Linearization and domain decomposition methods for porous media flows

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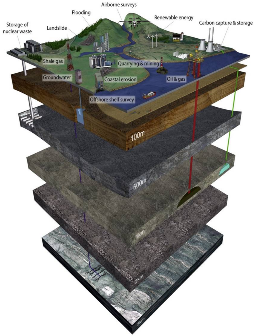
Motivation

Societal relevance

- Goundwater management;
- Geological CO_2 storage;
- Oil recovery.

Mathematical challenges

- Nonlinear, possibly degenerate equations;
- Heterogeneous media (jump-type discontinuities).



British Geological Survey
https://www.bgs.ac.uk/research/images/manageTheSubsurface.jpg

Outline

- Motivation
- Unsaturated/partially saturated flow
- Linear iterative schemes
- Domain decomposition

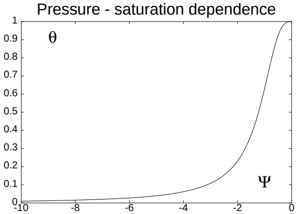
The Richards equation

$$\partial_t \theta = \nabla \cdot (k_r(\theta)\nabla(\psi + z))$$

*Typical** θ–ψ, $m \in (0, 1)$:

Leverett: $\theta = \begin{cases} |\psi|^{-m}, & \psi < -1 \\ 1, & \psi \geq -1 \end{cases}$

van Genuchten: $\theta = (1 + |\psi|^{1/(1-m)})^{-m}$

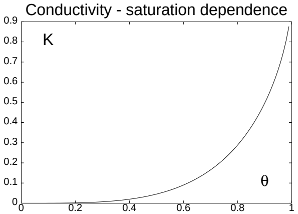


Typical k_r–(θ, ψ)

Brooks–Corey: $K = K_s \theta^{\lambda}$

Mualem:

$$K = K_s \theta^{\frac{1}{2}} \left[\int_0^{\theta} \frac{1}{\psi(s)} ds \bigg/ \int_0^{1} \frac{1}{\psi(s)} ds \right]^2$$



*Standard, equilibrium-based model; non-equilibrum features (hysteresis, dynamic capillarity) can be included

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Pressure based formulation

$$\partial_t \theta(\psi) - \nabla \cdot (k_r(\theta(\psi))\nabla(\psi + z)) = 0$$

Features:

- Doubly degenerate problem, slow and fast diffusion
  ($\theta' = 0,$ or $\theta' = \infty, \quad k_r = 0$)
- Change of type, free boundaries, lack in regularity, infinite gradients
- ψ-more regular than θ

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Results:

- Existence, uniqueness of weak solutions: van Duijn, Peletier '82, Alt, Luckhaus '83, Otto '96 [...]
- Numerical schemes*: Euler implicit + FEM/MFEM, FV, MPFA, DG, GDM

* Nochetto, Verdi '88, Jäger, Kačur '91, '95, Arbogast, Wheeler, Zhang '96, Riviere, Wheeler, Banas '00, '02, Radu, P., Knabner '04, '09, Wheeler, Yotov '05, Eymard, Vohralik, Hilhorst '06, Klausen, Radu '08, Bause '08, Vohralik, Wheeler '14, Cancés, P., Vohralik '14, Droniou et al. '20

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Linear iterative schemes

Pressure-based formulation

$$\partial_t \theta(\psi) - \nabla \cdot (k_r(\theta(\psi))\nabla(\psi + z)) = 0$$

Regularization parameter: $\varepsilon \geq 0$

$$\theta_\varepsilon(\psi) = \int_0^\psi \min\{\max\{\varepsilon, \theta'(z)\}, \frac{1}{\varepsilon}\} dz \text{ and } k_{r,\varepsilon}(\psi) = k_r(\psi) + \varepsilon$$

Note: If $\varepsilon > 0$ then $k_r \geq \varepsilon > 0$ and $0 < \varepsilon \leq \theta' \leq \frac{1}{\varepsilon} < \infty$

Linear iterative schemes

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Euler implicit discretisation:

$n \in \mathbb{N},\ \tau = T/n,\ t_k = k\tau,\ k = \overline{1, n},\ \psi(t_k) \approx \psi^k$

$$\partial_t \theta(\psi)(t_k) \approx \frac{\theta_\varepsilon(\psi^k) - \theta_\varepsilon(\psi^{k-1})}{\tau}$$

$$\theta_\varepsilon(\psi^k) - \theta_\varepsilon(\psi^{k-1}) \quad = \quad \tau \nabla \cdot \left(k_{r,\varepsilon}(\theta_\varepsilon(\psi^k))\nabla(\psi^k + z)\right)$$

$+$ your favourite spatial discretisation (FV, MPFA, FEM, MFEM, DG, GDM)

Iterative schemes: Newton

With $i > 0$, assume $\psi^{k,i-1}$ and ψ^{k-1} given. Solve

$$\begin{aligned}
&\theta_\varepsilon'(\psi^{k,i-1})(\psi^{k,i} - \psi^{k,i-1}) + \theta_\varepsilon(\psi^{k,i-1}) - \theta_\varepsilon(\psi^{k-1}) \\
&= \tau\nabla\cdot\big(k_{r,\varepsilon}(\theta_\varepsilon(\psi^{k,i-1}))\nabla(\psi^{k,i} + z)\big) \\
&\quad +\tau\nabla\cdot\big(k_{r,\varepsilon}'(\theta_\varepsilon(\psi^{k,i-1}))\theta_\varepsilon'(\psi^{k,i-1})\nabla(\psi^{k,i-1} + z)(\psi^{k,i} - \psi^{k,i-1})\big).
\end{aligned}$$

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&\theta'_\varepsilon(\psi^{k,i-1})(\psi^{k,i} - \psi^{k,i-1}) + \theta_\varepsilon(\psi^{k,i-1}) - \theta_\varepsilon(\psi^{k-1}) \\
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&\quad + \tau\nabla\cdot\left(k'_{r,\varepsilon}(\theta_\varepsilon(\psi^{k,i-1}))\theta'_\varepsilon(\psi^{k,i-1})\nabla(\psi^{k,i-1} + z)(\psi^{k,i} - \psi^{k,i-1})\right).
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[1] Radu, P, Knabner '05

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\end{aligned}$$

Rem: Quadratic convergence is guaranteed (mathematically rigorous) if[1]:

1. θ' is Lipschitz, the model is regularised, $\varepsilon > 0$,
2. the initial guess is $\psi^{k,0} = \psi^{k-1}$, and
3. the time step is subject to severe constraints, $\tau = O(\varepsilon^3 h^d)$ (h being the mesh size).

[1] Radu, P, Knabner '05

Iterative schemes: Picard-type

With $i > 0$, assume $\psi^{k,i-1}$ and ψ^{k-1} given. Solve[1]

$$
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&= \tau \nabla \cdot \big(k_{r,\varepsilon}(\theta_\varepsilon(\psi^{k,i-1}))\nabla(\psi^{k,i} + z)\big).
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[1]Celia et al. '90

[2]Radu, P, Knabner '05

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Iterative schemes: Picard-type

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\end{aligned}
$$

Rem: Convergence is linear. It is guaranteed (mathematically rigorous) if[2]:

1. the model is regularised, $\varepsilon > 0$,
2. the initial guess is $\psi^{k,0} = \psi^{k-1}$, and
3. the time step is subject to severe constraints, $\tau = O(\varepsilon^3 h^d)$ (h being the mesh size).

[1] Celia et al. '90

[2] Radu, P, Knabner '05

Iterative schemes: L-scheme

With $i > 0$, assume $\psi^{k,i-1}$ and ψ^{k-1} given. Solve[1]

$$L(\psi^{k,i} - \psi^{k,i-1}) + \theta(\psi^{k,i-1}) - \theta(\psi^{k-1}) = \tau \nabla \cdot \big(k_r(\theta(\psi^{k,i-1}))\nabla(\psi^{k,i} + z)\big),$$

where $L \geq \frac{1}{2} L_\theta$ (assume θ - Lipschitz and increasing, i.e. $0 \leq \theta' \leq L_\theta$)

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[1] P, Yong '97, P, Radu, Knabner '04, Radu, List '16, Radu et al '15, '18

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Rem: Convergence is linear: fixed point, less restrictive conditions!

1. The time step is subject to mild constraints, $\tau \leq C$ (not depending on h the mesh size!);
2. No regularisation needed, $\varepsilon = 0$;
3. No restriction for the initial guess;
4. Convergence rate $\rho = \|\psi^{k,i} - \psi^k\| / \|\psi^{k,i-1} - \psi^k\|$ decreases with L, $\rho = L/(L + C\tau)$;
5. Works for Hölder-continuous θ (e.g. $\theta(\psi) = C\psi^\alpha$ for some $\alpha \in (0,1)$), worse convergence rate.

[1] P, Yong '97, P, Radu, Knabner '04, Radu, List '16, Radu et al '15, '18

Iterative schemes: Modified L-scheme

Assume that $\Lambda > 0$ exists s.t. $\|\psi^k - \psi^{k-1}\|_{L^\infty(\Omega)} \leq \tau\Lambda$. With $\psi^{k,0} = \psi^{k-1}$ and for $i > 0$, given $\psi^{k,i-1}$ and ψ^{k-1}, solve[1]

$$L_k^i(\psi^{k,i} - \psi^{k,i-1}) + \theta_\varepsilon(\psi^{k,i-1}) - \theta_\varepsilon(\psi^{k-1}) = \tau\nabla\cdot\big(k_{r,\varepsilon}(\theta_\varepsilon(\psi^{k,i-1}))\nabla(\psi^{k,i} + z)\big),$$

where $L_k^i = \max\{\theta'_\varepsilon(\psi^{k,i-1}) + \mathcal{M}\tau, 2\mathcal{M}\tau\}$ and $\mathcal{M} \geq \Lambda\|\theta''\|_\infty$.

[1] Mitra, P. '19

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Lemma: Assume $\theta \in C^2$, smooth initial data and that the problem is regular(ized). One has $\|\psi^{k,i} - \psi^k\|_{L^\infty(\Omega)} \leq \tau\Lambda$ for all i.

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Lemma: Assume $\theta \in C^2$, smooth initial data and that the problem is regular(ized). One has $\|\psi^{k,i} - \psi^k\|_{L^\infty(\Omega)} \leq \tau\Lambda$ for all i.

Theorem: The modified L-scheme converges linearly in H^1 with convergence rate

$$\rho = \sqrt{\mathcal{M}/(\mathcal{M} + C_1)},$$

for some $C_1 > 0$. In the non-degenerate case a $C_2 > 0$ exists s.t.

$$\rho = \min\{\rho, C_2\tau\}.$$

[1] Mitra, P. '19

Numerical example

Richards equation (2D)

- Domain and coefficient functions ($m = \frac{2}{3}$, $n = \frac{1}{1-m}$):

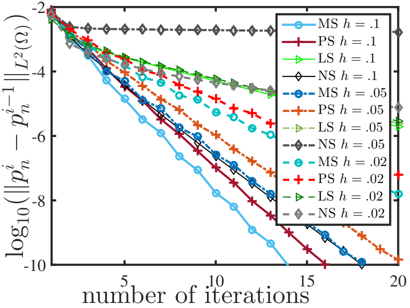
$$\Omega = (0,1) \times (0,1),\ k_r(\theta) = \theta^{1/2}\left(1 - (1 - \theta^{1/m})^m\right)^2,\ \theta(\psi) = \begin{cases} (1 + (-\psi)^n)^{-m}, & \psi < 0, \\ 1, & \psi \geq 0. \end{cases}$$

- Boundary/initial data and source term s.t. the solution is

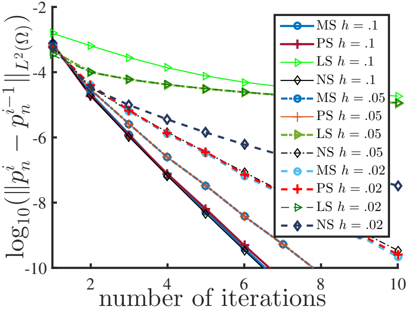
$$\psi(t,x,y) = 1 - (1 + t^2)(1 + x^2 + y^2).$$

Influence of the spatial mesh

- For $t = 0.5$, $\mathcal{M} = 10$, $L = 1$



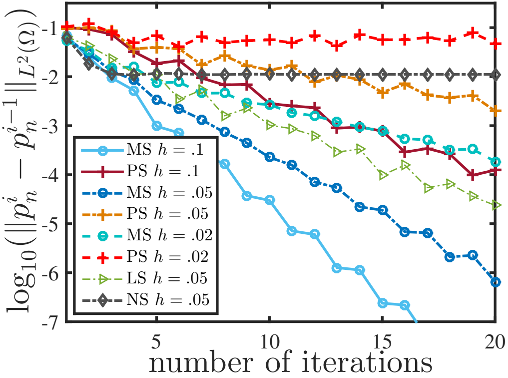
$\tau = 0.01$



$\tau = 0.001$

Influence of the spatial mesh

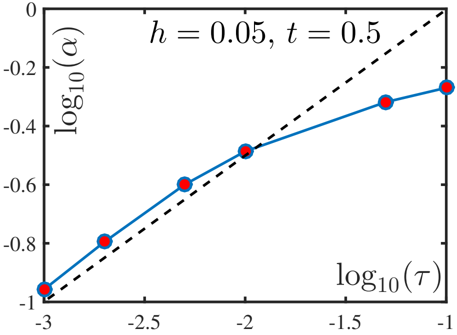
- For $t = 0.5$, $\mathcal{M} = 10$, $L = 1$



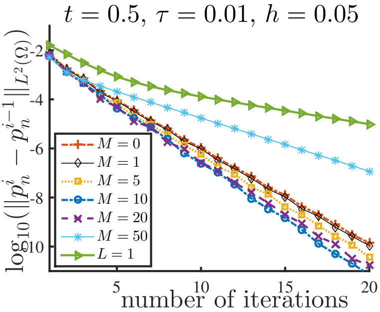
$\tau = 0.1$

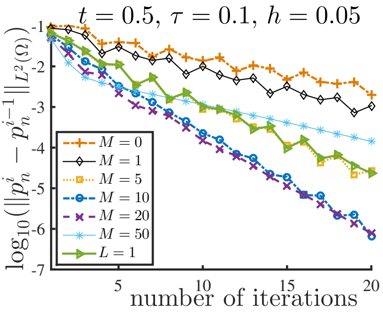
Influence of the timestep

- For $t = .5$, $h = 0.05$, $\mathcal{M} = 10$



Influence of $\mathcal{M}$





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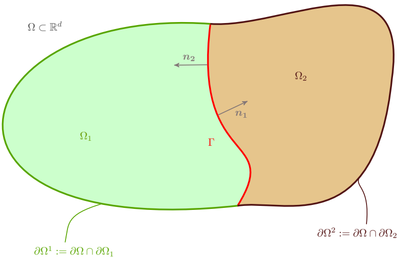
Domain decomposition

Why?

- *Same model* but different properties (e.g. $k_{r,1} \neq k_{r,2}$)
- Different models on $\Omega_1 \neq \Omega_2$
- Parallelization
- Boundary/interface conditions: possible discontinuities in the capillary pressure.



Wheeler, Yotov '98, Gander, Vanderwalle '07, Skogestad, Keilegavlen, Nordbotten '13, 16, Berninger et al. '15

Simplified problem, two sub-domains

Find a pair (ψ_1, ψ_2) satisfying

$$\begin{array}{rcll} \partial_t \theta_\ell(\psi_\ell) & = & \nabla \cdot (k_{r,\ell}(\theta_\ell(\psi_\ell)) \nabla \psi_\ell) & \text{in } \Omega_\ell \times [0, T] \\ k_{r,1}(\theta_1(\psi_1)) \nabla \psi_1 \cdot \mathbf{n_1} & = & -k_{r,2}(\theta_2(\psi_2)) \nabla \psi_2 \cdot \mathbf{n_2} & \text{on } \Gamma \times [0, T] \\ \psi_1 & = & \psi_2 & \text{on } \Gamma \times [0, T] \\ \psi_\ell & = & 0 & \text{on } \partial\Omega_\ell \times [0, T] \end{array}$$

and the initial conditions $\psi_\ell(\cdot, 0) = \psi_{\ell,0}$ on Ω_ℓ $(\ell = 1, 2)$.

Time discretization

Let $n \in \mathbb{N}$, $\tau := \dfrac{T}{n}$, $t^k := k\tau$, $k \in 0, 1, .., n$

ψ_ℓ^k: pressure on Ω_ℓ at time-step t^k

Time discretization

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ψ_ℓ^k: pressure on Ω_ℓ at time-step t^k

Semi-discrete problem

Given $(\psi_1^{k-1}, \psi_2^{k-1})$, find (ψ_1^k, ψ_2^k) s.t.

$$
\begin{aligned}
\theta_\ell(\psi_\ell^k) - \theta_\ell(\psi_\ell^{k-1}) - \tau\nabla\cdot\left(k_{r,\ell}(\theta_\ell(\psi_\ell^k))\nabla\psi_\ell^k\right) &= 0 && \text{in } \Omega_\ell \\
k_{r,1}(\theta_1(\psi_1^k))\nabla\psi_1^k\cdot\mathbf{n_1} + k_{r,2}(\theta_2(\psi_2^k))\nabla\psi_2^k\cdot\mathbf{n_2} &= 0 && \text{on } \Gamma \\
\psi_1^k &= \psi_2^k && \text{on } \Gamma \\
\psi_\ell^k &= 0 && \text{on } \partial\Omega_\ell
\end{aligned}
$$

Challenge: nonlinear, coupled problems

Decoupling[1] ($\lambda > 0$)

Assume ψ_ℓ^{k-1} given. Let

$$\begin{aligned} \psi_\ell^{k,0} &:= \psi_\ell^{k-1} \\ g_\ell^{k,0} &:= -\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^{k-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k-1} - \lambda\psi_\ell^{k-1} \end{aligned}$$

Iteration over i:

For $i \in \mathbb{N}$ given $\{\psi_\ell^{k,j}\}_{j=0}^{i-1}$ and $\{g_\ell^{k,j}\}_{j=0}^{i-1}$. Find $(\psi_1^{k,i}, \psi_2^{k,i})$ s.t. $(\ell = 1, 2)$

$$\begin{aligned} \theta_\ell(\psi_\ell^{k,i}) - \theta_\ell(\psi_\ell^{k-1}) &= \tau\nabla \cdot \left(k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i}))\nabla\psi_\ell^{k,i}\right) && \text{in } \Omega_\ell \\ -\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} &= \lambda\psi_\ell^{k,i} + g_\ell^{k,i} && \text{in } \Gamma \\ \text{where } g_\ell^{k,i} &= -2\lambda\psi_{3-\ell}^{k,i-1} - g_{3-\ell}^{k,i-1}. \end{aligned}$$

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[1] Lions '88

Linearization (L^{i}DD scheme)

Add the ("asymptotically zero") term

$$L_\ell^{k,i}\psi_\ell^{k,i} - L_\ell^{k,i}\psi^{k,i-1} \to 0 \quad \text{if} \quad \psi_\ell^{k,i} \to \psi_\ell^k$$

with $L_\ell^{k,i} = \max\{\theta'_\ell(\psi_\ell^{k,i-1}) + \mathcal{M}\tau, 2\mathcal{M}\tau\}$.
Note: In the LDD-scheme $L_\ell^{k,i} = L_\ell \geq L_\theta$ (constant w.r.t. k, i) and $\mathcal{M} = 0$, while $g_\ell^{k,i} = -k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} - \lambda\psi_\ell^{k,i}$

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Linear iteration (iteration index i)

For $i \in \mathbb{N}$ given $\{\psi_\ell^{k,j}\}_{j=0}^{i-1}$ and $\{g_\ell^{k,j}\}_{j=0}^{i-1}$, find $(\psi_1^{k,i}, \psi_2^{k,i})$ s.t. $(\ell = 1, 2)$

$$\begin{aligned}
L_\ell^{k,i}\psi_\ell^{k,i} - \tau\nabla\cdot\left(k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\nabla\psi_\ell^{k,i}\right) &= L_\ell^{k,i}\psi_\ell^{k,i-1} - (\theta_\ell(\psi_\ell^{k,i-1}) - \theta_\ell(\psi_\ell^{k-1})),\\
-\tau^{\frac{1}{2}}k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} &= \lambda\psi_\ell^{k,i} + g_\ell^{k,i},\\
\text{where } g_\ell^{k,i} &= -2\lambda\psi_{3-\ell}^{k,i-1} - g_{3-\ell}^{k,i-1}.
\end{aligned}$$

Here $\psi_\ell^{k,0} := \psi_\ell^{k-1}$, $g_\ell^{k,0} := -\tau^{\frac{1}{2}}k_{r,\ell}(\theta_\ell(\psi_\ell^{k-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k-1} - \lambda\psi_\ell^{k-1}$

Convergence idea

If $\psi_\ell^{k,i} \to \psi_\ell^k$ and $g_\ell^{k,i} \to g_\ell^k$ then

$$\begin{aligned} L_\ell^{k,i}\psi_\ell^{k,i} - \tau\nabla\cdot\left(k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\nabla\psi_\ell^{k,i}\right) &= L_\ell^{k,i}\psi_\ell^{k,i-1} - (\theta_\ell(\psi_\ell^{k,i-1}) - \theta_\ell(\psi_\ell^{k-1})), \\ -\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} &= \lambda\psi_\ell^{k,i} + g_\ell^{k,i}, \\ \text{where } g_\ell^{k,i} &= -2\lambda\psi_{3-\ell}^{k,i-1} - g_{3-\ell}^{k,i-1}. \end{aligned}$$

Convergence idea

If $\psi_\ell^{k,i} \to \psi_\ell^k$ and $g_\ell^{k,i} \to g_\ell^k$ then

$$\begin{aligned} L_\ell^{k,i}\psi_\ell^{k,i} - \tau\nabla\cdot\left(k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\nabla\psi_\ell^{k,i}\right) &= L_\ell^{k,i}\psi_\ell^{k,i-1} - (\theta_\ell(\psi_\ell^{k,i-1}) - \theta_\ell(\psi_\ell^{k-1})), \\ -\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} &= \lambda\psi_\ell^{k,i} + g_\ell^{k,i}, \\ \text{where } g_\ell^{k,i} &= -2\lambda\psi_{3-\ell}^{k,i-1} - g_{3-\ell}^{k,i-1}. \end{aligned}$$

$$\Big\downarrow \; i \to \infty$$

$$\begin{aligned} \theta_\ell(\psi_\ell^k) &= \theta_\ell(\psi_\ell^{k-1}) - \tau\nabla\cdot\left(k_{r,\ell}(\theta_\ell(\psi_\ell^k))\nabla\psi_\ell^k\right), \\ -\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^k))\partial_{\mathbf{n}_\ell}\psi_\ell^k &= \lambda\psi_\ell^k + g_\ell^k, \\ \text{where } g_\ell^k &= -2\lambda\psi_{3-\ell}^k - g_{3-\ell}^k. \end{aligned}$$

Interface conditions

$$g_1^k = -2\lambda\psi_2^k - g_2^k$$

$$g_2^k = -2\lambda\psi_1^k - g_1^k$$

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Adding the above gives

$$g_1^k + g_2^k = -\lambda(\psi_1^k + \psi_2^k)$$

Finally, since $-\tau^{\frac{1}{2}} k_{r,\ell}(\theta_\ell(\psi_\ell^{k,i-1}))\partial_{\mathbf{n}_\ell}\psi_\ell^{k,i} = \lambda\psi_\ell^{k,i} + g_\ell^{k,i}$,

$$-\tau^{\frac{1}{2}} k_{r,1}(\theta_1(\psi_1^k))\partial_{\mathbf{n_1}}\psi_1^k - \tau^{\frac{1}{2}} k_{r,2}(\theta_2(\psi_2^k))\partial_{\mathbf{n_2}}\psi_2^k = \lambda(\psi_1^k + \psi_2^k) + (g_1^k + g_2^k) = \mathbf{0}$$

Convergence result: LDD

Theorem[1]: Assume that

1. (ψ_1^n, ψ_2^n) exists[2],
2. $\psi_\ell^n \in W^{1,\infty}(\Omega_\ell)$[3],
3. $k_{r,\ell}$ and θ_ℓ are Lipschitz continuous with Lipschitz constants $L_{k,\ell}$ and $L_{\theta,\ell}$, and $k_{r,\ell}(\cdot) \geq m > 0$

Then for $L_{\theta,\ell} < 2L_\ell$, $M \geq \|\nabla \psi_\ell^n\|_{\mathbf{L}^\infty} + 1$ and τ so that

$$\tau < \frac{2m}{L_{k,\ell}^2 M^2}\left(\frac{1}{L_{\theta,\ell}} - \frac{1}{2L_\ell}\right)$$

one obtains the following:

1. Existence and uniqueness of a solution for each L-scheme iteration step.
2. The sequence $\{\psi_\ell^{n,i}\}_{i\in\mathbb{N}}$ converges to ψ_ℓ^n $(\ell = 1, 2)$.

---

[1] Seus et al. '18

[2] Jäger, Kutev '98, Eymard et al. '14, Berninger '14, P. et al. '17, List et al. '20

[3] Cao, P. '15

Convergence result: L^{i}DD

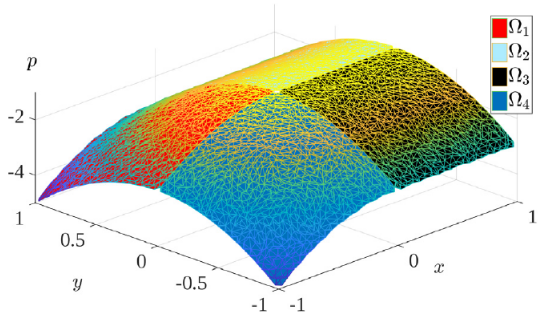
Theorem: Assume that

1. (ψ_1^k, ψ_2^k) exists,
2. $\psi_\ell^k \in W^{1,\infty}(\Omega_\ell)$,
3. $k_{r,\ell}$, θ_ℓ and θ'_ℓ are Lipschitz continuous and $k_{r,\ell}(\cdot) \geq m > 0$,
4. $\|\psi_\ell^k - \psi_\ell^{k-1}\|_{L^\infty(\Omega_\ell)} \leq \tau\Lambda_\ell$ for some $\Lambda_\ell > 0$.

Then for τ small enough and with $L_\ell^{k,i} = \max\{\theta'_\ell(\psi_\ell^{k,i-1}) + \mathcal{M}_\ell\tau, 2\mathcal{M}_\ell\tau\}$ and $\mathcal{M}_\ell \geq \Lambda_\ell\|\theta''_\ell\|_\infty$ one obtains the following:

1. Existence and uniqueness of a solution for each L^i-scheme iteration step.
2. The sequence $\{\psi_\ell^{k,i}\}_{i\in\mathbb{N}}$ converges to ψ_ℓ^k $(\ell = 1, 2)$.

Numerical example: domain



Numerical example: Richards equation

- Domain and coefficient functions ($\ell = 1, \ldots, 4$):

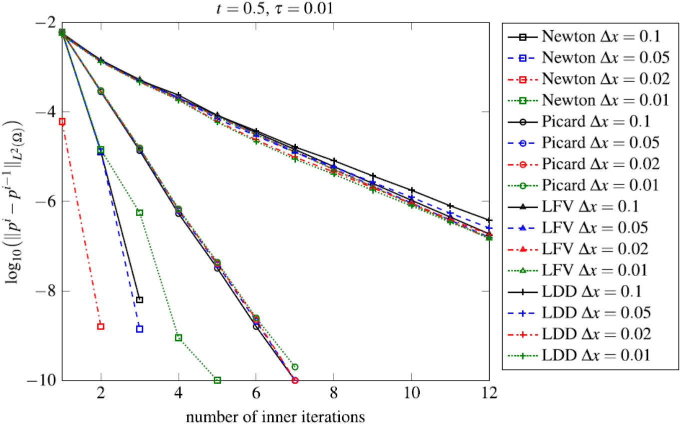
$$k_{r,\ell}(\theta) = \theta^{\ell},\ \theta_{\ell}(\psi) = \begin{cases} (1-\psi)^{-\frac{1}{\ell+1}}, & \psi < 0, \\ 1, & \psi \geq 0. \end{cases}$$

- Boundary/initial data and source term s.t. the solution is

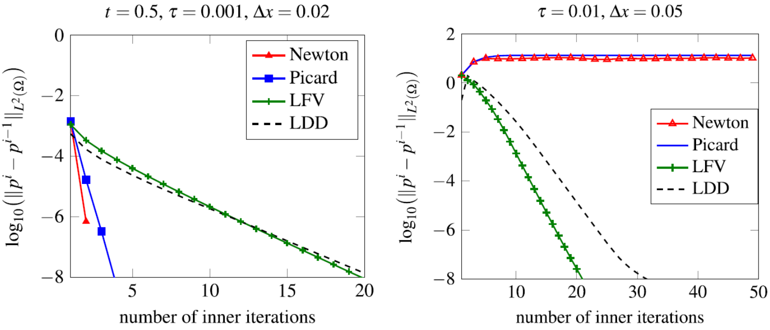
$$\psi(t,x,y) = \begin{cases} 1-(1+t^2)(1+x^2+y^2), & x \in (-1,0), y \in (-1,1), \\ 1-(1+t^2)(1+y^2), & x \in (0,1), y \in (-1,1). \end{cases}$$

Comparison of different schemes

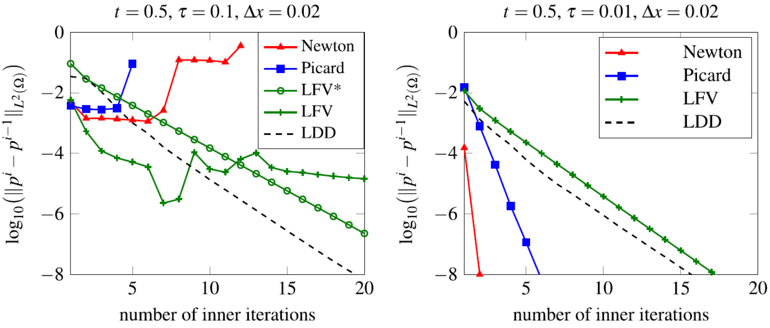
Convergence behaviour of the LDD scheme compared with the monolithic versions of the L-scheme, Newton, or Picard, and for different meshes



Effect of the time step and of L



Effect of the initial guess



Computational time

	1 CPU	2 CPUs	4 CPUs
Time	753s	411s	207s
Speedup	-	1.83	3.64

Speedup for 2/4 CPUs. Here $L_\ell := 0.25$, $\lambda_\ell = 4$, $h = 0.05$, $\tau = 0.001$. Stopping criterion is $\|\psi(t_k) - \psi^k\_L^2 \leq 10^{-6}$ at $t = 1$.

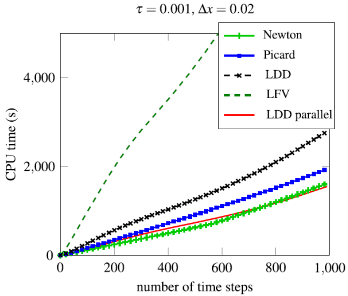
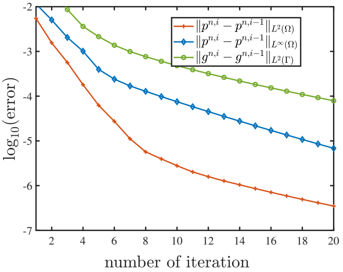
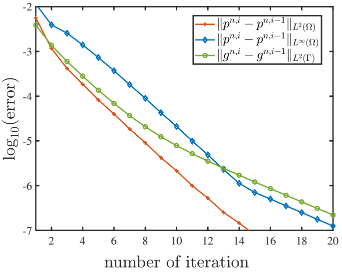


Fig. 8. Time performance of the LDD, L-FV and the Newton-FV schemes.

Numerical example: comparison LDD-L^{i}DD





$h = 0.05$, $\tau = 0.01$, $t = 0.2$, $L = 0.25$ (left) resp. $M = 1$ (right)

Numerical example: comparing schemes

timestep (τ)	LDD	LiDD	Newton DD iteration	Newton total iteration
0.01	77	20	53	295
0.05	62	46	53	316
0.10	67	70	56	360

$h = 0.05$, $\mathcal{M} = 20$, $L = 0.25$, $\lambda = 4$ (LDD and Newton)

Stopping criterium: $\sum_{\ell,p} \|g_{\ell,p}^k - g_{p,\ell}^k\|_{L^2(\Gamma_{p,\ell})} \leq 10^{-6}$ (DD);
$\|\psi_\ell^{k,i} - \psi_\ell^{k,i-1}\|_{L^2(\Omega_\ell)} \leq 10^{-6}$ (Newton).

Non-standard models

Non-equilibrium effects in the $\psi - \theta$ relationship

$$\psi = \underbrace{p_c(\theta)}_{\text{"Standard" capillary pressure}}$$

[1]Bottero, 2009, PhD thesis

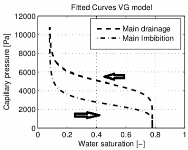
[2]Bottero et al., 2011, Water Resour. Res.

[3]Lunowa et al. '19, '20

Non-standard models

Non-equilibrium effects in the $\psi - \theta$ relationship

$$\psi \in \underbrace{p_c(\theta)}_{\text{"Standard" capillary pressure}} + \underbrace{\gamma(\theta)\mathrm{sign}(\partial_t \theta)}_{\text{Play-type hysteresis}}$$



Fitted van Genuchten model
for $\psi - \theta$ measurements[1]

Results[3]: Rigorous convergence proof for two LDD-type schemes.

[1]Bottero, 2009, PhD thesis

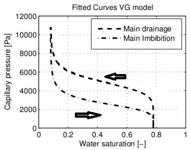
[2]Bottero et al., 2011, Water Resour. Res.

[3]Lunowa et al. '19, '20

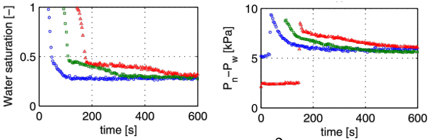
Non-standard models

Non-equilibrium effects in the $\psi - \theta$ relationship

$$\psi \in \underbrace{p_c(\theta)}_{\text{"Standard" capillary pressure}} + \underbrace{\gamma(\theta)\mathrm{sign}(\partial_t \theta)}_{\text{Play-type hysteresis}} + \underbrace{\tau(\theta)\partial_t \theta}_{\text{Dynamic effects}}$$



Fitted van Genuchten model for $\psi - \theta$ measurements[1]



Non-equilibrium experiments[2] (at fixed location)

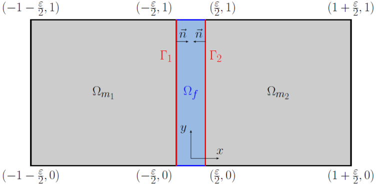
Results[3]: Rigorous convergence proof for two LDD-type schemes.

[1]Bottero, 2009, PhD thesis

[2]Bottero et al., 2011, Water Resour. Res.

[3]Lunowa et al. '19, '20

Fractured media: thin, equidimensional fracture

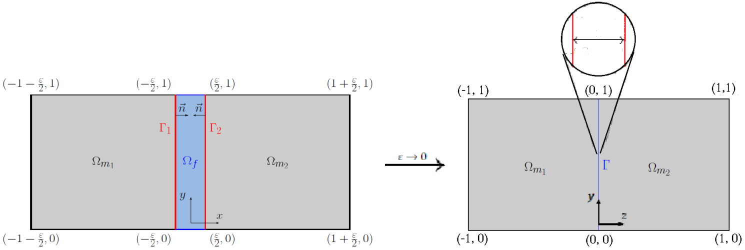


$$\begin{cases} \partial_t \theta_m(\psi^\varepsilon_{m_j}) - \nabla\cdot\left(k_{r,m}(\theta_m(\psi^\varepsilon_{m_j}))\nabla\psi^\varepsilon_{m_j}\right) = f_{m_j} & \text{in } \Omega^T_{m_j}, \\ \partial_t(\varepsilon^\kappa \theta_f(\psi^\varepsilon_f)) - \nabla\cdot\left(\varepsilon^\lambda k_{r,f}(\theta_f(\psi^\varepsilon_f))\nabla\psi^\varepsilon_f\right) = f^\varepsilon_f & \text{in } \Omega^T_f, \\ v^\varepsilon_{m_j}\cdot\mathbf{n} = v^\varepsilon_f\cdot\mathbf{n} & \text{on } \Gamma^T_j, \\ \psi^\varepsilon_{m_j} = \psi^\varepsilon_f & \text{on } \Gamma^T_j, \end{cases}$$

*Pressure continuity: Martin, Jaffré & Roberts '05, Knabner & Roberts '14
Nonlinear transmission condition (entry pressure models): van Duijn et al ('95, '02, '16), Schweizer et al. ('07, '11), Cances et al. ('10, '12), Jäger & Simon '02, P, Bogers, Kumar '17

Reduced dimensionality: Morales, Showalter '10, '12, Ahmed et al. '17, Kumar et al. '20, List et al. '20

Fractured media: reduced dimensionality

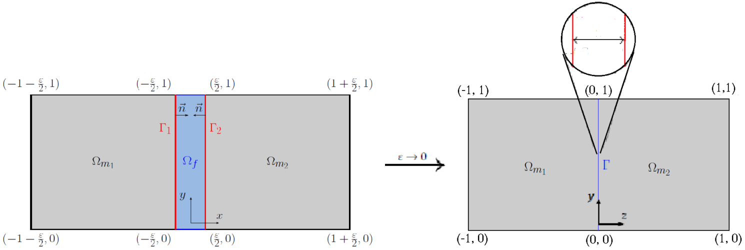


$$
\begin{cases}
\partial_t \theta_m(\Psi_{m_j}) - \nabla \cdot \left(K_m(\theta_m(\Psi_{m_j}))\, \nabla \Psi_{m_j} \right) = f_{m_j}, & \text{in } \Omega_{m_j}^T, \\
\partial_t \theta_f(\bar{\Psi}_f) - \partial_y \left(K_f(\theta_f(\bar{\Psi}_f))\, \partial_y \bar{\Psi}_f \right) & \\
\qquad = \left(K_m(\theta_m(\Psi_{m_1}))\nabla \Psi_{m_1} \cdot \mathbf{n}_{m_1} + K_m(\theta_m(\Psi_{m_2}))\nabla \Psi_{m_2} \cdot \mathbf{n}_{m_2} \right) \Big|_{\Gamma}, & \text{on } \Gamma^T, \\
\Psi_{m_j} = \bar{\Psi}_f, & \text{on } \Gamma^T,
\end{cases}
$$

Note: Results depend on κ and λ; here $\kappa = \lambda = -1$ (rigorous convergence proof[1])

[1] List et al. '20

Fractured media: reduced dimensionality

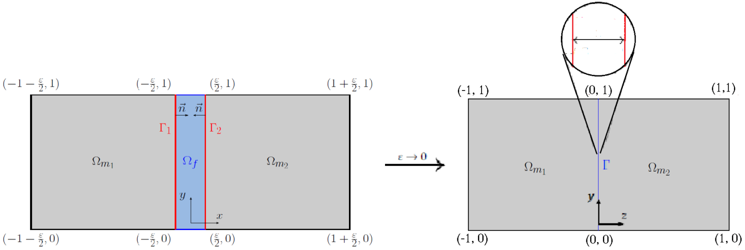


$$\begin{cases} \partial_t \theta_m(\Psi_{m_j}) - \nabla \cdot \left(K_m(\theta_m(\Psi_{m_j}))\, \nabla \Psi_{m_j} \right) = f_{m_j}, & \text{in } \Omega_{m_j}^T, \\ \partial_t \theta_f(\bar{\Psi}_f) & \\ \qquad = \left(K_m(\theta_m(\Psi_{m_1}))\nabla\Psi_{m_1} \cdot \mathbf{n}_{m_1} + K_m(\theta_m(\Psi_{m_2}))\nabla\Psi_{m_2} \cdot \mathbf{n}_{m_2}\right)\Big|_{\Gamma}, & \text{on } \Gamma^T, \\ \Psi_{m_j} = \bar{\Psi}_f, & \text{on } \Gamma^T, \end{cases}$$

Note: Results depend on κ and λ; here $\kappa = -1 < \lambda$ (rigorous convergence proof[1])

[1] List et al. '20

Fractured media: reduced dimensionality



$$\begin{cases} \partial_t \theta_m(\Psi_{m_j}) - \nabla \cdot \left(K_m(\theta_m(\Psi_{m_j}))\, \nabla \Psi_{m_j} \right) = f_{m_j}, & \text{in } \Omega_{m_j}^T, \\ \quad -\partial_y \left(K_f(\theta_f(\bar{\Psi}_f))\, \partial_y \bar{\Psi}_f \right) & \\ \qquad = \left(K_m(\theta_m(\Psi_{m_1}))\nabla\Psi_{m_1} \cdot \mathbf{n}_{m_1} + K_m(\theta_m(\Psi_{m_2}))\nabla\Psi_{m_2} \cdot \mathbf{n}_{m_2} \right) \big|_\Gamma, & \text{on } \Gamma^T, \\ \Psi_{m_j} = \bar{\Psi}_f, & \text{on } \Gamma^T, \end{cases}$$

Note: Results depend on κ and λ; here $\kappa > \lambda = -1$ (rigorous convergence proof[1])

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[1] List et al. '20

Conclusions and perspectives

- Mathematical model for unsaturated porous media flows
- Euler implicit approach
- Linear domain decomposition, incorporating a robust linearization - (modified) L-scheme:
  - robust scheme (no restrictions on the initial guess);
  - mild condition on the time step;
  - applicable in combination with (M)FEM, FV, MPFA, dG, GDM...
- Convergence for Hölder continuous, non-decreasing functions
- More complex models, multi-phase flows

Joint work with

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J. Bogers, ASML

D. Seus, Stuttgart
C. Rohde, Stuttgart
F. Radu, Bergen
K. Kumar, Bergen

Thank you!

fwo Research Foundation Flanders
Opening new horizons

ODYSSEUS
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