1}) \cdot \frac{X_i - X_{i-1}}{l_{i-1}},
\]

\[
f(\varepsilon_i) = \frac{\kappa}{2} \left( \varepsilon_i + \frac{1}{2} - \frac{1}{2(\varepsilon_i + 1)^2} \right),
\]

\[
X_i(0) = X_{0i}, \quad \frac{dX_i(0)}{dt} = V_{0i}, \quad i = 0, 1, \cdots, N - 1.
\]

For the partial differential equation model, we suppose that the function \( f: \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous. Let \( N \to \infty \) in Model 1. And add the effect of keeping straightness \( \gamma u_{xxxx} \). Then, we get the problem P.
Model 2 (Initial and boundary problem P)

\[
\rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left( f(\varepsilon) \frac{\partial u}{\partial x} \right) = 0, \quad \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \text{ on } Q(T), \\
\frac{\partial^i}{\partial x^i} u(t, 0) = \frac{\partial^i}{\partial x^i} u(t, 1) \text{ for } 0 \leq t \leq T \text{ and } i = 0, 1, 2, 3, \\
u(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = v_0(x) \text{ for } 0 \leq x \leq 1.
\]

Definition of weak solutions of problem P

If a function \( u \) satisfies the following condition, we call \( u \) a weak solution of problem P.

\( u \in W^{1,\infty}(0, T; H) \cap L^{\infty}(0, T; V), u(0) = u_0, \) and

\[-\rho \int_{Q(T)} u_t \cdot \eta_t \, dx \, dt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} \, dx \, dt + \int_{Q(T)} f(\varepsilon) u_x \cdot \eta_x \, dx \, dt = \int_{0}^{1} v_0 \cdot \eta(0) \, dx,\]

for any \( \eta \in W^{1,2}(0, T; H) \cap L^{2}(0, T; V) \) with \( \eta(T) = 0. \)
Proof of main theorem (PDE)

Main theorem

Let $H := L^2(0, 1)^2$, $V := \{ z \in W^{2,2}(0, 1)^2 | z(0) = z(1), z_x(0) = z_x(1) \}$.

**Definition of weak solutions of problem P**

If a function $u$ satisfies the following condition, we call $u$ a weak solution of problem P.

$$u \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), u(0) = u_0, \text{ and }$$

$$-\rho \int_{Q(T)} u_t \cdot \eta_t dx dt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dx dt + \int_{Q(T)} f(\varepsilon) u_x \cdot \eta_x dx dt = \int_{0}^{1} v_0 \cdot \eta(0) dx,$$

for any $\eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$ with $\eta(T) = 0$.

**Theorem 1.**

If $f: \mathbb{R} \to \mathbb{R}$ ; Lipschitz continuous, monotone increasing and $f(0) = 0$, $u_0 \in V$ and $v_0 \in H$, then problem P has a weak solution.
(Sketch of the proof) Apply the Galerkin method.

(Step 1) $V$ is the separable Hilbert space. So, we can choose $\{\varphi_n\}_{n \in \mathbb{Z}_0^+}$ as the complete orthonormal system of $V$ normalized in $H$. Let $V_n$ be the closed linear subspace generated by $\varphi_1, \varphi_2, \cdots, \varphi_n$. We consider the following approximation problem $P_n$.

**Approximation problem $P_n$**

Suppose $u_{0n} \in V_n$, $v_{0n} \in V_n$, $u_{0n} \to u_0$ in $V$, $v_{0n} \to v_0$ in $H(n \to \infty)$.

Find $u_n = \sum_{k=1}^{n} a_k^{(n)}(t) \varphi_k(x)$ satisfying for each $j = 1, 2, \cdots, n$,

$$
\int_0^1 u_{ntt}(t) \varphi_j dx + \gamma \int_0^1 u_{nx}(t) \varphi_{jx} dx + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \varphi_{jx} dx = 0, \quad \forall t \in [0, T]
$$

$u_n(0) = u_{0n}$, $u_{nt}(0) = v_{0n}$, $\varepsilon_n = |u_{nx}| - 1 (n = 1, 2, \cdots)$.

To prove the existence of a solution of the problem $P_n$, we consider the following initial value problem $I_n$ for the ordinary differential equation.
In the approximation problem $P_n$, we put $u_n = \sum_{k=1}^{n} a_k^{(n)}(t) \varphi_k(x)$, then we get the following initial value problem $I_n$.

**Initial value problem $I_n$**

$$
\rho \frac{d^2 a^{(n)}}{dt^2} = -F(a^{(n)}) - G(a^{(n)}), \quad a^{(n)}(0) = a_0^{(n)}, \quad \frac{da^{(n)}}{dt} = b_0^{(n)}.
$$

For any $n \in \mathbb{Z}_{>0}$, we can prove the existence and uniqueness of a solution for the problem $I_n$ by using Banach’s fixed point theorem. Therefore, $P_n$ has the unique solution $u_n \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$.

**Remark 1.**

$$
\int_0^1 u_{ntt}(t) \varphi_j dx + \gamma \int_0^1 u_{nxx}(t) \varphi_{jxx} dx + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \varphi_{jx} dx = 0, \quad \forall t \in [0, T]
$$

$$
\iff
$$

$$
\int_0^1 u_{ntt}(t) \eta dx + \gamma \int_0^1 u_{nxx}(t) \eta_{xx} dx + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \eta_x dx = 0, \quad \forall \eta \in V_n
$$

(\#)
(Step 2) Put $\eta = u_{nt}$ in (⑵), then, for any $n \in \mathbb{Z}_{>0}$, the solution $u_n$ of approximation problem $P_n$ has the following preserved energy

$$\frac{\rho}{2} \int_0^1 u_{nt}^2(t) \, dx + \frac{\gamma}{2} \int_0^1 u_{nxx}^2(t) \, dx + \frac{1}{2} \int_0^1 \hat{g}(u_{nx}^2(t)) \, dx,$$

where $\hat{g}$ is the primitive of $f$ and satisfies

$$\hat{g}(1) = 0, \quad \hat{g} \geq 0.$$

Since $f$ is Lipschitz continuous, monotone increasing and $f(0) = 0$, by using the preserved energy, we get the following uniform estimates.

$$\exists C > 0 \text{ s.t. } |u_{nxx}(t)|_H \leq C, \quad |u_{nt}(t)|_H \leq C, \quad |u_{nx}(t)|_H \leq C \text{ for } \forall t \in [0, T], \forall n \in \mathbb{Z}_{>0}.$$

Hence, $\{u_n\}$ is bounded in $L^\infty(0, T; V)$ and $\{u_{nt}\}$ is bounded in $L^\infty(0, T; H)$. 
Proof of main theorem (PDE)

(Step 3) We put $X = \{ z \in W^{1,2}(0,1)^2 | z(0) = z(1) \}$. Then $V \subset X \subset H$, the imbedding $V \subset X$ is compact, and $V$ and $H$ are reflexible. By the uniform estimates and applying the Aubin compact theorem, we get the following convergences.

$\exists \{ n_j \} \subset \{ n \}$, $\exists u \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$ s.t.

$u_{n_j} \to u$ weakly in $L^2(0, T; V)$, in $L^2(0, T; X)$,

$u_{n_j t} \to u_t$ weakly in $L^2(0, T; H)$ as $j \to \infty$.

Since $u_{n_j}$ satisfies the definition of the weak solution for the problem $P_{n_j}$, we have

$$\rho \int_0^1 u_{n_j tt} \eta dx + \gamma \int_0^1 u_{n_j xx} \eta_{xx} dx + \int_0^1 f(\varepsilon_{n_j}) u_{n_j x} \eta_t dx = 0 \quad (\forall \eta \in V_{n_j}, \forall t \in [0, T]).$$

By integrating both sides on $[0, T]$, for any $\eta \in L^2(0, T; V_{n_j})$

$$\int_0^T \rho \int_0^1 u_{n_j tt} \eta dx dt + \int_0^T \gamma \int_0^1 u_{n_j xx} \eta_{xx} dx dt + \int_0^T \int_0^1 f(\varepsilon_{n_j}) u_{n_j x} \eta_t dx dt = 0. \quad (*)$$
Remark 2.

\( \forall \eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V) \) with \( \eta(T) = 0 \) \( \exists \{\eta_{n_j}\} \subset W^{1,2}(0, T; V) \) s.t.

\( \eta_{n_j} \in L^2(0, T; V_{n_j}), \eta_{n_j}(0) \to \eta(0) \text{ in } H, \eta_{n_j}(T) = 0 \) and

\( \eta_{n_j t} \to \eta_t \text{ in } L^2(0, T; H), \eta_{n_j} \to \eta \text{ in } L^2(0, T; V) \) \( (j \to \infty) \).

We put \( \eta = \eta_{n_j} \) in (*)

The 1st term: Integration by parts implies the following equation

\[
\rho \int_0^T \int_0^1 u_{n_j tt} \eta_{n_j} \, dx \, dt = \rho \int_0^1 \left\{ [u_{n_j t}(t, x) \eta_{n_j}(t, x)]^1_0 - \int_0^T u_{n_j t} \eta_{n_j t} \, dt \right\} \, dx
\]

\[
= \rho \int_0^1 \left\{ u_{n_j t}(T, x) \eta_{n_j}(T, x) - u_{n_j t}(0, x) \eta_{n_j}(0, x) - \int_0^T u_{n_j t} \eta_{n_j t} \, dt \right\} \, dx
\]

\[
= -\rho \int_0^1 u_{n_j t}(0, x) \eta_{n_j}(0, x) \, dx - \int_0^1 \int_0^T u_{n_j t} \eta_{n_j t} \, dt \, dx.
\]

In the second equality, the first term vanishes by using Remark 2, \( \eta_{n_j}(T) = 0 \).
By using Remark 2, \( \eta_n(j)(0) \to \eta(0) \) in \( H \) \((j \to \infty)\), we have

\[
\rho \int_0^T \int_0^1 u_{n_jtt} \eta_n \, dx \, dt = -\rho \int_0^1 u_{n_jt}(0, x) \eta_n(0, x) \, dx - \int_0^1 \int_0^T u_{n_jt} \eta_n \, dt \, dx \\
\to -\rho \int_0^1 v_0 \eta(0) \, dx - \rho \int_{Q(T)} u_t \eta \, dx dt \quad (j \to \infty).
\]

The 2nd term: By \( u_{n_j} \to u \) weakly in \( L^2(0, T; V) \), \( \eta_{n_j} \to \eta \) strongly in \( L^2(0, T; H) \) as \( j \to \infty \), \( |u_{n_j}(t)|_V \leq C \) \((\forall t \in [0, T], \forall j = 1, 2, \cdots)\), then, we have

\[
\gamma \int_0^T \int_0^1 u_{n_jxx} \eta_{n_jxx} \, dx \, dt \to \gamma \int_0^T \int_0^1 u_{xx} \eta_{xx} \, dx \, dt \quad (j \to \infty).
\]

**Remark 3.**

\[
|u_{n_jx}(t, x)| \leq |u_{n_j}(t)|_V \quad (\forall j = 1, 2, \cdots, \forall t \in [0, T], \forall x \in [0, 1]),
\]

\[
|u_x(t, x)| \leq |u(t)|_V \quad (\forall t \in [0, T], \forall x \in [0, 1]).
\]
The 3rd term: By \(|u_{njx}(t,x)| \leq |u_{nj}(t)|_V (\forall j = 1, 2, \cdots, \forall t \in [0, T], \forall x \in [0, 1])\),
\(|u_x(t,x)| \leq |u(t)|_V (\forall t \in [0, T], \forall x \in [0, 1])\), \(u_{nj} \to u\) in \(L^2(0, T; X)\) \((j \to \infty)\) and Remark 2, we have the following convergence;

\[
\int_0^T \int_0^1 f(\varepsilon_{nj}) u_{njx} \eta_{njx} dx dt \to \int_0^T \int_0^1 f(\varepsilon) u_x \eta_x dx dt (j \to \infty).
\]

Thus, we have

\[
-\rho \int_{Q(T)} u_t \eta_t dx dt + \gamma \int_{Q(T)} u_{xx} \eta_{xx} dx dt + \int_{Q(T)} f(\varepsilon) u_x \eta_x dx dt = \rho \int_0^1 v_0 \eta(0) dx.
\]

Hence, \(u\) satisfies the definition of the weak solution of problem P. \(\square\)

In this way, the existence of a weak solution is proved.
Future work

**Future work (PDE)**
- Prove the uniqueness of weak solutions.
- Observe the solution behavior by using numerical calculation.
- Prove the convergence of the numerical solutions.

**Future work (ODE)**
- Development of the proper scheme.
- To improve the accuracy rate of numerical calculations.

Thank you for your attention!