Designing Prediction Tasks in a Mathematics Software Environment

1 INTRODUCTION

The view of what it means to master mathematics has changed over the past decades, which is reflected in mathematics curricula in many countries (Goos, Galbraith et al. 2003). Alongside content knowledge, different competencies such as reasoning and communication are highlighted. Research shows that computers can be utilized to create learning situations where students are encouraged to practice such competencies (e.g. Goos, Galbraith et al. 2003, Healy, Hoyles 1999). However, there is a need for different kinds of task to promote such student activities (Doorman, Drijvers et al. 2012, Hitt, Kieran 2009, Laborde 2001).

This paper draws on a study which aimed to design and investigate teaching sequences consisting of a computer-based lesson and a follow-up classroom discussion. It focuses on one computer-based lesson where students are expected to use a piece of mathematics software when working in pairs. In particular, the aim of this paper is to pinpoint and elaborate on the key didactical variables that it proved necessary to consider in designing prediction tasks that foster student reasoning concerning exponential functions in a mathematics software environment. The study used a design-based research approach, and this paper focuses on a stage that other studies discuss little, the initial iteration of the design research. In this way the paper addresses the difficulty that has been identified in researchers gaining insight into the detailed practices of other research groups engaged in this kind of research (Bell 2004). To accomplish this, the design tool of didactical variables, suggested by Ruthven, Laborde, Leach and Tiberghien (2009) is employed in the processes both of design and of analysis.

2 THEORETICAL BACKGROUND

This section introduces the theoretical perspectives guiding the design rationale, taking account of literature concerning mathematical reasoning in a school context, and the specific topic of functions and graphs.

2.1 Mathematical Reasoning in a School Context

There is a wide agreement that the traditional content goals of mathematics education have to be supplemented by goals focusing on mathematical competencies. These goals have been expressed in different ways in the literature, e.g. *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000), *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford et al. 2001), and the Danish *KOM-project* (Niss 2003). All these documents, and many others, emphasize mathematical reasoning as an essential component in school mathematics.

Although there is a consensus concerning the importance of reasoning in mathematics, there is no agreement on its definition (Yackel, Hanna 2003). According to Kilpatrick et al. (2001), it has been common to view mathematical reasoning as deductive reasoning, especially in the form of formal proofs. However, they introduce the expanded notion of "Adaptive reasoning" and state that:

> Our notion of adaptive reasoning is much broader, including not only informal explanation and justification but also intuitive and inductive reasoning based on pattern, analogy, and metaphor (p. 129).

This paper uses a wide conception of mathematical reasoning in line with the one suggested by Kilpatrick et al. (2001).

Many researchers have elaborated on different aspects of reasoning in dynamic software environments (e.g. Arzarello, Olivero et al. 2002, Healy, Hoyles 2002, Fahlgren, Brunström 2014). In some of these studies, students were encouraged to investigate a mathematical situation to be able to formulate, verify and explain a conjecture. Moreover, several researchers suggest that a computer screen can serve as a common referent to enhance joint reasoning (Goos, Galbraith et al. 2003, Arzarello, Robutti 2010, Hennessy 1999). According to Marrades and Gutierrez (2000), the main advantage of dynamic software is that it provides students with a wide range of examples. This, in turn, promotes pattern recognition and conjecturing. Thus, dynamic software environments facilitate *inductive reasoning* (e.g. Ruthven, Hennessy et al. 2008), i.e. "…an inference which allows the construction of a claim generalizing from some particular cases" (Pedemonte 2007, p. 29). These particular cases could be displayed in different ways leading to different kinds of pattern, e.g. numeric pattern, visual pattern or structural pattern. If inductive reasoning is based on a visual pattern it also could be regarded as *visual reasoning*. According to Hershkowitz (1990), visual reasoning or visualization "generally refers to the ability to represent, transform, generalize, communicate, document, and reflect on visual information" (p. 75).

Natsheh and Karsenty (2014) elaborate on how visual reasoning interrelates with *conceptual reasoning.* They introduce the construct "visual inferential conceptual reasoning" and underline that generated inferences preferably should reflect conceptual understanding. Even if there is accumulating evidence concerning the link between visual reasoning and conceptual understanding (Natsheh, Karsenty 2014), a risk has been identified that powerful visual images sway students "to solve problems simply by perception, without mobilizing any mathematical knowledge" (Healy, Hoyles 1999, p. 60).

Swan (2007) elicits several design principles in his literature review of task design. One principle is that students should be encouraged to express their reasoning and make it public to other students and their teacher. However, Sinclair (2003) points out the tendency for students to give sparse responses when asked to explain their reasoning. As a way to encourage students to explain, Doerr (2006) suggests a task design principle implying "…that the response to the task will require students explicitly to reveal how they are thinking about the situation by documenting and representing their ideas."(Doerr 2006, p. 8).

Several studies emphasize the advantage of letting students predict an outcome before investigating the situation further (Arcavi, Hadas 2000, Kasmer, Kim 2012, Laborde 2001). Kasmer and Kim (2012) suggest prediction as a particular reasoning aspect with the potential to provide students with increased mathematical understanding. Further, they stress that predictions should be accompanied by reflections to increase the opportunities for students to resolve any conflicts that may arise between their predictions and the answers. Laborde (2001) introduces the notion of prediction tasks in a dynamic geometry environment. In these tasks students are prompted to make predictions about a mathematical situation before investigating it in a dynamic geometry environment (*Cabri*). In cases where students' predictions are inconsistent with the results achieved from their computer investigations, there are good opportunities for students to reflect and try to find an explanation (Laborde 2001). Arcavi and Hadas (2000) point out the importance of students expressing their predictions explicitly. In this way students are encouraged "…to be clearer of how they envision the situation they are working on …" (p. 26).

To summarize, the literature suggests that computers could be used to encourage students to investigate mathematical situations and to formulate, verify, refine and explain conjectures. Ways to encourage students to perform different aspects of reasoning might be to (a) foster collaboration and communication by using a shared screen, (b) use prediction tasks, where students are supposed to first predict an outcome and then investigate the situation further and (c) explicitly request students to explain their reasoning, preferably in writing.

2.2 Functions and Graphs

It is well documented that it is a challenge for many students to fully understand the topic of functions and graphs (Leinhardt, Zaslavsky et al. 1990, e.g. Knuth 2000). For students to comprehend the concept of functions it is essential to be able to move flexibly between different perspectives on functions (Moschkovich, Schoenfeld et al. 1993). The literature offers several dualities concerning perspectives on functions.

While one duality is that functions can be conceived both *structurally* as *objects* and *operationally* as *processes* (Moschkovich, Schoenfeld et al. 1993, Lagrange, Psycharis 2014, Even 1998), another is the *local/global* duality (Even 1998, Leinhardt, Zaslavsky et al. 1990). The local view is characterized by a point by point attention, where the focus is on specific values of a function. A global view, on the other hand, draws attention to global features of a function such as the shape of its graph or the structure of its closed form equation. The dualities mentioned so far are closely related in the way that a global approach supports the view of functions as objects.

A further duality is the distinction between a *correspondence* and a *covariation* approach to functions (Confrey, Smith 1994, Confrey, Smith 1995, Ellis, Ozgur et al. 2012, Lagrange, Psycharis 2014). A correspondence approach builds on a view of functions as rules that determine a unique y-value to each value of x, i.e. the focus is on the correspondence between x and y. In a covariation approach, on the other hand, the focus is on how the value of y changes and how this is related to the change in x-value, i.e. how y covaries with x.

Another important dimension is the connection between different representations of functions (Moschkovich, Schoenfeld et al. 1993, Leinhardt, Zaslavsky et al. 1990, Knuth 2000). For instance, Moschkovich et al. (1993) emphasize the importance for students to be "…able to move flexibly across representations and perspectives" (p. 97). Leinhardt et al. (1990) discuss a specific kind of tasks where students are supposed to translate between different forms of representation, i.e. algebraic, graphical, tabular and verbal representations. One challenge for students, pointed out by Leinhardt et al. (1990) and Knuth (2000), is the translation in the graph-to-equation direction.

An issue raised in relation to the increased use of graphing software is the one of scale and scaling of axes (Zaslavsky, Sela et al. 2002, Goldenberg 1988, Leinhardt, Zaslavsky et al. 1990). In textbooks, graphs often are presented as static diagrams with the coordinate axes graded in an appropriate way (Zaslavsky, Sela et al. 2002). In a software environment, on the other hand, the scaling of axes is often left for students. Goldenberg (1988) points out the necessity for students to be aware of the effects that a chosen scale has on the visual appearance on the screen. In connection to this, he points out that a "zoom" operation, i.e. changes of both axes by the same factor, is more intuitive than using different scaling factors on the axes.

The study reported in this paper focuses on exponential functions. Although, research in this domain is sparse, it does show that it is a challenge for students to fully understand exponential functions (Ellis, Ozgur et al. 2012, Castillo-Garsow 2013). One problem is that students tend to revert to linear functions when they are introduced to nonlinear functions (Leinhardt, Zaslavsky et al. 1990, Green 2008, Kasmer, Kim 2012). Confrey and Smith (1994, 1995) advise a covariation approach to support students' understanding of exponential functions, by making the 'rate of change' visible. To find the 'rate of change' pattern it is necessary to consider a multiplicative rate of change, i.e. consider the ratios between succeeding y-values (Confrey, Smith 1994, Confrey, Smith 1995, Green 2008). However, even if a covariation pattern is recognized, it is a challenge for many students to find the correspondence relation with a closed form equation (Confrey, Smith 1994, Doerr 2006).

In summary, to promote students' comprehension of the concept of functions, the literature suggests that students should be encouraged to (a) use different forms of representation of functions and make translations between them, (b) experience both a local and global view of functions, (c) use both a correspondence and covariation approach to functions and (d) pay attention to the effects of scaling of coordinate axes.

3 METHODOLOGY

This study uses a design based research approach where the label *design experiment* is used (Cobb, Gravemeijer 2008, Cobb, Confrey et al. 2003). A design experiment in educational research is characterized among other things by its close connection to an educational practice (Cobb, Confrey et al. 2003). Design experiments involve a cyclic process of (re)design and analysis of instructional activities. This paper, however, concerns only the first iteration in the design experiment that we undertook to design, trial and analyse a sequence of tasks aimed at encouraging student reasoning about exponential functions when working with a piece of dynamic mathematics software, in this case *GeoGebra*. In the following, this section first introduces the study participants, and then it describes the implementation of the three main phases of the design experiment; preparing for the experiment, conducting the experiment and, finally, retrospective analysis of the experiment (Cobb, Confrey et al. 2003, Gravemeijer, Cobb 2006).

3.1 Participants

Together with four upper secondary school teachers, the authors of this paper formed the research team. The teachers were working at two different schools but under similar conditions, e.g. both schools provide their students with a computer of their own. In total 39 pairs of students from four different classes participated in the study. They are all first year students at upper secondary school following the same mathematics course[1]. None of these students had previous experience of working with this kind of mathematics software. The teachers participated voluntarily and they received no compensation for the time they spent on the project. Therefore, it became an important condition that the study should be part of the regular curriculum.

3.2 Preparing for the Experiment

3.2.1 Basic Conditions

Both the research literature and our own experiences suggested three basic conditions for the experimental activity. First, students were to work in pairs with *one* computer per pair. The purpose of this is that the computer screen should provide a shared object for discussions

[1] Course 1b. Worth noting is that this course is not preparing for further studies in mathematics.

between students (Goos, Galbraith et al. 2003, Sinclair 2003). Second, the tasks were to be intertwined with computer instructions since students would be expected to do the constructions by themselves, i.e. they would not use prepared applets. The main reason for this choice is to enhance the scope for the mathematics software to become an instrument for the students (Verillon, Rabardel 1995). Finally, students were, when appropriate, to be expected to formulate their conclusion in writing (Doerr 2006).

3.2.2 Didactical Variables

To articulate the theoretical rationale for different design choices and to analyse them after empirical testing, the design tool of *didactical variables* was employed. Ruthven et al. (2009) show how design research has generated tools for "…the design of learning environments and teaching sequences informed by close analysis of the specific topic of concern…" (p. 329); these design tools provide a framework to identify and address specific aspects, both in the construction of an initial design and in subsequent revisions or refinements with respect to empirical findings. Ruthven et al. argue that didactical variables (Brousseau 1997) is a design tool which can be used independently of the theoretical framework in use. This study used the didactical variables tool to emphasize important choices that might affect students' reasoning. According to Ruthven et al. (2009) didactical variables are features of the task environment which act as "key levers to precipitate and manage the unfolding of the expected trajectory of learning" because they "significantly affect students' solving strategies" (p. 334). The identification of didactical variables "…starts from analysis of the knowledge available to students. Observations of how situations play out with students in the classroom may then reveal further variables not identified through prior analysis." (p. 334). The literature served as guidance in the *a priori* identification of the following seven didactical variables:

Didactical Variable 1 (DV 1): Choice of medium (i.e. paper and pencil or computer)
Didactical Variable 2 (DV 2): Prediction task or not
Didactical Variable 3 (DV 3): Ask for an explanation or not
Didactical Variable 4 (DV 4): Choice of representation form
Didactical Variable 5 (DV 5): Local or global approach
Didactical Variable 6 (DV 6): Correspondence or covariation approach
Didactical Variable 7 (DV 7): Specify scaling or not

3.2.3 Research Meetings

At the first research team meeting, the instructional goals and the starting points (Gravemeijer, Cobb 2006) were discussed and specified. The research team agreed that functions, especially exponential functions would be a suitable topic to utilize the affordances provided by a dynamic mathematics software environment. Next, the authors proposed an initial draft of a task sequence (in the format of a worksheet), which formed the basis for a subsequent planning meeting. At this meeting, each task was discussed and revisions were made based on feedback from the teachers who provided valuable information regarding the participating students' capabilities and their current practices.

The outcome from this planning meeting is what Gravemeijer (2004) designates as a *conjectured local instruction theory*. This includes not only the designed task sequence, and the tools that will be used, but also considerations on didactical variables and conjectures about student reasoning that might evolve when they engage with the tasks.

3.3 Conducting the Experiment

3.3.1 Data Collection

Three main sources of data were collected in the classrooms. First, the written responses in the worksheets from all students were copied. Second, video and screen recordings from one pair of students in each class were gathered. The camera was positioned at an angle behind the students to capture the computer screen and also student gestures, e.g. *if* and *how* they were pointing at the screen. Third, all teacher-student interactions during the lesson were audio recorded using a microphone attached to the teacher. In addition, field notes were taken to identify which students the teacher interacted with at specific moments. These data were, for instance, used in the analysis process to establish whether students written responses were influenced by the teacher. Besides, the planning meetings and the follow up meeting were audio recorded, and to capture the teachers' spontaneous reaction after the lesson, a short debriefing meeting with each teacher was held.

3.3.2 Data Analysis

The written responses from all pairs of students were transcribed to get an overview of this data. The video recordings of the target pairs allowed a more detailed observation of the student activities leading to their written responses. We initially watched and listened to the video recordings several times, and significant moments were identified, time coded, and then transcribed. Significant moments are connected sequences of student utterances and actions which reveal and illustrate significant results related to the conjectured local instruction theory.

Related to the local instruction theory, an interpretative framework, closely tuned to the prediction tasks, is developed. The framework describes and categorizes expected kinds of reasoning in this particular piece of work. When describing what kinds of anticipated student activity we count as reasoning, we distinguish between *visual reasoning* and *symbolic reasoning*. The notion symbolic reasoning embraces both a *numeric* and an *algebraic* approach, in line with Healy and Hoyles (1999). Further, we explicate what considerations we anticipate that students will base their reasoning on. The framework is reported for each prediction task under the sub-heading Aim and Anticipation in the Results and Discussion section.

In the analysis of the empirical data, the design choices and their associated expectations of student activity, made *a priori*, were evaluated and discussed. The interpretative framework was used to identify different kinds of student reasoning. The concept of didactical variables served as a tool in the interpretation of the means by which expected or non-expected student reasoning is supported or constrained, both in terms of referring to the variables identified *a priori*, and identifying further variables of this type.

During the analysis process some unexpected kinds of reasoning and four further didactical variables were identified. In some cases, suggestions for revisions or refinements of the tasks are made. In this way, the outcome from the experimenting phase is an improved local instruction theory, which offers a frame of reference to support teachers that may adopt or adapt the task sequence into their own practice (Gravemeijer, Cobb 2006) .

3.4 Conducting Retrospective Analysis

According to Cobb et al. (2003), a primary aim with the retrospective analysis is to "…place the design experiment in a broader theoretical context, thereby framing it as a paradigm case of the more encompassing phenomena specified at the outset." (Cobb, Confrey et al. 2003, p. 13). In the retrospective analysis the prediction tasks were reviewed to discern any patterns concerning the use of particular didactical variables and the kinds of reasoning obtained. This was accomplished to get an overview and coherence across the prediction tasks concerning the employment of the didactical variables. The Retrospective Analysis and Discussion section will provide the main conclusions of this paper.

4 RESULTS AND DISCUSSION

The overall aim of the task sequence in terms of developing mathematical content is to make students aware of the difference between linear and exponential growth. In this section we will introduce the first five tasks and elaborate on three prediction tasks (Task 2, Task 4 and Task 5[2]).

In Task 1, students are introduced to a real world context, which frames the whole task sequence: "The height of a sunflower is 50 cm when it is measured for the first time (June 1). After that the sunflower grows so that it becomes 30 % higher each week." The students are then instructed to calculate the height of the sunflower after one week (65 cm) and to insert the points, representing the height at the start and after one week, in *GeoGebra*. They are also encouraged to adjust the scale of the axes in an appropriate way. This is a prerequisite for students to be able to see the points and to perform the subsequent tasks. The main reason to use *GeoGebra* (DV 1) already at this stage is that these points, together with points inserted throughout Task 2-4, are essential in Task 5. Further, the link between graphical representation and ordered pair representation of points, and scaling issues that may occur might elicit some instructive reasoning.

4.1 Task 2

4.1.1 Aim and Anticipation

One reason to design the prediction task (DV 2) in Figure 1 is the anticipation that many students will guess that the height of the sunflower is 80 cm after two weeks, i.e. that it grows linearly. We want to create a situation where students who think 'linearly' will experience a conflict when they then check their prediction against the result of a calculation, and be encouraged to reflect on the disparity between their predicted and calculated answer (provided that their calculation is correct). To encourage students to elucidate their thinking we decided asking them to explain (DV 3) both their prediction (Task 2a) and any disparity (Task 2d).

[2] One additional prediction task is included in the task sequence. However, for reasons of space, this task is excluded from this paper.

2. Guess (without any calculations) where the point showing the height of the sunflower after two weeks will be.

Insert this point by using the tool:

a) Explain why you chose to place the point at this specific position. BOX [3]

b) Note the point's coordinates (in the Algebra View): ________________

c) *Calculate* the height of the sunflower after two weeks (round the answer to the nearest cm) BOX

Insert this point by entering its coordinates into the "**Input Bar**"

d) Compare the two points (the one you guessed and the one you calculated). If the points do not coincide, try to explain why. BOX

Figure 1 Task 2

The reason for choosing a graphical representation (DV 4) when asking students to guess, is to encourage a visual approach to prediction capable of disclosing the misconception that given two points any further point on the graph should lie on a linear extension (Leinhardt, Zaslavsky et al. 1990). Further, we think that a graphical representation might prevent students from using any kind of calculations. However, since the participating students are not used to making predictions in mathematics, we believe that some students will be hesitant and make a calculation anyway, because this is what they are used to doing. To further reduce this risk, we chose to add "without any calculations" within brackets.

There is a risk, though, that some students who guess 'linearly' will not notice the disparity between their guessed and calculated point, since the disparity is rather small. Further, their scaling of axes will influence how they experience the disparity. One possibility would be to specify an appropriate scaling of the axes (DV 7) in the instructions (before Task 2). However, the research team decided not to provide this scaffolding since it might be instructive for students to reason about scaling issues.

Expectations Task 2a. In this task the forms of reasoning that we expect students to use are Exponential[3] Numeric Reasoning (ENR), Additive Numeric Reasoning (ANR) or Linear Visual Reasoning (LVR). In outline, then, the expected forms which we count as displaying reasoning in student responses are as follows:

ENR: Students appreciate that 30% is not a fixed amount and explain by referring to the fact that it now is 30% of a higher number.

ANR: Students assume that 30% is a fixed amount and express this by stating that it grows 15 cm every week.

LVR: Students assume that the points should follow a line and thereby that it grows 15 cm every week.

[3] We anticipate that some students will appreciate that the growth is proportional to the height, i.e. exponential growth, while others just realize that 30 % of a higher number must imply a greater growth. We use the term "exponential" to cover both these kinds of reasoning.

Expectations Task 2d. Since it is desirable to put 'linearly thinking' students in a situation where they experience a conflict and also are given opportunities to resolve this conflict the responses given by 'linearly thinking' students are the most interesting. Although it might be the visual evidence of the difference between predicted and calculated points that makes the conflict visible for the students, we expect students that resolve the conflict to use ENR. We expect this reasoning to be of the same kind used by those students who made correct predictions in Task 2a.

4.1.2 Observations, Analysis and Discussion

In total, 20 out of 39 pairs made a guess that indicates 'linear' thinking. This shows that they needed an opportunity to investigate and elaborate on this topic. Hence, we argue that the choice to use a prediction task (DV 2) was an appropriate one.

It was observed from both the audio and the video recordings that it was a challenge for many students to scale the axes appropriately (before tackling Task 2). This was also pointed out by the teachers in the follow up discussion. This problem turned out to give opportunity for reflection, in particular when students could not find their points on the screen. In these cases they reflected on the reason why they could not see any points in the coordinate system even though they could see them in the algebra view. Hence, we stick to our decision to not specify scaling of axes in the instructions, even if teachers often had to remind students to adjust the scale of the axes and show how to adjust one axis at a time. To reduce this problem, we suggest that students are reminded of this instrumentation issue before working with the activity.

In Task 2a, half of the students (19 out of 39 pairs) provided one of the three expected kinds of reasoning. These are illustrated with examples of student responses below:

ENR: "*y* is a little more than 80 because it is 30% of a higher number than before"

ANR: "Because it should increase weekly by 30%, and 30% was 15 cm. So after a week the plant was 65 cm, then it should be 15 cm higher week two"

LVR: "Because it should be a straight line. It grows equally each week"

Some of the student explanations in the 'LVR category' indicate that these students might think that given two points any further point on the graph should lie on a linear extension (Leinhardt, Zaslavsky et al. 1990). Since we think that it is important to access this type of visual misconception among students and to get these students to experience a conflict, we argue that the choice of representation form (DV 4), i.e. graphical representation, is appropriate.

Among the rest of the student responses in Task 2a, mainly three categories of responses were discerned. First, six pairs calculated (instead of guessed) the height of the sunflower after two weeks and referred to this calculation in their explanations. This problem might be reduced by emphasizing the request "without any calculations", e.g. in the following way (changes in bold):

2. Without any calculations, guess where the point showing the height of the sunflower after two weeks will be.

An alternative or complement to this revision might be to make the numbers 'unfriendly' to mental calculations, e.g. initial height 42 cm and rate of growth 34 % per week. Hence, we suggest the choice of numeric values as a new didactical variable (DV 8). If one chooses to make such a change it is, of course, important to check what consequences this might have on the other tasks.

Second, there were three pairs that only described what the two coordinates of the specific point represent, e.g. they expressed that the value of x must be 2 since it is after two weeks. Maybe, it is the wording "place the point at this specific position" in Task 2a, which entails this focusing. To change these students' focus, we consider revising Task 2a in the following way:

2. a) Explain how you came up with your guess.

We suggest that the choice of wording to direct students' focus on what to explain is crucial, and therefore we consider this as a further didactical variable (DV 9).

Finally, there were 11 pairs that gave responses which evaluated the answer rather than explained it, e.g. "because it is right" or "it seemed reasonable". This problem, also identified by Sinclair (2003), indicates a need to discuss what it means to explain in mathematics in these classrooms. Another factor might be that some students find it difficult to put their mathematical thinking into words.

In Task 2d, the responses given by the 'linearly thinking' students are the most interesting. Among these, there are two categories that we count as reasoning, one of which is the expected one (ENR), illustrated by the following transcript:

> "Because we did not take 30% of 65 which was last week. Instead we increased by as much as the first week"

Just about one third of the 'linearly thinking' students gave a response belonging to this category. In the other kind of student response counted as reasoning (5 pairs) the students realize that their guess is incorrect. However, their explanations are about why they made a 'linear guess', and not about why it is a non-linear growth, i.e. they use ANR or LVR.

As expected, there were students that noticed a difference without giving any explanation, e.g. "It grew faster than we had expected". Further, some students just referred to inaccurate predictions, e.g. "It was as close as we could get without calculating". This result prompted the research team to consider how to improve the task to get more students to give an explanation in the 'ENR category'. Concerning the students that focused their explanations on the reason why they placed their predicted point as they did, i.e. more or less a response to Task 2a, we suggest a revision to change their focus (DV 9). One way to do this might be to revise Task 2d in the following way (changes in bold):

d) Compare the two points (the one you guessed and the one you calculated). If the points do not coincide, try to explain why **the sunflower grows more than you thought or less than you thought.**

The reason why this formulation might work better is that it highlights what the explanation should be about. It also moves the attention from the points to the growth of the sunflower, and thereby a covariation approach is emphasized (DV 6). This revision, with a more explicit formulation of the question, might also support students that did not provide any reasoning. To make questions more pointed as a way to avoid trivial student responses is also suggested by Sinclair (2003).

In Figure 2, Task 3 is introduced since it is a preparation task for both Task 4 and Task 5. It is also a confirmation task because it involves calculating the height of the sunflower after one and two weeks, i.e. the values students were supposed to calculate in Task 1 and Task 2c, respectively.

3. What is the height of the sunflower after 3 weeks? After 4 weeks? After 5 weeks?

Continue to fill in the table below (round the answers to the nearest cm):

Number of weeks	The height of the sunflower (in cm)	The corresponding point
0	50	(0,50)
1	$50 \cdot 1.3 = 65$	(1,65)
2	$50 \cdot 1.3^2 = 84.5 \approx 85$	(2,85)
3		
4		
5		

Insert the rest of the points (from the table) in the coordinate system by entering its coordinates into the "**Input Bar**".

Figure 2 Task 3

4.2 Task 4

4.2.1 Aim and Anticipation

At the planning meeting, we discussed whether there, at this stage, still are students who think linearly. We agreed that probably not all students have grasped how repeated percentage change works and decided to design this task (Figure 3) as a prediction task (DV 2) to give these students a further opportunity to disentangle any misconceptions. For the rest of the students, this task might work as a confirmation task.

4. Guess (without any calculations) where the point showing the height of the sunflower after six weeks will be.

Insert this point by using the tool: Change the color of the point!

a) Note the point's coordinates (in the Algebra View): ______________________

b) *Calculate* the height of the sunflower after six weeks: ______________________

Insert this point by entering its coordinates into the "**Input Bar**".

c) Compare the two points (the one you guessed and the one you calculated). Comments?

BOX

Figure 3 Task 4

The choice of graphical representation (DV 4) is made to encourage students to reflect on the shape of the graph which the points belong to, i.e. a global view (DV 5) of the underlying function. Even if we anticipate that most students will base their guess on the visual pattern in the graphical view, there might be students who focus on covariation in the table to find a numeric pattern.

We discussed that it might be difficult for students to determine what constitutes a ‘good enough' guess in Task 4c. Even for students who already think ‘nonlinearly’, the points will probably not end up in exactly the same position. The visual impression of the quality of the guess is highly influenced by the choice of scaling of the axes. However, we decided not to specify an appropriate scaling of the axes in the instructions (DV 7). This choice was based on the same assumption as in Task 2.

We also discussed whether to ask more explicitly for an explanation (DV 3) of any difference between the points or not in Task 4c. We agreed to just ask for comments, since it might be difficult for students who already realize the idea of exponential growth to know what to explain if they place the point a bit too high or too low. We stress that it is important that there is something to explain when asking for an explanation. Otherwise, students might get a wrong impression of what it means to explain in mathematics. Further, just asking for comments might also open up for other kinds of answer based on more reflective thinking. Thus, we decided to make it a more open question and just ask for comments.

Expectations Task 4. Since we decided not to ask students to explain their prediction (DV3), the only empirical evidence we got concerning reasoning behind their guess is from the video recording. We expect most of the students to use Pattern-Based Visual Reasoning (PBVR), even if there might be students that use Pattern-Based Numeric Reasoning (PBNR). In outline, then, the expected forms which we count as displaying reasoning in student activities are as follows:

PBVR: Students realize that the growth of the sunflower must increase by reflecting on the shape of the trace of points discerned on the screen

PBNR: Students realize that the growth of the sunflower must increase by reflecting on a numeric pattern in the table

Expectations Task 4c. Since we decided to just ask for comments in this task, we anticipate that most students just will give short answers, without reasoning, like “we guessed too little”. However, since the question is rather open, there might also be students that provide more comprehensive responses.

4.2.2 Observations, Analysis and Discussion

About half of the 35 pairs that performed this task made a guess with the *y*-value in the interval 236 – 246, i.e. 241 ± 5. Although, it is sometimes hard to interpret from the written data if students think linearly or not, several answers indicate that some students still think linearly, e.g. 9 pairs made a guess with *y*-value less than 226. Hence, we suggest that this task is important and that it was appropriate to use a prediction task (DV 2).

Data from the video recordings show that at least two of the four video recorded pairs still thought that the sunflower grows constantly each week. Both these pairs seem to use a kind of local view, where they focus only on a few points, when making their guess. This is illustrated in the following excerpt from one of these pairs (Eric and Sarah):

Eric: We are supposed to **guess** its position
Sarah: But now the spaces between them are quite equal
Eric: Where should we guess?...because the jumps are around 40 all the time, I would guess somewhere here [points at the screen and inserts a point at (6,220), see Figure 4]

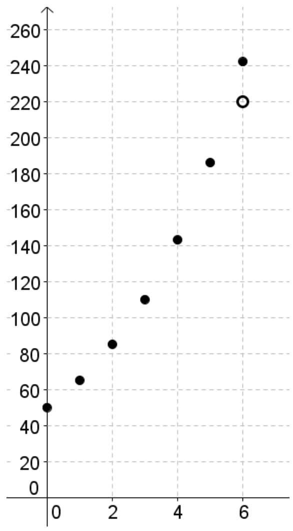

Figure 4 The screen image, obtained by Eric and Sarah, with both the guessed (small circle) and the calculated point (6,241).

The other pair of students even moves the coordinate system on the screen so that only parts of the *y*-axis are shown. Even if both pairs focus on just some of the points, we interpret their activities, i.e. utterances and gestures, as PBVR although they recognize a *linear* pattern. Even if the differences between the *y*-values of the guessed and the calculated point are 21 cm (Eric and Sarah) and 11 cm respectively, both pairs were satisfied with their guess. Eric and Sarah expressed in Task 4c that they “ended up close enough”.

All four video recorded pairs based their guess on the graphical representation. However, only one pair (John and Alex) focused on a visual pattern between *all* the points, i.e. used a more global approach. They used the computer cursor and pointed through all the points and thereby obtained a trajectory. Hence, we interpret that these students acted as anticipated, i.e. used PBVR. The guessed point from these students has the coordinates (6,245), i.e. the difference between the *y*-values of the guessed and the calculated point is 4 cm. When these students were asked to compare the guessed and the calculated point (Task 4c), they used the zoom tool to better see the difference (in the graphics view). Their response in Task 4c was “we guessed a higher number again”. It is notable that Eric and Sarah seemed more satisfied with their guess than John and Alex, even though John and Alex made a much better guess.

As expected, most student responses in Task 4c were rather short, commenting that they were satisfied with their guess and/or that there is a difference between the points. However, some students gave rather more extensive answers, often in the way that they explained how they came up with their guess. In this way we also got some written empirical data concerning the thoughts behind student guesses. Some of these answers indicate that students based their guess on covariation in the table, e.g. “we checked the increase between week 4 and week 5 and added a couple of cm for week 6”. We interpret this unexpected response as ENR, since they base their guess on the assumption that the growth of the sunflower is increasing. Other answers indicate that some students based their guess on a global graphical pattern, e.g. “I looked at the bend and tried to estimate by eye”. That is, they acted as anticipated (PBVR).

Among the nine pairs that made a guess with the *y*-value less than 226, just one pair gave an answer that clearly reveals that they have resolved a conflict and now understands how it works: “the number gets bigger and bigger and that is why it increases more and more". This response is an explanation of *why* the growth of the sunflower increases, which we interpret as ENR. Since we regard this response as desirable, the question is how to revise this task to get more 'linearly thinking' students to disentangle the misconception.

We suggest that the main reason why the task turned out to be less successful was the choice of graphical representation (DV 4), where students were supposed to use a global view (DV 5). Thus, a substantial revision is required. It is important to create a situation where students who think linearly perceive that a linear guess is inappropriate. Further, to unravel the misconception they must be encouraged to reflect on *why* the sunflower does not grow equally every week. To do this we suggest directing students' attention to the difference between subsequent y-values in the table in Task 3 (Figure 2). In this way students would be encouraged to focus on the *growth* instead of the *height* and use a covariation approach (DV 6). This implies a change of representation form (DV 4), from a graphical representation to a tabular one.

To direct students' focus on past growth before they make the guess, we choose to add a question, asking for the sunflowers' growth between Week 4 and Week 5.The students are then asked to guess how much the sunflower grows between Week 5 and Week 6, and thereafter to check their prediction by calculation. This decision was made to encourage students to use a local approach (DV 5). Another possibility would have been to ask students to calculate the weekly growth for each week to direct their focus on the covariation pattern (DV 6). This would support the idea of exponential variation in which the local rate of change increases, in contrast to linear variation in which the local rate of change is constant. In this way probably more students would make a guess that is close to the calculated value. However, there is a risk that students who still think linearly will make a close guess without reflecting on the underlying mathematical reasons. If these students instead have calculated the growth from Week 4 to Week 5 merely, they probably will guess that it grows equally between Week 5 and Week 6. As well as not breaking the linear mental set, this will just help students find a better basis for linear prediction. In this way there is a better chance that they will experience a conflict when they compare their guess with the calculated value.

To reduce the risk that students are 'satisfied' with their guess despite a relatively large difference between their guessed and calculated value, or 'unsatisfied' despite a relatively small difference, we suggest specifying a numerical value for an acceptable deviation. If the difference is more than 8 cm[4], we ask for an explanation instead of just comments (DV 3). The choice to offer students guidance or not concerning what constitutes a good enough guess, we suggest as a new didactical variable (DV10).

The suggested revision implies a shift of medium (DV 1). In the first version, students were intended to make their prediction in *GeoGebra*. In the revised version, on the other hand, students are supposed to write their prediction on the worksheet. Concerning the reasoning aspects, we suggest that the revision will encourage students to use ENR, and that more 'linearly thinking' students will both experience and resolve any conflict.

4.3 Task 5

4.3.1 Aim and Anticipation

By this stage we think students are prepared to produce an algebraic representation (DV 4) that describes how the sunflower grows, i.e. a closed form equation. The table already

[4] The choice of 8 cm as the limit is just a suggestion based on our anticipation that students who have grasped how repeated percentage change works will make a guess in the interval 55 ± 8 cm.

created in Task 3 provides a basis for this task (Figure 5). However, it is documented in the literature (Confrey, Smith 1994) that it is a challenge for many students to find the correspondence relation with a closed form equation from a tabular representation, most often associated with a covariation approach. To help students to see the pattern in the calculations in the table in terms of the correspondence relation (DV 6), it is important that each calculation is expressed in a way which relates it both to the initial value (50) and to the scaling factor (1.3); otherwise it is likely that students will perform iterative calculations ($y_{n+1} = y_n \cdot 1.3$) and perceive the pattern accordingly. To direct their focus towards the correspondence relation, we chose to explicitly refer to the correspondence calculations in the table. This choice to explicitly refer to proceeding tasks, we suggest as a further didactical variable (DV 11). Task 3 also serves as a base for Task 5 in such a way that students could use the points which they were instructed to insert in *GeoGebra* to check if a suggested formula results in a graph that follows the points. Since there now are good opportunities for students to guess and check what the formula looks like, a prediction task is suggested (DV 2).

5. The height of the sunflower *y* (in cm) after *x* weeks follows a certain pattern that can be described by a *formula*. Study your calculations in the table (Task 3) and try to figure out what this formula could be like.

When you think you have found the formula for the sunflower: Try it by inserting your suggestion into the **"Input Bar"**.

a) The formula for the sunflower: ______________________

b) Explain how you know that you have the right formula. BOX

Figure 5 Task 5

The overall aim of this task is to encourage students to make connections between different representations. They start by finding patterns in the calculations in the table, and then they are supposed to give an algebraic representation that can be used to produce a graphical representation. They are then supposed to observe the connection between the continuous graph obtained and the discrete graph already on the screen.

Expectations Task 5a. Since we decided not to ask students to explain how they came up with their predicted formula (DV3), the only empirical evidence we got concerning this is from the video recordings. When predicting a closed form equation, we expect students first to use Pattern-Based Numeric Reasoning (PBNR) to discern the numeric pattern in the table. However, for students to be able to make the translation from this pattern to the corresponding algebraic representation, they need to use Generalization-Based Algebraic Reasoning (GBAR). In outline, then, the expected forms which we count as displaying reasoning in student activities are the following:

PBNR: Students notice that 50 and 1.3 are constant and that the exponent increases by one every week by referring to a numeric pattern in the table

GBAR: Students translate the numeric pattern into a closed form equation and realize what *x* and *y* represent

Expectation Task 5b. We anticipate that students will use Visual Reasoning based on 'Graphical Knowledge' (GKVR) in the way outlined below:

GKVR: Students explain that they have got the right formula by referring to the connection between the discrete graph and its corresponding continuous graph.

4.3.2 Observations, Analysis and Discussion

As many as 32 pairs, out of the 35 pairs that suggested a formula, gave a correct formula in Task 5a. However, the written data do not show how many attempts students made before they came up with the right formula. Further, the audio recordings reveal that several pairs asked the teacher for help with this task. This was also pointed out by the teachers in the follow up discussions.

Two of the video recorded pairs did not display any reasoning when predicting the closed form equation. The other two pairs, on the other hand, discussed a lot on this issue. Both these pairs performed the expected kind of reasoning, PBNR, and in a quite straight forward manner. However, the other expected kind of reasoning, GBAR, turned out to be more challenging, as illustrated in the excerpts below.

Jim: …*x* must be weeks, or? ...*y* cm after *x* weeks
Roger: Thus, it is the height of the sunflower *y* cm times *x* raised to …no
Jim: But how far do they want it to go then, in weeks?
Roger: It is not known, that is why we call the weeks *x*.
Jim: Yes, then I understand!
[…]
Roger: Yes, the height of the sunflower is *y* cm ... it becomes a calculation like this: *y* times ...

The students test their suggestion in *GeoGebra* but receive "invalid data" as feedback.

Jim: It must be wrong, delete the *y*
Roger: Then, it must be 50 as I said before
Jim: mm
Roger: Let´s test it…then it becomes 50 times…1 point 3
Jim: Raised to *x*.

The excerpts above also reveal a common misconception, observed several times in both the audio and the video recordings. Many students tend to write "y * ..." (e.g. "y * 1.3^x") instead of "y = ...". We suggest that one reason for this might be that they are stuck in an iterative way of thinking, i.e. scaling up the previous *y*-value step by step ($y_{n+1} = y_n \cdot 1.3$). We discussed two possible revisions to scaffold students and make the correspondence view even more visible (DV 6). One possibility would be to make the following addition in Task 5a (changes in bold):

a) The formula for the sunflower **is *y* =** ______________________________

Another revision would be to change the table in Task 3 in the following way, see Table 1.

Table 1 Suggested changes in bold

Number of weeks *x*	The length of the sunflower (in cm) *y*	The corresponding point **(*x*,*y*)**
0	**$y = 50$**	**(0,50)**
1	**$y = 50 \cdot 1.3 = 65$**	**(1,65)**
2	**$y = 50 \cdot 1.3^2 = 84.5 \approx 85$**	**(2,85)**

In both cases, the chance that students will predict the correct formula might increase. However, one risk with too much scaffolding is that misconceptions remain invisible and thereby unsolved for students. Further, as in the case with Jim and Roger, misconceptions of this kind turned out to result in some instructive discussions triggered by computer feedback. If students resolve a conflict on their own, as in the case with Jim and Roger, the chance for understanding and established knowledge might increase. After careful consideration, we decided not to make any modifications.

The written data from Task 5b reveal that just over one third of the students (13 out of 32 pairs) responded as anticipated (GKVR), e.g. "The curve produced follows our points perfectly". Since an intention with this task is to draw students' attention to the connection between the points (the discrete graph) and the continuous graph, we suggest the following revision (changes in bold) to direct students' focus (DV 9) on the graphical representation (DV 4).

b) Explain how you, **by looking at the graph, can be sure** that you have the right formula.

Among the rest of the student responses almost one third displayed an unexpected kind of reasoning by justifying why the formula must be correct. We denote this kind of reasoning as Justifying Algebraic Reasoning (JAR). The following transcript illustrates this kind of reasoning.

> "Our constant number is always 50 cm which is the starting height. It increases by 30% every week, i.e. 1.3. But what is changing is the number of weeks, thus raised to x and the answer is always different as shown on the y-axis."

Since we regard this unexpected kind of explanation as valuable reasoning, we suggest adding one further question, in which students are expected to explain their formula, i.e. explain their prediction (DV 3).

5 RETROSPECTIVE ANALYSIS AND DISCUSSION

The prediction tasks are characterized by three main student requirements: (a) make a prediction, (b) investigate the situation under consideration and (c) reflect on the outcome of the investigation in relation to the prediction. In the tasks used in our study, the investigations in (b) were rather straight-forward, consisting of a routine calculation or computer testing. Accordingly, the most interesting outcomes regarding reasoning are student responses in (a) and (c).

To develop an overview of the different kinds of reasoning discerned in the three prediction tasks, we arrange them into categories. Besides the distinction between visual and symbolic reasoning, we found it useful to categorize the reasoning obtained into either inductive reasoning or conceptual reasoning[5]. Accordingly, the different kinds of reasoning are organized into four main categories as follows:

- *Inductive Visual Reasoning (IVR)*
  Pattern-based visual reasoning (PBVR)
- *Conceptual Visual Reasoning (CVR)*
  Linear visual reasoning (LVR) and visual reasoning based on graphical knowledge (GKVR)
- *Inductive Symbolic Reasoning (ISR)*

[5] Reasoning based on assumptions about mathematical concepts and/or relations.

Pattern-based numeric reasoning (PBNR) and generalization-based algebraic reasoning (GBAR)

- *Conceptual Symbolic Reasoning (CSR)*
  Exponential numeric reasoning (ENR), additive numeric reasoning (ANR) and justifying algebraic reasoning (JAR)

In Table 2, an overview of the prediction tasks (regarding prediction and comparison activities) in relation to different didactical variables and types of reasoning is introduced.

Table 2 An overview of the four prediction tasks. An arrow (→) indicates a suggested revision.

TASK (predict/ compare)	**DV 1** (Computer /Paper)	**DV 3** (Explain: Yes/No)	**DV 4 data**[6] (Graph/ Tabular/ Numeric)	**DV 4 response**[7] (Graph/ Numeric/ Algebraic)	**DV 5** (Global/ Local)	**DV 6** (Correspond/ Covariation)	**DV 7-11** Scaffolding discussed	**Expected Reasoning**	**Enacted Reasoning**
2 p	C	Y	G	G	-	-	DV 7, 8, 9	CSR/CVR	CSR/CVR (19 out of39)
2 c	C	Y	G	G	-	- → cov.	DV 9	CSR	CSR/CVR (11 out of 20[8])
4 p	C→P	N	G→N	G→N	gl. → lo.	- → cov.	DV 7	IVR/ISR→ CSR	IVR[9]
4 c	C→P	N→Y	G→N	G→N	gl. → lo.	- → cov.	DV 7, 10	- →CSR	(CSR/IVR) (8 out of 35)
5 p	P	N→Y	T	A	-	corr.	DV 11	ISR→ ISR/CSR	ISR[10]
5 c	C	Y	G	G	- → gl.	-	DV 9	CVR	CVR/CSR (19 out of 32)

Some tendencies discerned from Table 2 will be discussed under the following sub-headings: *Didactical variables related to explanations*, *Didactical variables related to functions* and *Didactical variables related to scaffolding issues*. Finally, in this section, the general applicability of the didactical variables is discussed.

5.1 Didactical Variables Related to Explanations

The didactical variables concerning explanations (DV 3 and DV 9) turned out to be crucial in the design of prediction tasks. DV 3 was discussed in *all* prediction tasks, and in the revised version we suggest adding some further requests for explanations. Arcavi and Hadas (2000) point out that it is important that students express their predictions explicitly. We agree and argue that adding a request for an explanation of how students came up with their prediction could be instructive.

Having made comparisons, students are expected to make reflections on them (Kasmer, Kim 2012). When asking students to reflect on the comparison, we argue that it is important to consider whether to ask for an explanation or not. If not, it might be appropriate to ask for a description or just comments. Thus, the choice whether to ask for an explanation or not was essential both in the prediction part and the comparison part of each prediction task.

[6] "DV 4 data" signifies the form of representation of the data that students were expected to base their prediction on.
[7] "DV 4 response" signifies the form of representation that students were expected to use in their response.
[8] Only 6 out of the 20 'linearly' thinking pairs displayed that they have resolved the conflict.
[9] Only empirical data from video recording.
[10] Only empirical data from video recording.

Our results indicate that the wording is crucial when formulating questions where students are asked for explanations. This implied the identification of a new didactical variable regarding how to direct students' focus on *what* to explain (DV 9). Actually, all requests for explanations in our study turned out to require some refinement concerning this didactical variable. In the revised versions, we endeavor to make the requests more pointed, as suggested by Sinclair (2003).

5.2 Didactical Variables Related to Functions

The didactical variable concerning what form of representation to use (DV 4), both for data and response is elaborated on in *all* prediction tasks. For instance, if the aim is to provoke a known misconception as in Task 2, it is important to reflect on whether this misconception interrelates to a specific form of representation. Another example that illustrates the importance of what form of representation to choose is in Task 4. Our initial choice to use a graphical representation and emphasize a global view (DV 5) turned out to be unsuccessful. In the revision of this task another form of representation was suggested to direct students' focus on both a more local and a covariation view of functions (Confrey, Smith 1994, Confrey, Smith 1995).

The didactical variable concerning the choice of covariation or correspondence approach (DV 6) turned out to be essential. In several cases revisions were made to emphasize a covariation approach. The main reason for these revisions was to encourage students to reflect on the mathematics behind exponential growth and thereby promote conceptual reasoning. However, in other situations it is important to emphasize a correspondence approach. According to Confrey and Smith (1994), it is a challenge for many students to move from a covariation to a correspondence approach. This was observed in Task 5 when students were expected to find a closed form equation. Even if we tried to emphasize a correspondence approach when designing Task 5, the empirical data indicates that several students were stuck in a covariation approach.

The didactical variable concerning whether to emphasize a local or a global view (DV 5) was not discussed in many cases in our study. However, it became essential in the revision of Task 4 since the linear misconception still was frequent. To enhance 'linearly thinking' students to experience a conflict and use conceptual reasoning instead of inductive reasoning to resolve this conflict a local view was promoted in the suggested revision.

5.3 Didactical Variables Related to Scaffolding Issues

Scaffolding issues were elaborated on in *all* prediction tasks and actually all the new didactical variables identified during the design process are more or less related to these issues. One of them, DV 9, has already been discussed in a preceding paragraph, and still another, DV 8, will be discussed later on in this section. The other new didactical variables concerning scaffolding are DV 10 and DV 11. DV 10 was introduced in the revision of Task 4, as students were given a numeric value of what to count as a good enough guess. DV 11 was identified when we made the choice to explicitly refer to preceding tasks to scaffold students in Task 5.

One further didactical variable related to scaffolding concerns the question whether to support students by specifying the scaling of axes or not (DV 7). Our basic position has been not to specify scaling since it might be instructive for students to reflect on scaling issues. Our confidence in this approach was strengthened when we observed on several occasions that discussions concerning scaling involved fruitful reasoning.

In summary, an issue that frequently appeared when designing the tasks was how much scaffolding to offer. Without scaffolding, many students might get stuck and the situation could be cumbersome. On the other hand, too much scaffolding reduces the opportunities for student reasoning, and the risk that misconceptions remain invisible and unsolved increases. These findings are in line with the dilemma of determining which details or questions to include in a task, pointed out by Sinclair (2003). Making too much available, she argues, can reduce the motivation for students to make investigations while on the other hand, providing too little, can make the task to difficult.

5.4 General applicability of the Didactical Variables

Two didactical variables are closely linked to computer environments, DV 1 and DV 7. When designing a task sequence to be used in a mathematics software environment, it is not always obvious in which medium a particular task should be performed to optimize the opportunities for reasoning and learning (DV 1). For instance, in Task 4 the use of the computer medium to guess the position of a point turned out to imply some problems even if this choice of medium worked rather well in another situation when students were expected to perform a similar task (Task 2).

Besides being linked to computer environments, DV 7 is also linked to functions and graphs, i.e. it is a domain specific didactical variable. Two further didactical variables closely linked to functions and graphs are DV 5 and DV 6. The choice of representation form (DV 4) was elaborated on in all prediction tasks. One might argue that this didactical variable also is tuned to the domain of functions and graphs. However, different kinds of representation are central in mathematics and therefore this didactical variable also becomes important in other domains.

Another didactical variable to consider that might influence students' reasoning is the choice of using 'friendly' or 'unfriendly' numbers (DV 8). For instance, to prevent students making mental calculations instead of predictions based on reasoning, 'unfriendly' numbers could be used. On the other hand, 'friendly' numbers might reduce the cognitive load and facilitate student reasoning. We suggest this didactical variable as a valuable tool that often remains unnoticed.

Since prediction tasks most often includes one or more requests for explanations, the didactical variables related to explanations (DV 3 and DV 9) became important in these tasks. However, we suggest these didactical variables together with DV 4, DV 8 and DV 11 as the most general since they could be used beyond the domain of functions and graphs and are not tied to prediction tasks or computer environments.

6 CONCLUDING REMARKS

The main aim of this paper is to pinpoint and elaborate on key didactical variables to consider when designing prediction tasks that foster student reasoning in relation to exponential functions in a mathematics software environment. A further aim is to provide a detailed description of the design and analysis processes by using the Didactical Variables tool.

The use of didactical variables as a design tool was helpful in several ways. Foremost, it facilitated the identification of important choices to consider in the design process. The use of this tool encouraged us to explicate and argue for different kinds of choice made to enhance particular learning trajectories and to elicit student reasoning. This, in turn, provided useful guidance in the analysis process since it pinpointed what to pay special attention to.

To summarize, there was lots of reasoning of different kinds going on in the classrooms during the computer activity lesson. However, the student responses reveal that there was a need for several kinds of revision addressing different didactical variables. For instance, there was a need to revise Task 4 towards conceptual reasoning making students reflect on the mathematics behind exponential growth, and thereby prevent students making conclusions simply by visual perception (Healy, Hoyles 1999). This implied reconsideration of as many as six didactical variables. Another palpable result is the frequent need to direct students' focus on *what* to explain. This result indicates the importance of careful considerations of wording when formulating prediction tasks.

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