Redesigning task sequences to support instrumental genesis of the use of movable points and slider bars

This paper examines the process of instrumental genesis through which students develop their proficiency in making use of movable points and slider bars – two tools that dynamic mathematics software provides for working with variable coordinates and parameters in the field of functions. The paper analyses students' responses to task sequences designed to support a planned instrumental genesis and then examines how features of these task sequences might be modified accordingly to improve such support. Evidence about student responses was collected during the first iteration of a design based research study conducted in collaboration with four upper secondary school teachers. Three task sequences were trialled with four 10th grade classes, involving a total of 85 students. Orchestration and examination of the instrumental genesis were guided by the identification of key elements – both conceptual and technical – of the instrumented action schemes involved.

1 INTRODUCTION

Nowadays many schools provide students with a personal computing device (Valiente, 2010), which opens up new possibilities for the teaching and learning of school subjects, not least mathematics. Although the literature shows some evidence of successful use of technologies in mathematics classrooms, the integration of digital tools into school mathematics is still sparse (Hoyles, Noss, Vahey, & Roschelle, 2013). Researchers point out that one reason for this is that the complexity of learning to making effective use of digital tools is often underestimated, so that insufficient attention is given to supporting the process of *instrumental genesis* by which a basic artefact becomes a functioning instrument for a user (e.g. Artigue, 2002; Trouche, 2004). Different aspects aiming at supporting this process has been extensively studied. However, less attention has been paid to the use of specific dynamic software tools to manipulate functional variables and parameters.

In particular, there is a need to give closer attention to how to deliberately orchestrate this process of instrumental genesis through designing learning materials that guide students towards effective pattern of use of different types of digital tool. This paper concerns the design – and projected redesign after an initial trial – of a set of task sequences intended to support the instrumental genesis of specific dynamic software tools – movable points and slider bars – for manipulating functional variables and parameters.

2 THEORETICAL BACKGROUND

This section first introduces the literature concerning a functional approach to algebra followed by a description of research related to technology in the field of functions and graphs, particularly literature concerning movable points and slider bars. Then the section provides aspects, relevant to the study, concerning the instrumentation theory and concludes with the aim of the paper.

2.1 A functional approach to algebra

Central in the teaching and learning of algebra is appreciation of the various uses of letters. Among beginning algebra students the perception of letters as unknowns dominates

(Ely & Adams, 2012; Trigueros & Jacobs, 2008). This use of letters relates to problem solving (Usiskin, 1988) where letters stand for particular numerical values which have to be found, most often by solving an equation (Ely & Adams, 2012). The use of letters as variables is more complex since it takes place in several different ways (Trigueros & Jacobs, 2008), e.g. as generalized numbers connected to pattern and structure or as related quantities in functional algebra (Ely & Adams, 2012; Usiskin, 1988). Moreover, besides not being a mathematically well-defined concept, a 'variable' plays different roles in different settings which can be confusing for students (Trigueros, Jacobs). The literature describes mainly three different approaches to algebra: generalization, problem-solving and functional approach (Drijvers, 2003; Usiskin, 1988). For instance, to emphasize the interpretation "…of letters as variables rather than unknowns,…" (p. 132), Chazan and Yerushalmy suggest a function-based approach to algebra (2003).

Besides the use of literal symbols as unknowns and variables, another usage of letters is as *parameters*. Researchers point out the importance of being able to distinguish parameters from unknowns and variables (Bloedy-Vinner, 2001), which has proven difficult for students. There are two main reasons for this difficulty (Bloedy-Vinner, 2001). First, the role of letters depends on the context (Bloedy-Vinner, 2001; Ely & Adams, 2012). For example, out of context an expression such as $y = kx$ could represent an equation with 3 unknowns or a function with 2 variables (Ely & Adams, 2012).The second difficulty relates to the notion of parameter and its contradictory epistemic nature (Bardini et al., 2005) causing students to experience a conflict. It is the apparent contradiction between a parameter as a constant but one that varies that causes this conflict (Bardini et al., 2005; Bloedy-Vinner, 2001). The reason for this is that parameters can play roles as unknowns and variables, however, at a higher level (Bloedy-Vinner, 2001).

A further inherent difficulty concerning the role of a letter is that the terms in which it is framed often change in the course of a solution procedure). To illustrate this, Usiskin (1988) uses the traditional problem of finding an equation for a line given the slope and a point that the line goes through. In this particular example, the role of b changes from being a parameter in the equation $y = mx + b$ to being an unknown to be solved for as soon as the given data – the value of the slope m and the coordinates of the point – are substituted into the equation.

To summarize, in teaching and learning of algebra, letters are used as variables and parameters. Furthermore, both variables and parameters could play the role as unknowns or as general numbers, however, at different levels.

2.2 Technology in the field of functions and graphs

Within the domain of functions and graphs, the connection between algebraic and graphical representations is central (Leinhardt, Zaslavsky, & Stein, 1990). The affordance of technology to provide access to multiple representations of functions is well documented (Ferrara, Pratt, & Robutti, 2006). With dynamic mathematics software it is possible to make direct manipulation of dynamically linked representations of functions, e.g. algebraic and graphic representations (Ferrara et al., 2006).

2.2.1 Manipulating Variable Coordinates

The literature identifies some student difficulties in making connection between symbolic and graphical representations (Hennessy, 1999). For students to be able to make this connection, they need to understand the notion of *ordered pairs of numbers* represented by points in the coordinate systems (Goldenberg, 1988; Hennessy, 1999). According to Hennessy, students "…fail to grasp the association between an ordered pair and corresponding *x*- and *y*- values;" (Hennessy, 1999, p. 27). This difficulty, in turn, might explain the recognized problem that students have in realizing that a continuous line consists of discrete points (Goldenberg, 1988, p. 158). Furthermore, Goldenberg, Scher and Feurzeig (2008) point out how students' inaccessibility of controlling the variable x in a graphical software environment made them confused since "It was called 'the variable', but they never varied it!" (p. 77). On the contrary, students could change the values of the parameters in a polynomial function, such as $y = ax^2 + bx + c$, which in turn, might give the impression that the variables are a, b and c, and not x (Goldenberg, 1988).

One defining feature of dynamic interactive geometry software environments (DGE) is the ability to drag points in geometric constructions and manipulate them dynamically (Goldenberg et al., 2008). According to Goldenberg et al. software designers and researchers have strived to develop analogue dynamical tools aimed at investigating variable coordinate-pair. Researchers working with DGE for investigating Euclidean Geometry use the notion of draggable points (e.g. Arzarello, Olivero, Paola, & Robutti, 2002) but this notion is seldom connected to an algebraic representation. In this paper the notion 'movable point' is used when referring to a draggable coordinate-pair.

2.2.2 Control of a Parameter through a Slider Bar

Drijvers (2003) and Zbiek and Heid (2001) have used a functional approach to investigate students' use of technology to develop the understanding of parameters. Although if they use different technology in their studies, the particular tool they use, termed a *slider bar*, is provided by both of these software environments (computer algebra system (CAS) and DGE respectively). This tool, which enhances the manipulation of parameters, gives students the opportunity to examine the visual effect on a graph while changing the value of a parameter (Drijvers, 2003; Zbiek & Heid, 2001). However, in an overview, Zbiek et al. (2007), raises the following question: Does the physical sense of moving a slider obscure rather than enhance the desired cognition of the connection between the parameter value and a salient visual consequence? (p. 1177) According to them, besides the symbolic and graphical representation of the function itself, the slider bar could be regarded as a third representation for students to focus on while investigating the effect of different values of a parameter.

A further risk with the use of slider bars, identified by Drijvers (2003) is that students tend to examine the effect of the 'sliding parameter' superficially, and thus not bother about reasons behind a particular behavior. Moreover, the students show a lack of capability in using natural language to express their observations of the effect that a sliding parameter has on a graph (Drijvers, 2003).

2.3 Instrumentation theory

This study draws on the instrumental approach with a particular focus on the process of instrumental genesis, which originates in the work by Verillon and Rabardel (1995). According to them, an *instrument* encompasses both an *artefact* and associated *utilization schemes*. An artefact is an object, material or abstract, upon which a subject acts with a certain objective in mind. During this action, the subject develops utilization schemes associated to the artefact. Thus, an artefact does not initially constitute an instrument for a user but becomes so through the process of instrumental genesis (Verillon & Rabardel, 1995). The idea of instrumental genesis has been adopted by researchers focusing on the integration of technology into mathematics education (Artigue, 2002; Trouche, 2004). In this field, many researches use the notion of *instrumented action schemes* when referring to utilization schemes (Drijvers & Gravemeijer, 2005; Trouche, 2004).

Since this is a cognitive construct, not visible for observation, it is the observable part of instrumented action schemes which could be investigated (Drijvers, 2003; Guin & Trouche, 2002). That is, it is the technical activity undertaken by a subject performing a certain task which could be the object of investigation. These activities, which involve both conceptual and technical knowledge, are referred to as *instrumented techniques* (Artigue, 2002) or just *techniques* (Lagrange, 1999). In other words, the instrumented action schemes involve an intertwinement of both technical and conceptual aspects (key elements) and it is the observation of techniques that is the "…gateway to the analysis of instrumental genesis." (Drijvers & Gravemeijer, 2005, p. 169).

One example of a basic instrumented action scheme that students have to develop concerns the framing of the viewing window to obtain an appropriate visible appearance of a function graph. These schemes, which Artigue (2002) term "framing schemes" include both conceptual and technical knowledge. Artigue illustrates this with a particular example in a CAS environment in which students were asked to make conjectures on the properties of the function $f(x) = x(x + 7) + 9/x$ by examining the graphical representation of the function in a CAS environment (or a graphical calculator). Although the participating students were expected to master the technical skills required to be able to make conjectures about the properties of the particular graph, most of them lacked these skills. However, these identified difficulties need not be due to the students' lack of conceptual understanding because, Artigue argues, all the students in the study were clearly aware that the graph they obtained in the standard window was not correct. This example shows the complexity of the instrumental genesis, which Artigue points out, has been underestimated and thus an obstacle in the integration of technology as a pedagogical tool in mathematics classrooms.

Another related example of an instrumented action scheme, highlighted by Drijvers and Gravemeijer, is Goldenberg's (1988) example concerning scaling of the viewing window of a graphical calculator (Drijvers & Gravemeijer, 2005). In line with the observation made by Artigue, Goldenberg (1988) discusses the difficulty students have of setting an appropriate viewing window. However, this difficulty might be because of students' lack of awareness that the viewing window only displays a small part of the Cartesian plane. That is, in contrast to the case reported by Artigue (2002). Difficulties that users seem to have with the technology might depend on lack of conceptual knowledge (Drijvers & Gravemeijer, 2005). In this way, Drijvers and Gravemeijer argue, the technology makes the lack of underlying conceptual knowledge visible for the teacher, which in turn, could turn technical obstacles into teaching and learning opportunities (Drijvers & Gravemeijer, 2005).

2.4 The aim of the study

This paper concerns the design – and projected redesign after an initial trial – of a set of task sequences intended to support particular aspects of students' instrumental genesis process. These aspects concern the use of particular dynamic mathematics software tools related to variables and parameters. The literature served as guidance in the *a priori* identification of key elements (KE) of instrumented action schemes that we seek to develop, which in turn provided a basis for an initial version of the task design. In order to provide suggestions for redesign to enhance the scope for students' acquisitions of the instrumented action schemes, this paper analyses and reflects on student responses to the key elements identified *a priori*, and also to identify some further key elements *a posteriori*. This leads to the following research question: How could the identification of key elements of instrumented action schemes related to movable points and slider bars for manipulating functional variables and parameters be used in the (re)design of tasks to support students' instrumental process? The main theoretical frames guiding design and analysis in the study are the functional approach to algebra and the instrumental approach. Particularly, the construct of key elements of an instrumented action scheme, adopted from Drijvers and Gravemeijer (2005), served both as a design tool and as an analytical tool.

3 THE METHOD

The study is embedded within an iterative form of design experiment, which creates the possibility of revising the tasks to be undertaken by students so as to support a smoother acquisition of instrumented action schemes. This paper focuses on this aspect of the first iteration of a design experiment.

3.1 Participants

Two researchers and four upper secondary school teachers formed the research team. The teachers were working at two Swedish schools admitting both male and female students of all abilities. Both schools provide their students with a computer of their own. Altogether, 85 students participated in the study. The participating classes were in the tenth grade, with no previous experience of working with either dynamic software or graphical calculators. The mathematical course they were following is part of the regular curriculum, although not intended to prepare students for further studies in mathematics.

3.2 Procedure

The three teaching sequences which were the focus of design and experimentation each consist of a computer lesson and a follow-up classroom discussion. Each computer lesson was guided by a worksheet consisting of a sequence of related tasks. In total, three worksheets, one for each lesson, were designed and trialed in classrooms. The first and the third worksheets are about exponential functions, while the second worksheet concerns linear functions and inequalities. The students were to work in pairs with *one* computer per pair. The purpose of this is that the computer screen should provide a shared object for discussions between students (Brunström & Fahlgren, 2015). Moreover, the students were to be expected to formulate their conclusions in writing. Although the researcher were responsible for designing the task sequences (TS), the teachers provided valuable information regarding the participating students' capabilities and their current practices. We, the researchers and teachers, had meetings before and after each teaching sequence.

3.3 Materials

3.3.1 Key elements in the initial version of the task design

Figure 1 shows a screenshot of the (GeoGebra-based) dynamic environment with which students were to engage. Note, in particular the appearance of the movable point on a graph, in this case the general exponential function $f(x) = C \cdot a^x$, and the slider bars which are the main features of the environment referred to in the list of key elements below. The user interacts with these tools by pointing on and dragging the 'tool-points'. The tools are created by selecting a specific Tool in the Toolbar at the top of the GeoGebra window and then following the instructions provided by the environment. The design process called for the identification and analysis of the forms of interaction expected of students.

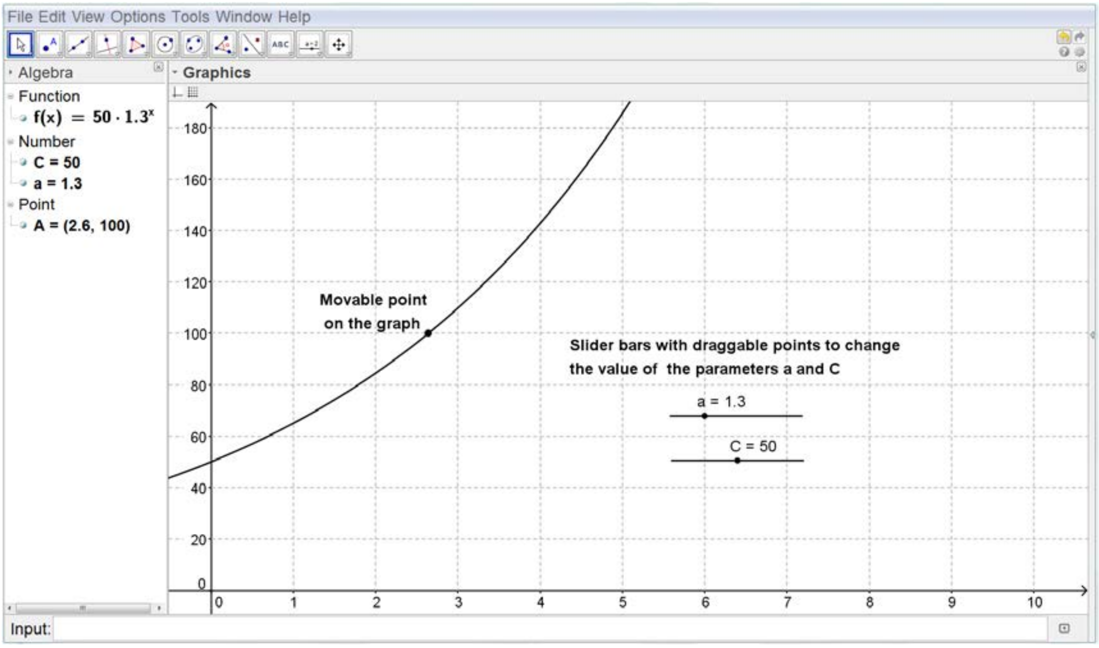


Figure 1 A screenshot of the dynamic software environment

Below are the concepts and techniques, in terms of *a priori* identified key elements, that students have to master in order to successfully complete each task sequence introduced. Thus, if students do not have this pre-knowledge, they are expected to develop it in the course of the sequences. Although, the conceptual and the technical aspects are separated out, they are closely related since they address the same construct.

Key elements (KE) – conceptual (C) aspects

KE-C1: Understand ordered pairs of numbers as represented by points

KE-C2: Interpret a variable coordinate as an unknown in an functional relationship, e.g. by constructing and solving equations

KE-C3: Interpret a variable coordinate as a general number in a functional relationship

KE-C4: Understand that, although open to variation, a functional parameter can act as a fixed value.

KE-C5: Interpret a parameter as an unknown to be found to solve a linear inequality

KE-C6: Interpret a parameter as an unknown to be found to solve a problem given some specific conditions

KE-C7: Realize that different values of a functional parameter affect the location and/or shape of a corresponding graph

Key elements (KE) – technical (T) aspects

KE-T1: Enter a point into the (a) Input bar and (b) coordinate system

KE-T2: Create a movable point on a graph

KE-T3: Solve equations graphically by using a movable point

KE-T4: Create a slider bar

KE-T5: Use a slider bar to set different values of a parameter

KE-T6: Use a slider bar to solve a linear inequality by moving the slider until a specific condition is fulfilled

KE-T7: Use a slider bar to solve an exponential inequality by moving the slider until the conditions are fulfilled

KE-T8: Use the slider tool to investigate parameters as changing quantities

Thus Table 1 provides an overview of the planned instrumental genesis conceived in terms of a functional approach to algebra, over the course of the three task sequences, in terms of the instrumented action schemes that we seek to develop.

Table 1 The planned instrumental genesis throughout the three task sequences

Constructs	TS 1	TS 2	TS 3
Graph coordinates	KE-C1 KE-T1		
Function variables	KE-C2	KE-C3 KE-T2 KE-T3	
Function parameters -as 'fixed' general numbers		KE-C4 KE-T4 KE-T5	
-as unknowns		KE-C5 KE-T6	KE-C6 KE-T7
-as changing quantities			KE-C7 KE-T8

Besides distinguishing between variables and parameters as general numbers and unknowns, a further distinction between parameters as general numbers is used. In this way, three roles of parameters are the focus of attention in this study (see Table 1). First, a parameter as a 'fixed' general number closely connected to the placeholder view of a parameter, described by Drijvers (2003) as "…an empty place into which numerical values can be inserted…"(p. 68). Although, the parameter can have different numerical values, the focus is on one value at a time. Second, a parameter as unknown selects the value of the parameter that fulfils a specific condition, e.g. solving a context problem. Finally, the role of a parameter as changing quantity is the one closely associated with the slider bar (Drijvers, 2003; Zbiek & Heid, 2001).

3.3.2 Principles behind design of the task sequences

On the worksheets, one for each TS, the tasks were intertwined with computer instructions. The reason for this was to enhance the scope for the dynamic mathematics software to become an instrument for the students alongside their development conceptual knowledge about functional relationships.

Our approach to design stipulated that a task sequence should be framed by a theme to which many tasks can be related. In this way students are provided with opportunities to make reflections and connections between different tasks. The theme should relate to a context to which the students could relate, i.e. a context that is experientially real for them. In addition, by framing the tasks within a context, the role of a specific parameter is clarified (Bloedy-Vinner, 2001; Ely & Adams, 2012). Moreover, the tasks in the task sequence are of two kinds; paper-and-pencil tasks and tasks deliberately designed for dynamic mathematics software. One reason for this is to provide opportunity for students to compare paper-and-pencil and instrumental solution techniques, which have different epistemic value (Artigue, 2002). Another reason is to enhance the relation between algebraic and graphical representations (Leinhardt et al., 1990).

3.4 Data collection and analyses

The empirical data used in this paper were collected during the three computer lessons, which lasted about 60 minutes, in the four classes. The data are of three different kinds; (a) in each class one pair of students was video recorded, (b) all teacher-student interactions during the lesson were audio recorded using a microphone attached to the teacher, and (c) copies were made of the written responses from all students.

In the analysis process, the video recordings become the primary source of data since they made it possible to observe students' instrumented techniques which are the observable part of the instrumented action schemes (Drijvers & Gravemeijer, 2005). The video recordings were watched and listened to several times to identify significant moments of students' utterances and activities concerning the planned instrumental genesis in terms of the *a priori* identified key elements (see Table 1). To illustrate the analytical process more in detail, the analytical steps are described below by using the example reported in Section 4.1 concerning two basic key elements related to ordered pairs of numbers in the coordinate system (denoted KE-C1 and KE-T1 in Table 1).

First, the data from the video recording of four pair of students working on the initial tasks in TS 1 were analysed to identify technical strategies used as well as verbally articulated mathematical thinking. Then, video recording was triangulated against the audio recordings and the written records from the larger number of students to ensure that the observed behaviour was typical among the participating students. In this particular example, while the written data indicated that students have performed the initial task as expected, the video recordings made student difficulties concerning the KE-C1 visible. In this way, the audio recordings confirmed that the difficulties observed among the video recorded pair of students were not quite usual. The written data that include student responses in terms of their explanations of the observation they made added valuable insight to the analysis process. The results reported in Section 4 are illustrated, when suitable, with student responses.

4 RESULTS

The conceptual structure of Table 1 provides the framework for this section. The way we have chosen to organize (and highlight findings) is in terms of the underlying concepts being developed about coordinates and variables: these are introduced in TS1 and developed a little bit further in TS2, and then the attention shifts to parameters in TS2 and TS3. Thus, because there is quite a strong relationship between the flow of ideas from TS1 to TS2 to TS3, this approach also provides an effective way of presenting our analysis of the task sequences.

This section examines empirical data concerning students' responses to relevant parts of the initial task design. Results from the analyses are then used to identify some further key elements of the instrumented action schemes developed by students. On this basis, any proposals for revision of the initial task design are considered. To facilitate the reading, key elements in the Table 1, when being addressed, are shown within brackets throughout this and the next section.

4.1 Graph coordinates

The first task was intended to be a routine paper-and-pencil task which introduces the context of TS1: “The height of a sunflower is 50 cm when it is measured for the first time (June 1). After that the sunflower grows so that it becomes 30 % higher each week. Calculate the height of the sunflower one week after the first measurement.” The task is followed by conceptual guidance and computer instructions (Figure 2) showing how to use the software to enter the point representing the initial height of the sunflower.

When the first measurement is performed, x = 0 (since 0 weeks have passed) and y = 50. The corresponding point in a coordinate system is (0,50).

- Insert this point by entering (0,50) into the ”Input Bar”: Input: **(0,50)**

NOTE! To be able to see the point you must adjust the scale on the y-axis. This can be done by ”dragging” the y-axis. (first mark)

- Insert the point that shows the height of the sunflower after one week (due to your calculation in task 1) by entering its coordinates into the ”Input Bar”.

Figure 2 Conceptual and technical guidance and instructions between Task 1 and Task 2 in TS1

Although most students managed to enter the first point, i.e. (0,50), into the “input bar”, both the video and audio recordings reveal that some students encountered problems when entering the calculated point (1,65) (KE-T1). Below is an excerpt showing how the computer feedback “invalid data” prompted one pair of students, having tried to enter first “65cm” and then just “65”, to read the instruction once again:

Student 1: By entering its coordinates. [reads the instructions aloud with an emphasis on the 's' in the word ‘coordinates’]

Student 2: Should it be several?

Student 1: But it is just one.

Student 2: Is it?

Student 1: Yes, it is only one height.

The students seemed to ignore the value of *x* by only focusing on the height of the sunflower, i.e. the value of *y* (KE-C1).

4.2 Function Variables

Later on in TS1, to introduce the use of variables as general numbers in a functional relationship, students were encouraged to produce a closed form equation, describing how the sunflower grows. Then, the students were required to use the Graphic view to estimate (a) the height of the sunflower after three and a half weeks and (b) after how many weeks the height of the sunflower is 160 cm. That is, in (a) the value of the variable *y* is unknown and in (b), the value of the variable *x* is unknown. One interesting observation was made among one of the video recorded pairs of students; To facilitate the task, they created a movable point on the graph (KE-T2). Then, they solved the two problems of estimation, by moving this point along the graph and reading off its coordinates (KE-T3). In this way the students used the movable point to manipulate variables as unknowns (KE-C2).This result prompted us to redesign TS1 so as to introduce the movable point earlier. In a revised version of the task design, we add the requirement to create a movable point (KE-T2), to be used to investigate variables as unknowns.

The second task sequence, TS2, introduces a new context – a class selling candy. Initially, the students were encouraged to formulate a formula for a linear relationship (in this case $y = 100 + 40x$) to enter into the software. Then, the students were instructed to create a movable point (KE-T2) on the graph and move it along the graph while observing how its coordinates change, i.e. by focusing on variable coordinates as general numbers (KE-C3). In the subsequent task, to emphasize the variable coordinates as unknowns, students were encouraged to describe how they could use the movable point to check their calculation made in the first task (KE-T3). In this case, this means moving the point so that its x-coordinate becomes 10 and read off the corresponding value of y.

Both the audio and video recordings reveal some instrumentation issues in relation to the use of a movable point. For example, students' tendency to use the zoom operation technique to change the scales of the axes (instead of changing one axis at a time) often ended in a Graphics View like the one in Figure 3. This, in turn, made it impossible for students to use a movable point and read off the corresponding value of y when $x = 10$.

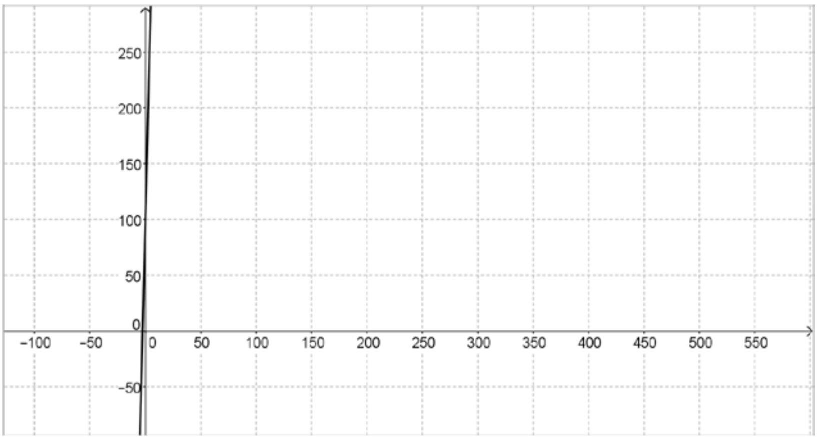


Figure 3 One example of a Graphics View obtained by students working on Task 2 in TS2

Another instrumentation problem observed was that several students did not create a movable point *attached to* the graph (KE-T2), although they believed had done so. When solving the problems, they tried to move a free point along the graph. This prompted us to add a notice about this instrumentation issue in the revision to the task.

The written data indicate that the request of using the movable point to graphically solve an equation (KE-T3) where the value of the variable x is unknown turned out to be straightforward for the participating students. Further, the video recordings show that students really used the movable point to solve this kind of elementary equation graphically. However, when students in the subsequent task were asked to solve the same problem by constructing and solving an equation by paper and pencil, only half of the students solved this task, although, it was regarded by the teachers as a routine task for the participating students (KE-C2).

4.3 Function parameters as ‘fixed’ general numbers

Later on in TS2, students were asked to formulate and enter the linear function representing the income as a function of the amount of candy sold (in kg) if the price is set to 50 SEK per kg, i.e. $y = 50x$. In this way the students obtain two linear graphs in the coordinate system; one representing the costs and one representing the income. By this stage, the slider bar is introduced to make it easier for students to change the price of the candy by changing the numerical value of the parameter p in the general formula representing the income, $y = p \cdot x$ (KE-T4). Students were provided a detailed description, including pictures, of how to construct a slider bar with specific settings such as the minimum and maximum value and the increment. Both the video and audio recordings imply that students seemed to understand the idea behind the use of the slider bar, i.e. as an easy way to change the numerical value of a parameter (KE-C4/T5).

4.4 Function parameters as unknowns

After being introduced to the slider bar, the students were asked to define the price per kilo needed to make a profit if they buy and sell more than 4 kg. That is, they have to define the value of the parameter given a particular condition of the variable *x.* This is an example where the parameter plays the role of an unknown. Although we anticipated that this use of the slider bar would be more demanding for students than the preceding one, it turned out to be no problem for those students (about half of the students) who performed the task (KE-C5/T6).

The last task in TS3 introduces a new context problem in which one of the parameters and the relation between two pairs of numbers are known: “The value of a car drops from 100 000 SEK to 50 000 SEK in two years. What is the annual decrease in percentage if the value of the car is decreasing exponentially?” In this example, one of the parameter plays the role as an unknown. The students were explicitly encouraged to use the slider bar to solve the problem.

About half of the students performed this task. Most of them explained in their written responses that the graph must go through the two points (0,100) and (2,50), which indicates an understanding of the relationship between ordered pairs of numbers and the corresponding points (KE-C1). The video recordings show how students first entered a point with the coordinates (2,50) (KE-T1) and then dragged the slider until the graph was going through that point (KE-T7). Finally, they read off the value of the slider (KE-C6). Below is an excerpt showing the discussion between one pair of students:

Student 1: Let’s drag this one [starts to drag the *y*-axes to change the scale] so that we can see 50 000… there it is.

[…]

Student 1: And then …hmm … find “appropriate values of the sliders.” [reads the instruction loud]. How do you set the sliders then?

Student 2: You should set *C* on 100 000.

Student 1: [drags the slider to its maximum value; 100] How can we do that?

Student 2: You must change so that it is possible to set it on 100 000…or just set it on 100 and then we can take this away [starts to re-change the scale on the *y*-axes].

Student 1: Why should you do that? Why do you do like that?

Student 2: Because … Why should we set it on 100 000 when we can set it on 100 instead … it will be the same calculation.

Student 1: Yes, yes.

Student 2: And then these [the sliders] will work.

It is interesting to notice how the need for rescaling the *y*-axis and changing the maximum value of the slider *C* seemed to make the students conscious about the possibility to use another unit on the *y*-axis.

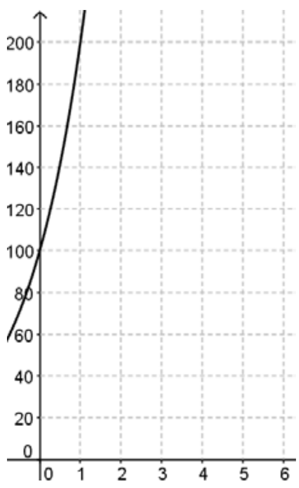
4.5 Function parameters as changing quantities

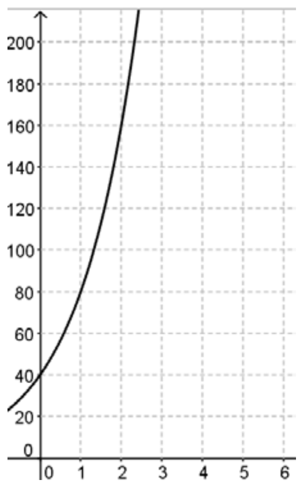
The third task sequence introduces a general formula representing an exponential function ($y = C \cdot a^x$). Students were encouraged to view parameters as changing quantities by examining how the parameters *C* and *a*, respectively, affect the graphical representation of a general exponential function such as $f(x) = C \cdot a^x$ (KE-T8).

To examine the influence of the parameter *C*, students were instructed to set the slider *a* to 2, and then they were asked to: "Drag the slider *C* so that the value of *C* varies. Describe in your own words how to see the value of *C* in the graph." To facilitate the discovery that the value of *C* can be found where the graph intersects the *y*-axis, we decided to give the hint that the value of *C* could be seen "in the graph". Furthermore, since it is recognized in the literature that students might have difficulties in using natural language to express how different values of a parameter affect the graph (Drijvers, 2003), we decided to add "in your own words".

The written data show that most students made the connection between the parameter *C* and the intersection with the *y*-axis (KE-C7). Many students commented on it by using phrases like "it is where the graph starts" or "*C* is the starting value of the graph". Although, we did not ask explicitly about which effect different values of *C* has on the shape of the graph, some students also recognized this effect and tried to describe it. However, both the video recordings and the written data show that there were students commenting that they think that the parameter *C only* shows were the graph starts and does not affect the shape of the graph. This misconception was found a few times in the written data by student comments such as "Whatever the value of *C*, the graph will increase at the same rate and is not affected by the starting value *C*". The misconception was also observed among one of the video recorded pair of students, which allowed us to make the observation below.

The students, by dragging the slider *C* back and forth several times, obtained the graphic views in Figure 4a-c. Through studying these views together, it might be easier to recognize that the shape of the graph is affected. However, the dynamic property of the slider bar implies that the views only are observable one at time.





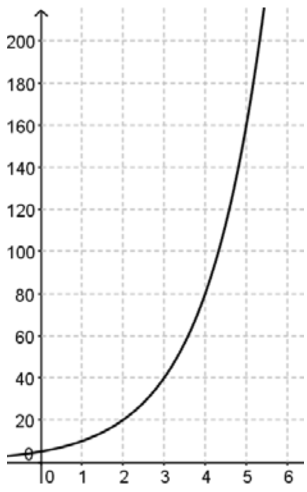


Figure 4a $C = 100$ Figure 4b $C = 40$ Figure 4c $C = 5$

A further reason why students think that the parameter C affects only the intersection with the y-axis might be their earlier experience with the corresponding algebraic representation. In the algebraic expression $y = C \cdot a^{x}$, students might regard the value of C as the 'beginning value' of the function, i.e. the value of y when x equals 0. To increase the chance that more students will discover how the parameter C affects both the location and the shape of the graph, and to reduce the risk of the observed misconception, we suggest reformulating the student request in the following way: "Describe in your own words how the value of C, in the formula $f(x) = C \cdot a^x$, influences the location and the shape of the graph". Further, we suggest adding a request for a mathematical explanation of students' observations.

Besides the identified misconception discussed above, a further observation among the video recording data made us conscious about a risk with the use of a slider bar as a changing quantity (KE-T8). For instance, one student commented: "If it is 50 here, it is 50 there", while at the same time pointing to the screen, first at the slider and then at the intersection with the y-axis. The student's focus seemed to be on the graph and the icon (representing the slider bar) in the Graphic View without paying any attention to the corresponding symbolic expression in the Algebraic View.

This issue was further confirmed in the subsequent tasks where students were expected to examine what will happen for values of the parameter a between 0 and 1, i.e. exponential *decay*. The students were encouraged to make a prediction of the shape of the graph of a particular function ($f(x) = 80 \cdot 0{,}5^x$) by making a paper-and-pencil sketch in a coordinate system. Below is an excerpt from the video recordings showing some students' discussions.

Student 1: It goes down then.

Student 2: Why?

Student 1: Because 1 is straight, 0.5 should go down.

Student 2: Yes, it should… but no, we do not have a slider here. If *a* was 0.5 [with emphasis on *a*], then it would go down but we have no sliders here, now we just have a formula ... a function. Should we ask for help, or…?

Student 1: No, we should guess; we do not need to be right.

The utterance "we do not have a slider here" reveals the students' difficulty in making a connection between the slider bar and the corresponding parameter in the formula. In revising the task design, we suggest inserting a copy of the functional expression close to the slider bar icon in the Graphic view. This implies a new technical key element of the instrumented action schemes to consider.

5 FEED FORWARD FOR RE-DESIGN

This section provides an overview of the analysis and reflection on student responses reported in the previous section. The observation of students' instrumented techniques, i.e. observable parts of their instrumented action schemes, in relation to a particular version of the task design, provided guidance for redesigning the task sequences to support smoother student acquisitions of the instrumented action schemes. During the analysis process some further key elements were identified. These elements are introduced in section 5.1 and a revised version of the planned instrumental genesis is provided in the Table 2.

In the subsequent sub-sections, results relating to the two tools, movable point and slider bar, respectively, are discussed to address the research question posed in section 2.4:

> How could the identification/awareness of key elements of instrumented action scheme related to movable points and slider bars for manipulating functional variables and parameters be used in the (re)design of tasks to support students' instrumental process?

5.1 Key elements in a revised version of the task design

Besides the suggested revisions reported in the results section, e.g. the introduction of the KE-T2 already in TS1, some new key elements both of conceptual and technical character were identified:

KE-C10: Realize that a graph consist of (infinity) many points

KE-C11: Realize when it is appropriate to change the scale of the *x*-axis to obtain an appropriate visible appearance of the object(s).

KE-C12: Be able to define a suitable domain for a parameter, appropriate to a particular situation

KE-C13: Realize that a slider bar corresponds to a parameter in a functional formula

KE-T10: Be able to change the scale of the axes by adjusting one axis at a time

KE-T11: Be able to adjust the settings of a slider bar

KE-T12: Know how to place the formula expression close to the slider bar icon in the Graphic view

Table 2 A revised version of the planned instrumental genesis throughout the three task sequences with changes in bold

Constructs	**TS 1**	**TS 2**	**TS 3**
Graph coordinates	KE-C1		
	KE-T1		
Function variables	KE-C2	KE-C3	
	KE-C10 **KE-C11** **KE-T2** **KE-T10**	KE-T3	

Function parameters -as 'fixed' general numbers	KE-4	
	KE-T4 KE-T5	
-as unknowns	KE-C5	KE-C6 **KE-C12**
	KE-T6	KE-T7 **KE-T11**
-as changing quantities		KE-C7 **KE-C13**
		KE-T8 **KE-T12**

5.2 The use of a movable point in manipulating variable coordinates

The findings concerning the key elements associated with ordered pair of numbers (KE-C1/T1) show how the use of technology revealed some lack of conceptual knowledge. There is a conceptual problem underpinning the fact that students enter "65 cm" or just "65" instead of (1,65). This example illustrates the intertwined relationship of technical and conceptual knowledge. The computer feedback "invalid data" indicates a technical obstacle (KE-T1), which in this case was shown to depend on lack of conceptual knowledge (KE-C1). In this way, the technology made the conceptual problem visible, which accords with the result reported by Drijvers and Gravemeijer (2005). The computer feedback draws the students' attention to a misunderstanding likely otherwise to have been overlooked. Furthermore, it helps teachers to become aware of students' lack of conceptual knowledge.

A movable point, then, can be developed to form two kinds of instrument. First, it can serve as an instrument for direct manipulation of variable coordinates to investigate their corresponding relationship in functional relationships (KE-C3/T2). This might enhance the opportunity for students to experience a graph as describing the relationship between variables and as consisting of infinitely many points (KE-C10), which are identified in the literature to be hard for students to grasp (Goldenberg, 1988; Hennessy, 1999). Therefore, we suggest this conceptual aspect as a further key element associated with the movable tool (see Table 2).

Second, a movable point can serve as an instrument for solving equations graphically, where the values of the variables y and x are unknown (KE-C2/T3). The results indicate that this kind of usage was rather straightforward for students. However, the associated difficulty encountered by several students was in creating and solving the corresponding algebraic equation when x is the unknown. This was unexpected since the construction and solution of the equations in this study was regarded by the teachers to be pre-knowledge among the participating students. Artigue (2002) emphasizes the distinction between "paper & pencil techniques and instrumented techniques" (p. 259) and argue that they have different epistemic value. We argue that our decision to intertwine the computer work with paper and pencil work might deepen students' understanding.

Finally, we want to emphasize that the scaling of the axes, particularly the scaling of one axis at a time (KE-T10), is an important technical key element of instrumented action schemes which needs to be taken into account in the orchestration of students' process of instrumental genesis in dynamic mathematics software environments. The conceptual counterpart to this key element is that students need to be able to realize when an unequal change of the scales of the axes is necessary to obtain an appropriate visual appearance of the object (KE-C11). Thus, in a revised version of the task design, this issue has to be addressed in some ways.

5.3 The use of slider bars in controlling parameters

As described in the planned instrumental genesis (Table 1), three distinct usages of a slider bar for manipulating parameters have been examined, depending on the view of parameters under consideration. Initially, the slider bar is used as a tool for easily changing the numerical value of a particular parameter in a functional formula (KE-T5). In this way, the parameter takes on different values, as a so-called 'fixed' general number, but the focus is on one value at a time (KE-C4). The results indicate that students rather easily adopted this technique and that most students successfully made the necessary translations between the graphical representation and the symbolic representation obtained in the Algebraic view. These findings are consistent with Drijvers' (2003) assertion that this view of a parameter is the basic level of understanding of a parameter.

There were in total four *a priori* identified key elements associated with the view of parameters as unknowns. Two of them, KE-C5 and KE-T6 concerned linear inequalities and the other two, KE-C6 and KE-T7, concerned exponential equations. Although, we have distinguished these according to the type of functional relationship they address, they might, in more general terms, be regarded as the same.

Concerning the technical aspects of manipulation of parameters as unknowns, our findings indicate a need for instrumentation knowledge about how to adjust the setting of a slider bar. The initial task design offered all necessary technical instructions (and related conceptual knowledge) over the course of the task sequences. However, the findings show occasions where it might be appropriate to address this as something for students to find out by themselves. For instance, the last task in TS3 could be revised in such a way that the solution of the problem requires an adjustment of the domain of the parameter *C*. This means that for students to be able to solve the problem they need to know how to adjust the setting of the parameter in a proper way, e.g. both how to do it technically and conceptually to estimate what would be a suitable domain. Thus, we suggest these as two interrelated key elements (KE C12/T11) to consider in a revised version of the task design (see Table 2).

Not surprisingly, the use of a slider bar for dynamically changing the numerical value of a parameter while investigating the features of the corresponding graph and formula turned out to be the most challenging for students (KE-C7/T8). Three main obstacles have emerged which are worth highlighting. First, in line with the observation made by Drijvers (2003), students' responses turned out to be rather superficial and they seemed to only recognize what happens without knowing why it happens. However, on reflection, this was not really a surprise because in the initial version of the tasks they were encouraged just to give descriptions, not to provide any explanations. In the revised version of the tasks students are encouraged to examine and explain the effect of giving different values to a linear equation parameter. Second, since this use of a slider bar provides many snapshots of the graph, there is a risk that some salient features of the graph remain hidden for students. We discussed some technical possibilities to address this challenge but since these might hide other important aspects of the parameter, we decided not to revise the tasks where this issue arose except for some minor reformulations of the text.

Finally, the findings show how students encountered difficulties in making connections between the slider bar and the corresponding parameter in a function expression. This confirms the concern raised by Zbiek et al. (2007) that students might regard the slider bar as a third representation of a function, besides the graph and the algebraic expression. As noticed, the problem is that the parameter could not be manipulated directly. Instead, the slider bar provides a third representation not directly connected to either the graph or the formula. We found this problem as rather demanding to solve by revising the task design. Although, we have suggested one technical way to make the connection between the slider bar icon and the corresponding parameter more obvious (KE-C13/T12), we regard this issue as an important one worth paying attention to for software designers, as well as for researchers and teachers.

6 CONCLUSION

This paper has reported the design rationale, refined as a result of trialling, for a set of teaching sequences which support the process of instrumental genesis through which students develop their proficiency in making use of movable points and slider bars in the mathematical field of function and graphs. In particular, it details key elements, both conceptual and technical in character, forming such a progression.

Although these findings relate to the use of a particular piece of software within mathematics, we argue that they illustrate a much more widespread need for careful analysis of instrumental genesis, and demonstrates an approach in terms of key elements which is generic and, thus, worth considering when designing tasks for a wider range of computer packages. Thus the findings about the conceptual and technical components of the instrumented action schemes which students need to develop provide a useful mapping of the major lines of instrumental genesis that the use of these tools call for. For instance, the reciprocal relationship between key elements of technical and conceptual character clarifies how seemingly technical difficulties among students might depend on lack of conceptual knowledge or vice versa. Equally, the considerations informing the design and redesign of task sequences point to important dimensions of the orchestration of the development of the requisite action schemes by students.

While the study presented in this paper has followed a similar approach to that pioneered in the study of the use of computer algebra software to tackle algebraic equations reported by Drijvers and Gravemeijer (2005) it has introduced a variant approach. By giving attention to redesign of the task sequences, it has been able to provide not just an inventory of the conceptual and technical elements of instrumented action schemes but to bring related issues of task design to the fore and highlight the role of task sequences in this. This reinforces the suggestion by Drijvers and Gravemeijer (2005) that one advantage of reducing instrumented action schemes to a list of items is that it provides concise and concrete guidelines for designing tasks.

Ultimately, these kinds of result may also be important for software design. For instance, to cope with a problem such as the one identified about the way in which a slider bar allows for manipulation of a parameter, there is a need for cooperation between researchers, task designers, teachers and software designers.

REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245-274.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in cabri environments. *ZDM, 34*(3), 66-72.

Bloedy-Vinner, H. (2001). Beyond unknowns and variables-parameters and dummy variables in high school algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 177-189). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Brunström, M., & Fahlgren, M. (2015). Designing prediction tasks in a mathematics software environment. *International Journal for Technology in Mathematics Education, 22*(1), 3-18.

Chazan, D., & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In J. Kilpatrick, D. Schifter & G. Martin (Eds.), A research companion to the principles and standards for school mathematics (pp. 123−135). Reston, VA: NCTM.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter*. Dissertation, Utrecht: Freudenthal Institute, Utrecht University.

Drijvers, P., & Gravemeijer, K. (2005). Computer algebra as an instrument: Examples of algebraic schemes. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The didactical challenge of symbolic calculators turning a computational device into a mathematical instrument* (pp. 174-196). New York, NY: Springer.

Ely, R., & Adams, A. E. (2012). Unknown, placeholder, or variable: What is *x*? *Mathematics Education Research Journal, 24*(1), 19-38.

Ferrara, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education. past, present and future* (pp. 237-274). Rotterdam, The Netherlands: Sense Plublisher.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the education of graphical representation functions. *The Journal of Mathematical Behavior, 7*(2), 135-173.

Goldenberg, E. P., Scher, D., & Feurzeig, N. (2008). What lies behind dynamic interactive geometry software. In G. W. Blume, & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Syntheses, cases and perspectives* (pp. 53-87). Charlotte, NC: Information Age Publishing.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155-177.

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of intrumental orchestrations. *ZDM, 34*(5), 204-211.

Hennessy, S. (1999). The potential of portable technologies for supporting graphing investigations. *British Journal of Educational Technology, 30*(1), 57-60.

Hoyles, C., Noss, R., Vahey, P., & Roschelle, J. (2013). Cornerstone mathematics: Designing digital technology for teacher adaptation and scaling. *ZDM*, *45*(7), 1057-1070.

Lagrange, J. (1999). Complex calculators in the classroom: Theoretical and practical reflections on teaching pre-calculus. *International Journal of Computers for Mathematical Learning, 4*(1), 51-81.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Trigueros, M., & Jacobs, S. (2008). On developing a rich conception of variable. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 3-13). Washington, DC: Mathematical Association of America.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281-307.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford, & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 8-19). Reston, VA: NCTM.

Valiente, O. (2010). *1-1- in education: Current practice, international comparative research evidence and policy implications*. Paris, France: OECD Publishing.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77-101.

Zbiek, R. M., & Heid, M. K. (2001). Dynamic aspects of functions representations.In H. Chick, K Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra,* 682-689. Melbourne, Australia: University of Melbourne.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169-1207). Reston, VA: NCTM.