A Model for Task Design with Focus on Exploration, Explanation, and Generalization in a Dynamic Geometry Environment

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Abstract The increasing availability of new technologies in schools provides new possibilities for the integration of technology in mathematics education. However, research has shown that there is a need for a new kind of tasks that utilize the affordances provided by new technology. Numerous studies have demonstrated that dynamic geometry environments provide opportunities for students to engage in mathematical activities such as exploration, conjecturing, explanation, and generalization. In this paper, we introduce a model for design of tasks that promote these kinds of mathematical activities. Especially, tasks that foster students to make generalizations are emphasized. The model is primarily developed to suit geometrical locus problems. The construction of the model is based on previous literature. The model is demonstrated with a concrete example and tested out empirically in a case study with two doctoral students. Findings from the case study are used to revise the initial model. Finally, the revised model is introduced and discussed.

1 Introduction

The availability of different kinds of technologies in mathematics classrooms is increasing. In Sweden, more and more schools provide students with a computer of their own. This entails new possibilities for the integration of technology in mathematics education, but it requires change in teaching and learning practice (Drijvers et al. 2010; Pierce & Ball 2009). For example, there is a need for a new kind of tasks to utilize the features afforded by new technology (Doorman et al. 2007; Hitt & Kieran 2009; Laborde 2002).

It is well substantiated that dynamic geometry environments (DGE) provide opportunities for mathematical activities such as exploration, conjecturing, verification and explanation (e.g. Arzarello et al. 2002; Hanna 2000; Healy & Hoyles 2002). These kinds of activities have also been illuminated by several authors in relation to proof and proving (e.g. De Villiers 1990; Hanna & Jahnke 1996; Stylianides & Ball 2008). In particular, the link between the empirical and the theoretical domains in the proving process has been in focus (Baccaglini-Frank & Mariotti 2010; Hadas et al. 2000; Healy & Hoyles 2002; Hoyles & Jones 1998; Leung & Lopez-Real 2002; Olivero & Robutti 2007). In numerous of these studies, the proving process begins with a phase of *exploration* and ends with a phase that includes *explanation.* However, less attention has been paid to the issue of task design in these studies.

This study is situated in the context of task design. Especially, design of tasks appropriate for DGE, and that foster students' ability to explore, conjecture, verify, explain and make generalizations. Despite its importance, the formal proof construction is not our main concern in this paper. Instead, the focus is on the phases before and after the proof construction with emphasis on further generalizations.

In close connection to this field, Leung (2011) suggests an epistemic model of task design. His model is situated in a technology-rich environment, in particular a DGE. Leung draws attention to three aspects in relation to task design: exploration, re-construction (or re-invention) and explanation. According to him, tasks should enable students to engage in activities that blend these aspects. However, further generalizations are not considered in his paper.

In this paper, we suggest a model for design of task-situations[1] in the context of a DGE. The model is developed to suit geometrical locus problems. Previous literature serves as rationales for the construction of the model. We use a concrete example to demonstrate and examine the model. Then, the model is tried out empirically in a case study with two doctoral students in mathematics. We identify key findings in the case study and use them to improve the model. Finally, we propose and discuss a revised model for design of task-situations.

2 Theoretical Background

In this section the main notions that frames the model is introduced. The literature review is organized in the following way. The first part concerns activities related to the phase before a formal proof construction. The next part deals with further generalizations (after a proof construction), and the last part relates to task design.

2.1 From Exploration to Explanation

Researchers agree that one of the most appreciated affordances provided by DGE is the possibility to make explorations (e.g. Baccaglini-Frank & Mariotti 2010; Edwards 1997; Hanna 2000; Hoyles & Jones 1998; Santos-Trigo & Espinosa-Perez 2002; Öner 2008). Edwards (1997) used the metaphor “territory before proof” (p. 187) when discussing a comprehensive set of potential activities that could precede a formal proof construction.

> It is during this exploratory and testing phase that learners and expert mathematicians alike apply their intuitions in seeking patterns, follow hunches, testing ideas, and formulating generalizations that may become conjectures (Edwards 1997, p. 190)

DGE provide a wide range of tools for exploring different properties and relations between mathematical objects. First and foremost, the dragging function is emphasized as the defining feature of a DGE (Arzarello et al. 2002; Baccaglini-Frank & Mariotti 2010; Hölzl 2001; Laborde 2002; Leung 2011). Furinghetti and Paola (2003) reported on a case study with the aim to investigate how a DGE (*Cabri*) can influence students’ reasoning when they work with a conjecturing task. Initially, the students used the dragging tool. Furinghetti and Paola noticed that the students’ mode of dragging shifted from dragging more or less randomly to a more focused way of dragging.

Hölzl (2001) mainly discussed two different ways in which dragging can be used: drag to check, e.g. whether a construction has a desired property, and drag to search for new properties. The latter is a way to use a DGE to explore mathematical relations. The purpose of this kind of dragging is to search for

[1] A task-situation consists of a sequence of tasks. The notion “task-situation” is adopted from Kieran and Saldanha (2008).

regularities and invariants (Hölzl 2001; Olivero & Robutti 2007). Arzarello et al. (2002) introduced a hierarchy of different dragging modalities: *wandering dragging*, *bound dragging*, *guided dragging*, *dummy locus dragging*, *line dragging*, *linked dragging* and *dragging test*. They observed that students used different dragging modalities depending on purpose (Arzarello et al. 2002). For instance, to investigate and explore students used *wandering dragging, bound dragging* and *guided dragging*. By *wandering dragging*, a basic point is moved randomly on the screen "…in order to discover interesting configurations or regularities in the drawings." (p. 67). In *bound dragging*, the point that is dragged is a "semi-dragable point"[2]. *Guided dragging* is used in order to give a drawing a particular shape, by moving one of its basic points (Arzarello et al. 2002).

Baccaglini-Frank and Mariotti (2010) advance the research on different dragging modalities by introducing a model, *the maintaining dragging model*. In their research they focused on students' cognitive processes when they engage in explorations within a DGE (*Cabri*) during the production of conjectures. One of the dragging modalities utilized in this study is *dragging with trace activated.* Baccaglini-Frank and Mariotti refer to this form of dragging as a combination of two Cabri tools: ""dragging" plus "trace", which together constitute a *new* global tool that can be used in the process of conjecture-generation." (pp. 230-231). This tool is also discussed by Santos-Trigo and Espinoza-Perez (2002). They pointed out that "…a powerful tool to identify and explore mathematical relationships is to trace the locus of any specified object." (p. 47). They used three locus problems to show the power of a DGE as a means to "identify and examine geometric properties" (p. 38). However, dragging is not the only feature provided by a DGE in connection to exploration activities. For example, Olivero and Robutti (2007) discussed the affordances provided by different measuring tools.

When, for example a pattern has been noticed, it should be described in a way that could be shared by others, e.g. in words, mathematical notions, pictures or diagrams (Edwards 1997). If students expect that the pattern is valid in general, they are prepared to state a conjecture. Furinghetti and Paola (2003) found that the students in their study stopped to explore after a while and instead began to reflect on what they had done and then they formulated a conjecture. The possibility that DGE offer students to check their conjectures is commented on by several authors. For example, Marrades and Gutiérrez (2000) mentioned the possibility to get a wide range of example as a main advantage of DGE.

Convinced of the truth of their conjecture, the students in the study performed by Furinghetti and Paola (2003) were motivated to explain *why* the conjecture is true. To convince oneself about the truth of a statement before starting with the formal proof construction, also resembles how a mathematician works (De Villiers 1990; Hanna 2000). Further, to have a plan or an idea of *how* to construct the proof before starting with the proof construction is characterizing for a *semantic proof production* (Weber & Alcock 2004; Weber 2005). According to Weber and Alcock, this plan or idea could be found when using intuitive and informal representations to convince oneself *why* the statement is true. Analogically to semantic proof production, Raman (2003) discussed proofs based on *key ideas*. A key idea is an idea based on informal understanding which leads to a formal proof.

[2] Arzarello et al. (2002) use the notion "semi-dragable point" when they refer to a point that belongs to an object.

To find ways to explain *why* a conjecture is true may be a crucial step in the proving process. For example, Hölzl (2001) found that the main challenge for students was to find appropriate construction elements to add. He claimed that it is unlikely that only dragging objects around will give students ideas for explanations. The usefulness of auxiliary constructions to get ideas is also discussed in other studies (e.g. Santos-Trigo & Espinosa-Perez 2002).

Healy and Hoyles (2002) found that interacting with a DGE (*Cabri*) "…can help learners to explore, conjecture, construct and explain geometrical relationships, and can even provide them with a basis from which to build deductive proofs…" (p. 18). That DGE could provide a base for explanation has also been demonstrated in other studies (Edwards 1997; e.g. Jones 2000). For example, Furinghetti and Paola (2003) noted that students referred to their exploration on the computer when they constructed a formal proof in writing.

2.2 Further generalizations

In contrast to traditional teaching of geometry in which the deductive proof is the final goal, Chazan (1990a) suggested that students could continue to explore after they have completed a proof. He proposed some questions that students could be asked: "For what geometrical objects does it hold? Can it be generalized?" (p. 20) Furthermore, Chazan (1990b) referred to Brown and Walter's (1983) "what if not" strategy to help students make generalizations. With this strategy students are encouraged to systematically vary key aspects of geometrical or other mathematical situations.

Yerushalmy (1993) elaborated on the terms generalization, induction and conjecturing. For example, she pointed out that the term *generalization* is used both as a process and as the product of the process. Moreover, the product of an induction is also labeled a generalization. Conjectures then, "can be developed by both induction and generalization" (p. 57). However,

> In induction, the generalization is created from the examination of instances or examples. (…) Generalization, on the other hand, is a process acted out on a statement (either a conjecture or a proven statement). (…) Thus, in generalization, a more specific statement becomes a more general statement. (Yerushalmy 1993, p. 58).

Yerushalmy (1993) referred to a teaching experiment with two groups of students (ninth graders) working within a DGE (*Supposer*). One of the groups was taught how to make generalizations. The strategy used by the teacher was the one mentioned above, the "what if not" strategy introduced by Brown and Walter (Yerushalmy 1993). The findings of this experiment indicated a significant difference in the ability to generalize between the two groups.

Generalizations is also part of the last phase in the problem solving process, suggested by Pólya (1945). He emphasized that it is important for students not to abandon a problem when they think they have solved it. Instead they should utilize the opportunity to elaborate the problem further, and try to learn more from the result and the method they used. It is instructive for students to investigate if there are related problems and if it is possible to generalize the result. The importance of this phase is also illuminated by Mason, Burton and Stacey (2010), although they call it 'review'. They suggested that there is a great opportunity to pose questions, e.g. questions like "What if … ", when one just has solved a problem. This is closely linked to what Furinghetti and Paola (2003) found in their study. When the students had made a formal proof, they went back to the original problem and began a new process of exploration (in *Cabri*), now in a more systematic way.

2.3 Task Design

According to Hölzl (2001), the focus in many tasks presented in the context of DGE is to find ways of using DGE to verify a statement. He stated that "…learners are just supposed to vary geometric configurations and confirm empirically more or less explicitly stated facts." (p. 65). He discussed some frequently used examples and argued that the way these tasks are posed only promote limited exploration activities. Hölzl (2001) suggested "contrasting" uses of DGE. For example, he proposed tasks that enable students to experience moments of surprise or astonishment.

In several studies in the context of proof and proving, *different phases* in the proving process can be discerned (e.g. Christou et al. 2004; De Villiers 2004; Edwards 1997; Furinghetti & Paola 2003; Marrades & Gutierrez 2000; Olivero & Robutti 2007). For instance, Marrades and Gutiérrez (2000) utilized activities structured in three phases when they investigated ways of using DGE to improve students' proving skills. In the first phase students were asked to create a figure (in *Cabri*) and explore it. In the next phase, students generated conjectures and in the third phase they justified their conjectures. Marrades and Gutiérrez (2000) pointed out that a request for a justification includes an explanation of *why* the conjecture is true. In their conclusion they stressed, among other things, the importance of "carefully organized sequences of problems" (p. 120) to evolve students' abilities to make elaborated types of justifications. Furinghetti and Paola (2003) distinguished six different phases that students should pass through when they work with a (proof) problem. Principally, these phases include activities such as exploration, conjecturing, construction of a formal proof, and exploration for further generalizations.

Open problems are suggested to create teaching and learning environments that allow students to explore and produce conjectures (Baccaglini-Frank & Mariotti 2010; Furinghetti & Paola 2003; Mogetta et al. 1999). According to Mogetta, Olivero, and Jones (1999) some characterizing properties of an open problem are that they

> …usually consists of a simple description of a configuration and a generic request for a statement about relationships between elements of the configuration or properties of the configuration. (…) The requests are different from traditional closed expressions such as "prove that…", which present students with an already established result. (pp. 91-92)

One way to create open problems that exploit the features provided by DGE could be to turn traditionally closed tasks into open ones (Hadas et al. 2000; Mogetta et al. 1999).

Sinclair (2003) discussed the relation between the design of a task and the exploration process. In a case study, she used students' work with pre-constructed dynamic geometry sketches and accompanying lab sheets to find important design considerations. For example, Sinclair found that *open-ended* questions invite students to explore.

Kieran and Saldanha (2008) discussed how to design tasks for the co-development of conceptual and technical knowledge in CAS (Computer Algebra System) activities. Even if the task design that they elaborated concerned a CAS technology environment, there are some design principles that might be worthwhile to consider even in the context of a DGE. For instance, students were asked to interpret their work in writing to make it explicit for further reflection. Thus, the tasks contained a mix of paper-and-pencil work, and computer activities. "Reflection" is also highlighted by Lin et al. (2012) in their discussions about task design principles that promote conjecturing. By reflection, students get ideas for further explorations to improve their conjectures.

2.3.1 An Example of a Task Design Model

In a discussion paper, Leung (2011) proposes three aspects that could serve as guiding principles for task design in the context of technology-rich environments: *exploration*, *re-construction* and *explanation*. With the terminology *techno-pedagogic mathematics task design model,* Leung (2011) refers to a model of task design "…in which learners are empowered with amplified abilities to explore, re-construct (or re-invent) and explain mathematical concepts…" (p. 327). He proposes a task design model composed of three epistemic modes that reflect the aspects mentioned above. He designates these modes as *Establishing Practices Mode (PM), Critical Discernment Mode (CDM),* and *Establishing Situated Discourse Mode (SDM).*

Leung (2011) describes the *Establishing Practices Mode (PM)* as a mode of *exploration* in a technology-rich environment, in particular a DGE. One intention with this mode is to make students acquainted with the software and give them possibilities to develop interacting modalities with the DGE. The mathematical objects that are manipulated could be pre-designed or demanded to be constructed in the task. Leung (2011) states that "Constructing or manipulating virtual mathematical objects is a meaningful way to learn to turn virtual tools into pedagogical instruments." (p. 327).

Gradually, as the students become comfortable in using the software, their focus of attention may shift "…from establishing routine tool usage to meaning construction." (p. 328). This shift of attention will lead students to the next mode, the *Critical Discernment Mode (CDM).* In this mode

> …construction of mathematical meaning mediated by the instruments shaped in the Establishing Practices Mode takes centre stage, in another word, empirical experiences are being mathematized. (Leung 2011, p. 328)

This mode is characterized by activities such as observing and recording patterns of variation or invariants to get ideas for conjecture formation. This mode may encourage students to develop abilities to communicate their mathematical reasoning and argumentation (Leung 2011). In the next mode, the *Establishing Situated Discourse Mode (SDM)*, students are supposed to express what they have discovered in the preceding modes. The explorations in prior modes support an inductive approach. Thus, when entering this mode students may "Develop inductive reasoning leading to making generalized conjecture" (p. 328). Thereafter, students could "Develop discourses and modes of reasoning to explain or prove" (p. 328). Leung points out that this mode (SDM) may serve as a bridge between empirical experiences and formal mathematical reasoning.

4 Method

Our study consists of two phases. In the first phase, we constructed a model for design of task-situations based on previous literature. To illustrate and examine the model a concrete example was used. Each task in the example was examined, and predictions about student performances were made. The first phase is described in detail in section 5.

In the second phase, the example was tested out empirically in a case study. The case study participants were two Ph.D. students in mathematics, **C** and **E**. They were familiar with the mathematical content, in this case conic sections. However, the students were not used to work in a DGE. Therefore, they were introduced shortly to *GeoGebra*, the software used in the case study. In this introduction the most important tools that were needed to perform the task-situation in the example were introduced**.** The task-situation was presented on a worksheet and the participants were asked to work together and to write their answers on the worksheet. There was no time limit for the students' work with the task-situation.

To capture the students' work, we used video recordings. The position of the camera was at an angle behind the students to capture the screen and also how students pointed at the screen. All the *GeoGebra* files that were produced by the students were saved. In these files it was possible to follow the students' constructions step by step. The data were transcribed and the transcripts were used together with the video to analyze the students' performances.

Results from the case study are reported in section 6. Main findings from the case study were used to revise the initial model. According to Alcock and Inglis (2008), it is widely accepted to study the behavior of high achieving students for the design of mathematical activities. The revised model is introduced and discussed in the last section.

5 A Model for Design of Task-situations

5.1 Description of the Model

The following guiding principles were considered when the initial model was developed. First, students shall be asked to formulate their conclusions in writing. In this way their work can come to surface, and thus become object of reflection (Kieran & Saldanha 2008). Second, the tasks shall be sequenced in such a way that they provide students with a logical flow throughout the proving process, from exploration to explanation. Finally, students shall be encouraged to make further generalizations. Based on the latter principles, we have chosen to sequence the model as follows: *exploring and conjecturing, verifying, explaining, and generalizing.* Furthermore, the task-situation has to be introduced by a *description of the mathematical situation.* In the following, each part of the model is discussed and suggestions for formulations of the tasks are outlined.

5.1.1 Description of the mathematical situation

The description must include the conditions that define the mathematical situation under consideration. One way to construct suitable task-situations could be to reformulate existing problems. For instance, if a traditional proof task is used as a base, the explicit request "prove that…" could be excluded. In this way the original task is transformed into an *open problem* (Mogetta et al. 1999).

5.1.2 Exploring and conjecturing

To give students opportunities to get acquainted with both the DGE and the mathematical situation, students should make their own constructions, as far as possible. In the case of locus problems, it is particularly important for students to be aware of dependent and independent objects. Moreover, if students are supposed to make further investigations later on in the task-situation, it is important for them to know how to utilize the technology. When a construction is made, students can begin to explore the mathematical situation under consideration. In these explorations different dragging modalities provided by DGE might be helpful (Arzarello et al. 2002; Baccaglini-Frank & Mariotti 2010; Hölzl 2001; Laborde 2002; Leung 2011). In the case of geometrical locus problems the most useable dragging modalities are *bound dragging* and *dragging with trace activated* (Santos-Trigo & Espinosa-Perez 2002).

To encourage students to make a construction, initiate an exploration of the relationship between the objects involved, and formulate a conjecture we suggest the following task.

Task a) Make an appropriate construction in [the DGE, e.g. GeoGebra] [3] *and study the position of [a dependent object, e.g. a point], for different positions of [an independent object]. Make a conjecture.*

5.1.3 Verifying

Different features of the DGE can be used to verify (or refute) the truth of a conjecture. For example, the possibility to investigate several examples rapidly on the computer has been documented in the literature (Edwards 1997; Hadas et al. 2000; Hanna 2000; Marrades & Gutierrez 2000). To foster students to convince themselves of the truth of their conjecture before trying to prove it, the next task is proposed.

Task b) Use [the DGE] to support or refute your conjecture.

5.1.4 Explaining

Convinced of the truth of their conjecture, the students might be motivated to explain *why* it is true. Before a request for a formal proof, we suggest the following task.

Task c) Explain in your own words why your conjecture is true.

With the request to explain in "your own words", we intend to promote a semantic proof production where informal and intuitive representations are important (Weber & Alcock 2004; Weber 2005). Very likely there will emerge a need to add some more construction elements to explore and discover underlying properties that could be used to explain the conjecture (Hölzl 2001; Santos-Trigo & Espinosa-Perez 2002). When the students have found the *key ideas* (Raman 2003), they might be well prepared for the next task.

Task d) Construct a proof.

5.1.5 Generalizing

Traditionally, the proof construction is the final step in a proving process. However, it could be instructive to elaborate the problem further and try to generalize the result (e.g. Chazan 1990a; Furinghetti & Paola 2003; Mason et al. 2010). We want to foster students to explore the mathematical situation further by using the affordances provided by DGE. Therefore we propose the final task to be

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[3] The text within square brackets should be adapted to the circumstances under consideration.

Task e) Make new related investigations. Make conjectures, support or refute, explain and prove.

5.1.6 A summary of the model

The initial model is summarized in table 1.

Table 1 The initial model

Description of the mathematical situation. (a) Make an appropriate construction in [the DGE, e.g. GeoGebra] and study the position of [a dependent object, e.g. a point], for different positions of [an independent object]. Make a conjecture. (b) Use [the DGE] to support or refute your conjecture. (c) Explain in your own words why your conjecture is true. (d) Construct a proof. (e) Make new related investigations. Make conjectures, support or refute, explain and prove.

5.2 An Example

To demonstrate how the model could be utilized, we introduce a concrete example. As a basis, we used a proof task formulated in a traditional way, i.e. "prove that…" followed by a statement to prove. The proof task, a geometrical locus problem, was originally formulated in the following way:

Let P be an arbitrary point on the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and let F(ae,0) be a focus. Let M be the midpoint of FP. Prove that the locus of M is an ellipse.*

When the initial model (table 1) is used to reformulate the original proof task, the task-situation in table 2 is obtained.

Table 2 An example of a task-situation

Let *P* be an arbitrary point on an ellipse. Let *M* be the midpoint between P and one of the foci. (a) Make an appropriate construction in *GeoGebra* and study the position of point *M* for different positions of point *P*. Make a conjecture. (b) Use GeoGebra to support or refute your conjecture. (c) Explain in your own words why your conjecture is true. (d) Construct a proof. (f) Make new related investigations. Make conjectures, support or refute, explain and prove.

In the following, the tasks are elaborated and predictions about student performances are made.

5.2.1 Task a: Make an appropriate construction in GeoGebra and study the position of point M for different positions of point P. Make a conjecture.

In this example the construction is straightforward because there are special tools available both for conic sections and midpoints in *GeoGebra.* Dragging modalities that could be used to study what happens with the position of *M* as *P* is dragged along the ellipse are *bound dragging* and *dragging with trace activated*. When *P* is dragged along the ellipse, it can be observed that *M* seems to move along an ellipse as well. With trace activated, this is made quite obvious (figure 1).

In *GeoGebra* the locus of *M* could also be obtained by using the inbuilt locus tool. Thus, the last part of this task: "Make a conjecture" is fairly simple to perform, and a conditional statement could be expressed as follows: If *P* is an arbitrary point on an ellipse and *M* is the midpoint between *P* and one of the foci *F*, then the locus of *M* is an ellipse. However, there are further properties to discover, e.g. which are the foci of this (new) ellipse?

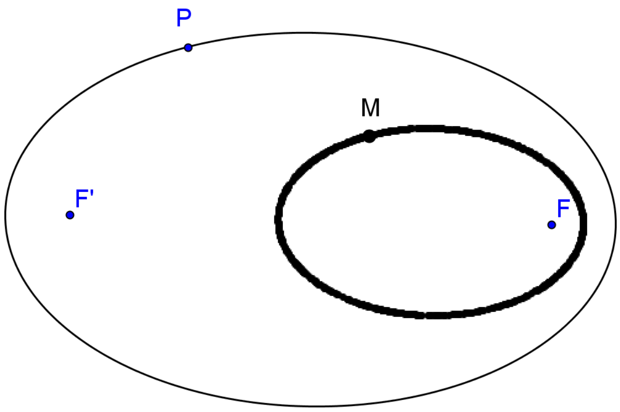

Fig. 1 The locus of *M* seems to be an ellipse

5.2.2 Task b: Use GeoGebra to support or refute your conjecture.

One way to verify the truth of the conjecture could be to construct some more ellipses and use the dragging tool in the same way as in task (a). Another way could be to construct a new smaller ellipse (in the same window) by selecting two foci, *A* and *B*, and one point *C* on the ellipse. By moving the points *A*, *B* and *C*, it is possible to make the new ellipse coincide with the locus of *M* (figure 2). This kind of *guided dragging* (Arzarello et al. 2002) can be compared to what Lopez-Real and Leung (2006) calls "drag to fit".

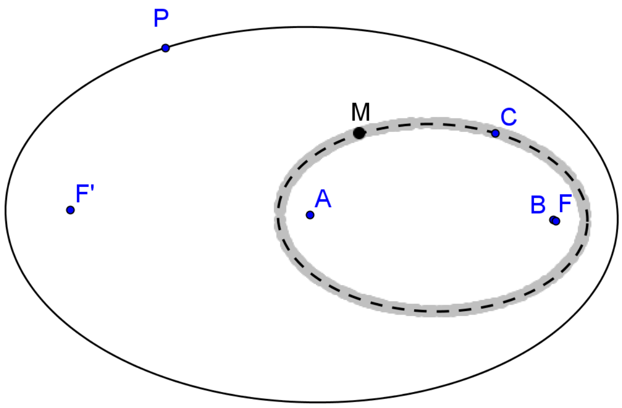

Fig. 2 The locus of *M* (thick grey line) coincide with the ellipse (dotted line) with the foci *A* and *B*, and the point *C* on it

If students make the construction illustrated in figure 2, they might discern that the foci of the new ellipse are the points *F* and the midpoint between *F* and *F'*. However, without the construction in figure 2, it could be a challenge to discover this property. In case students find this property and make a refined conjecture, further confirmation of the truth of the conjecture could be made. For instance, the conjecture could be supported by measuring distances (by using the tool for distance measurement) and using the distance property of an ellipse[4].

5.2.3 Task c: Explain in your own words why your conjecture is true.

The performance of this task depends on what has occurred in the preceding tasks. If students want to use the distance property to explain their conjecture, they need the foci of the small ellipse. In case the refined conjecture described in task (b) not has been found yet, the challenge to find these foci remains. If students search actively to find these foci, it should not be too difficult to hypothesize that the point *F* is a focus point of the small ellipse as well (see figure 1). If the position of one focus point is known it should be possible to estimate the position of the other one, and further to hypothesize that it is the midpoint between *F* and *F'*.

If students manage to find the new foci and want to use the distance property, they might obtain a drawing like the one in figure 3a. Then, it might be natural to come up with the idea to construct the corresponding segments to *AM* and *MF* in the initial ellipse, i.e. *F'P* and *PF* (figure 3b). Further, it should be possible to imagine that a segment between *F'* and *F* would give two similar triangles (figure 3c). This is rather easy to confirm since both *M* and *A* are midpoints.

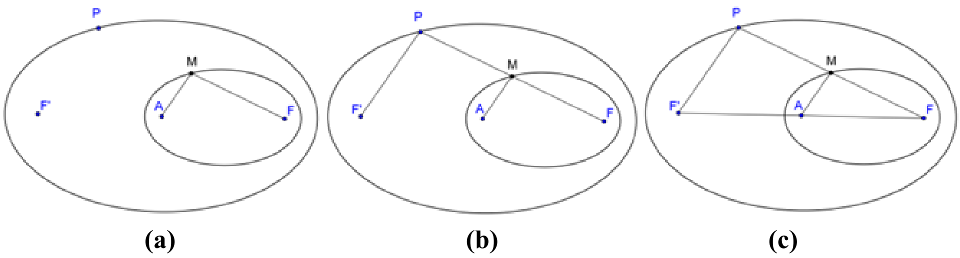

Fig. 3 The emergence of two similar triangles

5.2.4 Task d: Construct a proof

Our expectation for this task is that the final proof construction will be influenced by the informal reasoning in the preceding task and the explorations in the DGE. In this example, the key idea is the similarity described above.

5.2.5 Task e: Make new related investigations. Make conjectures, support or refute, explain and prove.

In this example there are several opportunities to generalize the conjecture. In table 3 suggestions of questions that might be posed are introduced.

Table 3 Questions of the type "what if…" that could be posed to find further generalizations

(i)	What if M is not the midpoint between P and F, i.e. $PM/MF \neq 1$?

[4] The sum of the distances from a point on the ellipse to the two foci is independent of the choice of point, i.e. the sum is constant.

(ii) What if P is a point on another conic section?
(iii) What if M is a point between P and an arbitrary point (instead of F)?

In the following the questions in table 3 are elaborated one by one.

(i) What if M is not the midpoint between P and F?

We assume that students will try to use *GeoGebra* to investigate this question. Most probably they will begin to investigate the case when *FM* < *FP* (figure 4a). To do this in *GeoGebra* they first must construct a segment or a line between the points *F* and *P* and then place the point *M* somewhere between *F* and *P* on this segment/line. If they chose to use a line there is a good chance that they also will investigate the case when *M* is placed outside the original ellipse, i.e. *FM* > *FP* (figure 4b).

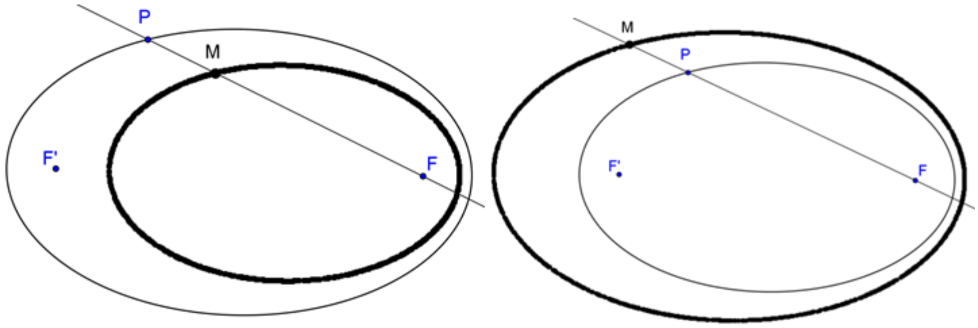

Fig. 4a *FM* < *FP* **Fig. 4b** *FM* > *FP*

If these investigations are performed, the following generalization of the conjecture could be stated: If *P* is an arbitrary point on an ellipse and *M* an *arbitrary point* on the line through *P* and one of the foci *F* (so that the ratio *FM/FP* is constant) then the locus of *M* is an ellipse[5]. To support this generalized conjecture *GeoGebra* can be used in the same ways as in task (b) above. The explanation and proof construction can be based on similar triangles in the same way as before.

(ii) What if P is a point on another conic section?

If students pose this question, it is straightforward for them to use *GeoGebra* to explore both hyperbolas and parabolas (figures 5a-b).

[5] The foci are *F* and a point *A* on the line through *F* and *F'*, where *FA/AF'=FM/MP*.

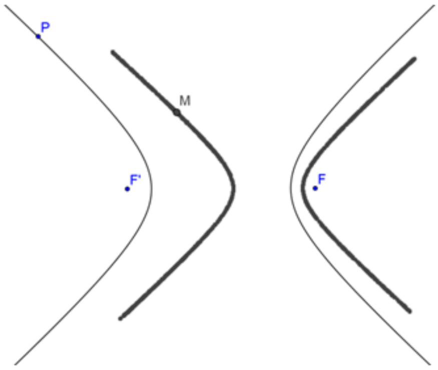

Fig. 5a The case of hyperbola

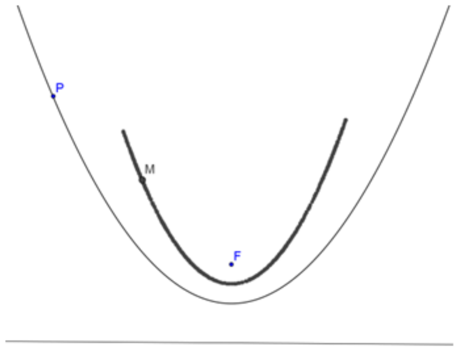

Fig. 5b The case of parabola

The outcomes from these investigations might result in the following generalization: If *P* is an arbitrary point on *a conic section* and *M* is an arbitrary point on the line through *P* and one of the foci *F* (so that the ratio *FM/FP* is constant) then the locus of *M* is *a conic section of the same kind*.

(iii) What if M is a point between P and an arbitrary point (instead of F)?

Perhaps it is less likely that students will pose this question. However, if they do they might be surprised by the result that *F* can be replaced by any point. This is straightforward to discover in *GeoGebra* (see figure 6). Thus, a further generalization can be stated: If *P* is an arbitrary point on a conic section and *M* is an arbitrary point on the line through *P* and *an arbitrary point Q* (so that the ratio *QM/QP* is constant) then the locus of *M* is a conic section of the same kind.

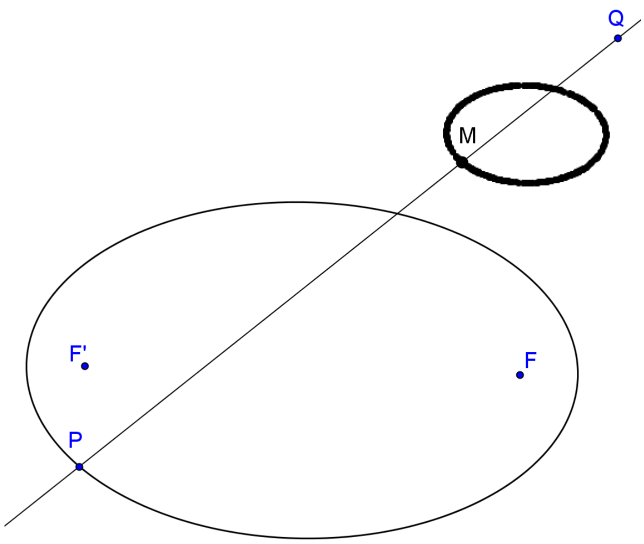

Fig. 6 *M* is an arbitrary point on the line through *P* and an arbitrary point *Q*

If a drawing like the one in figure 6 is obtained, it will probably throw a new light on the situation. Is this about conic sections at all? What if *P* is a point on another geometrical object? If students chose to investigate these questions, they will probably discover the most general case: If *P* is an arbitrary point on *an arbitrary geometrical object* and *M* is an arbitrary point on the line through *P* and an arbitrary point *Q* (so that the ratio *QM/QP* is constant) then the locus of *M* is similar to *this geometrical object*. In mathematical theory this is a transformation termed homothety[6]. In figure 7, an irregular polygon is used to illustrate the generalization above.

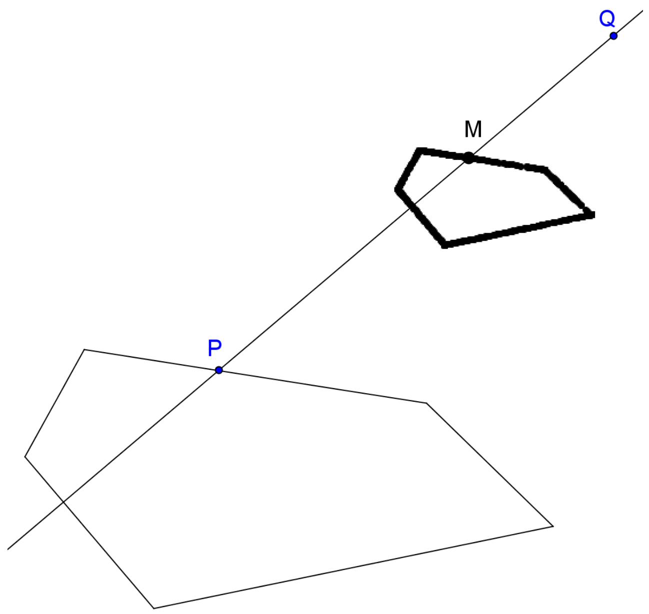

Fig. 7 An illustration of the generalized statement

6 The Case Study

In this section, we present the results of the case study.

6.1 Results

The students spent about one hour and a half working with the task-situation. Most of the time (about 70 minutes) was spent to find further generalizations. In the following we report the students' performance of each task one by one. Transcripts are used in order to highlight interesting sequences.

6.1.1 Task a: Make an appropriate construction in GeoGebra and study the position of point M for different positions of point P. Make a conjecture.

Before the students made the construction in *GeoGebra*, they guessed that the point *M* would follow the path of a circle or an ellipse. As was predicted, they used the tools for ellipse and midpoint to make an initial construction. They chose to put trace on the point *M* before they dragged the point *P* along the ellipse. Thus, they used the two dragging modalities that were predicted, i.e. *bound dragging* and *dragging with trace activated*, in order to explore the path that *M* follows as *P*

[6] A homothety is a transformation determined by a certain point *Q* that carries each point *P* into a point *M* on the straight line *PQ* in accordance with the rule *QM* = *kQP*, where k is a constant number, not equal to zero. The homothetic image of a figure *F* is the figure formed by the set of all points *M* homothetic to the points which constitutes the figure *F*. A homothety is a special case of a similarity.

is dragged. The students obtained the drawing in figure 8 and they stated that their initial guess was decent.

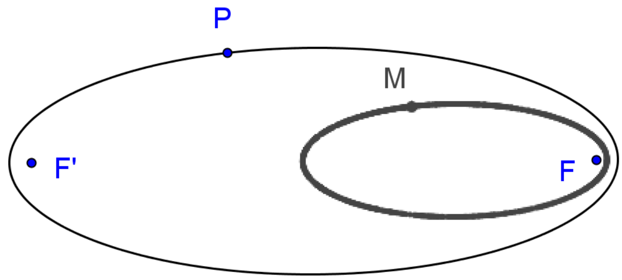

Fig. 8 The initial drawing obtained by the students

C suggested that the point *F* (figure 8) could be one of the foci (of the small ellipse), but the students were not convinced about this and left it for a while. Instead they changed the shape of the original ellipse (to get one more example) and verified that the locus of *M* still was an ellipse. However, they now rejected the suggestion that *F* is a focus point of the small ellipse. They read the task description once again and wrote on the worksheet: "Conjecture The locus of M is another ellipse!". The students only expressed the conclusion in their written conjecture, i.e. they did not express the premises in writing.

6.1.2 Task b: Use GeoGebra to support or refute your conjecture.

When **C** and **E** read the instruction for task (b), they noted that this was something that they already had done and they stated that they made the tasks (a) and (b) at the same time. They then proceeded with the next task (c). Even if the students began the speculation of whether *F* could be one of the foci of the small ellipse, they did not carry it out at this moment. Therefore, they had not refined their conjecture (i.e. hypothesized about the foci of the small ellipse) before they started to work on the next task.

6.1.3 Task c: Explain in your own words why your conjecture is true.

At this moment, they added some further construction elements with the motivation: "Just to see how things are about there" (minute 6:35). They constructed lines and midpoints between *F'* and *P* and *P* and *F* respectively. They activated trace on the midpoints and dragged point *P* and received the drawing in figure 9, i.e. they obtained two new ellipses. It was more or less a coincidence that they received two ellipses (instead of one).

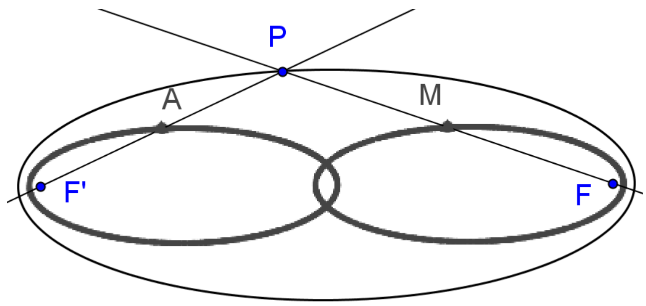

Fig. 9 The point *P* is dragged with trace activated on the two midpoints *A* and *M*

Their discussion proceeded as follows.

Transcript	Interpretation
(8:01) **E**: What is the defining characteristic of an ellipse? [**C** describes the distance property] **E**: That´s probably useful.	They referred to mathematical theory.
(8:53) **C**: I now still wonder if this can be the focus [**C** points to the point *F* in figure 9]	Again they reflected on the possibility for *F* to be a focus point of the small ellipse.
(9:04) **C**: I wonder if the midpoint between *F’* and *F* could be a common focus point of the two smaller ellipses.	A refinement of the initial conjecture was discussed.
(9:15) **C**: Maybe we can construct another ellipse that we can control with the slider and try to let the program construct those two, to support this or not. **E**: Clever idea.	To convince themselves of the truth of the refined conjecture they wanted to change the shape of the ellipse to examine several examples. However, they did not make this construction (with the slider tool) probably because they did not know how to do it in *GeoGebra*.
[They construct a new ellipse with a more circular shape than in figure 8]	
(9:50) **E**: Now it seems more likely, yes. Maybe the proportion fooled us in the previous one.	They supported the refined conjecture, by investigating a further example.

When **C** and **E** wanted to use the distance property to explain their conjecture they realized that they needed the new foci. They hypothesized about the positions of these foci but the utterance “it seems more likely” (9:50) indicates that they not were completely convinced at this moment. To find further support for their conjecture about the foci of the two small ellipses (figure 9), the students used a special case: the case of a circle. They changed the shape of the ellipse (by using *guided dragging*) into a circle and found that *F’* and *F* coincide into a single point. Then they noted that this point was the midpoint of both the original circle and the small circle (the locus of *A* and *M*). This fact made them convinced about the truth of their refined conjecture and they continued to think about how to explain *why* it is true. **E** came up with the idea of adding segments to see if they could “find some similarity properties of triangles or something” (12:28). Then they constructed a midpoint (*B*) between *F'* and *F* and segments *F'F* and *AB* (figure 10).

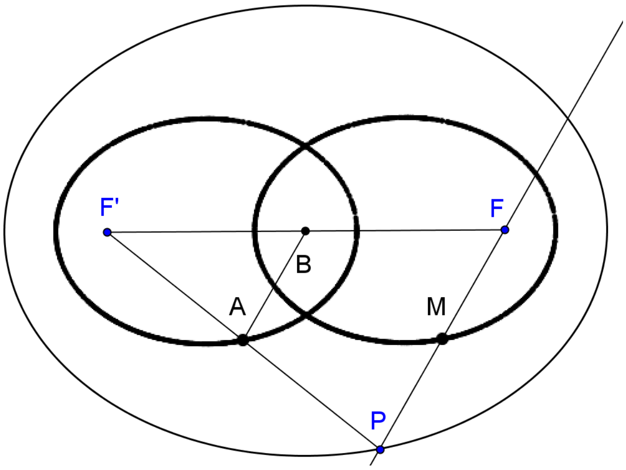

Fig. 10 Construction elements added to the drawing

They used the image on the screen (figure 10) to convince themselves that triangle *F'BA* is similar to triangle *F'FP* since the points *A*, *B* and *M* all are midpoints. This is in line with our prediction (see figure 3). Further, they stated that the ellipses are similar as well. At this moment they began to write on the worksheet. However, they were confused about what they were supposed to write in task (c) since they did not realize the difference between this task and the next one, task (d). The reason for this was probably that they felt that they already had proved the conjecture. Finally, they wrote: "Reason: similar triangles Two ellipses".

6.1.4 Task d: Construct a proof.

This task was straightforward for the students, and their proof construction was based on the similarity that was found in the preceding task. As anticipated, similarity was the key idea in this example.

6.1.5 Task e: Make new related investigations. Make conjectures, support or refute, explain and prove.

Rather soon, one of the students suggested that they should investigate a parabola since "a parabola is an ellipse with the other focus in infinity" (20:31). This suggestion corresponds to the second question that was predicted (see table 2). They opened a new window and used the inbuilt tools to construct a parabola, a point *P* on this parabola, and the midpoint *M* between *P* and the focus point. Then, they put trace on *M* and dragged *P* along the parabola. They obtained a drawing like the one in figure 5b, and it was obvious for them that the locus of *M* was a parabola. After this they opened a new window, and made the same investigation with a hyperbola. At this moment they began to formulate a generalized conjecture and wrote: "Conjecture: For any conic section (non-degenerate), the locus is another conic of the same type!". They stated that this would be straightforward to prove and opened a new window to construct a new parabola. However, this time they did not use the inbuilt tool. Instead, they made a construction based on the defining properties of a parabola to find *how* to explain the conjecture. In an analogical way to the case with the ellipse their explanation was based on similar triangles. As the students noted that they were supposed to prove their generalization, the observer gave them the instruction to give priority to further generalizations.

At this moment **E** stated "What we have done now is that we have checked the conic sections", and asked "What can we vary?" (39:22). "We can vary the position of *M*. It doesn't need to be the midpoint. It can be another point on the line, I guess" (39:35), **C** responded. This is coherent with the first question that was predicted (table 2). They performed, what **C** suggested, by going back to an earlier window (figure 10) and place a point *M* so that $FM/FP = 1/4$ (instead of one half). They dragged point *P* with trace activated on *M* and stated that *M* still follows an ellipse. They repeated this procedure with the point *M* placed randomly on the segment between the point *P* and one of the foci and noticed a second generalization. They realized that this generalization could be explained and proved in an analogical way as in earlier cases. This second generalization was formulated and explained, and the students continued their search to find further generalizations.

Transcript	**Interpretation**
(45:02) C: Can we do something more?	
(45:08) E: We have changed this guy [points at the word "ellipse" in the initial statement (on the worksheet)], and we have changed this guy [points at the word "midpoint"].	The students went back to the original description, and summarized the premises they had varied so far.
(45:19) E: The problem if we change this guy [points at the word "foci"] we probably will destroy our proof.	They begin to consider if M could be a point between P and another point (instead of F). Compare with the prediction in table 2.
(45:37) C: Ok, just for fun. [opens a new window and constructs an ellipse and an arbitrary point D inside the ellipse (figure 11), and drags the point P].	It seemed like C found it easy to make constructions to investigate different conditions within GeoGebra.
(46:35) C: Again, it looks like an ellipse. E: What peculiar.	The utterance "what peculiar" indicates that they have found something astonishing.

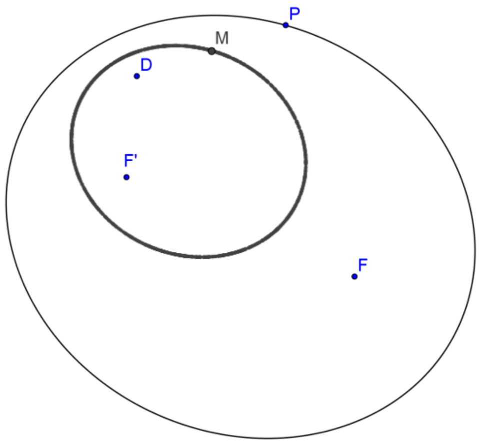

Fig. 11 A random point *D* instead of one of the foci

As predicted, a third generalization was found by the students. They noted that it still "looks like an ellipse" (46:35), when *M* is the midpoint between *P* and another point instead of *F*. Thereafter, **C** asked "What can we say about the foci of this ellipse?" (47:05). This question was their main concern during almost half an hour. They added further construction elements, investigated special cases, and went back and forth between different constructions (saved in different windows). Finally, they concluded that *E* (the midpoint between *D* and *F'*) and *Q* (the midpoint between *D* and *F*) must be the foci due to similar triangles (see figure 12).

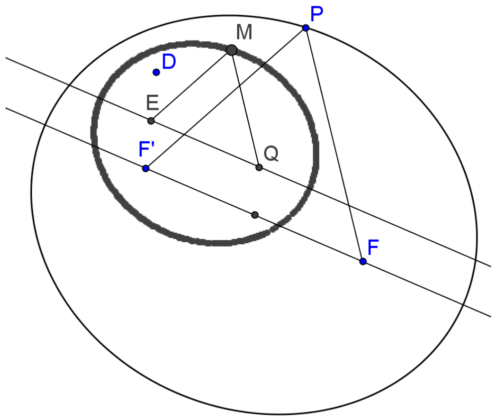

Fig. 12 The foci of the locus of *M* are *E* and *Q*

Next, they went back to the worksheet and began to formulate the third generalization in writing.

Transcript	Interpretation
(1:12:40) **C**: So the most general conjecture is that: If we choose a random point which is not necessarily the focus…inside the ellipse… actually what about outside? [laughing]	As they began to formulate the generalization in writing, they hesitated when they said "inside the ellipse" and reflected on what would happen if *D* is a point outside the ellipse.
(1:12:47) **E**: Just thought about it…when you said inside. So the most general conjecture will be a random point in the plane. [laughing]	They found this rather fun and it seemed that they did not take this idea seriously at the moment.
(1:12:53) **C**: Well, but let´s keep it inside.	They left this speculation for a while.
[…]	
(1:18:40) **E**: … so remaining problems… future work: Other non-degenerate conic sections and the big guy: outside ellipse [writes all this on the worksheet]	At this moment they began to reflect on future work. Once again, they reflected on what will happen if the random point *D* is placed *outside* the ellipse.
(1:19:21) **C**: Let´s just try outside ellipse, I´m curious. **E**: I´m curious as well. [they open a new window and make the construction in figure 13]	As noticed before (45:37), it seemed that **C** found it easy to make constructions to investigate different conditions within *GeoGebra*.

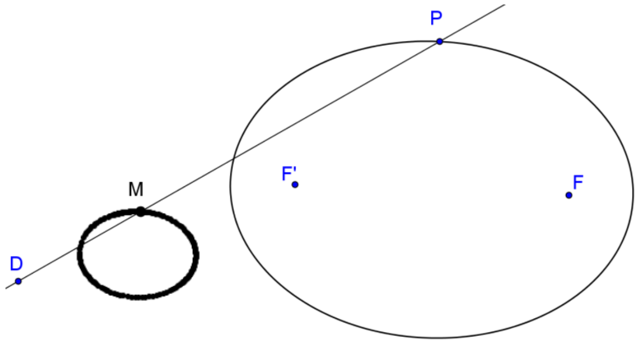

Fig. 13 The random point D placed *outside* the ellipse

At this moment, the students concluded that the locus of *M* still is an ellipse, even if the random point *D* is located outside the (original) ellipse. Next, they wrote on the worksheet: “Conjecture: Outside ellipse gives analogous results”. Thereby a refinement of the third generalization was stated. After a while, **C** stated that “This is just a transformation of the original curve”. Then, he pointed at the *GeoGebra* tool labeled with the following description: “Enlarge Object from Point by Factor”. **C** began to elucidate how the transformation works by using paper and pencil. He drew a triangle, a point and a smaller triangle, similar to the bigger one. He further said that the point directs everything. The following excerpt shows how they made connections between the drawing on the paper and an earlier construction in *GeoGebra* (figure 12).

Transcript		**Interpretation**
(1:27:56)	**C**: Is this the point *D*? Could it be the point *D*? [points at the point on the paper and compares it with the point *D* in figure 12] **E**: Maybe	C started to compare his sketch on the paper with the drawing in figure 12.
[…]		
(1:28:40)	**C**: It is really this mapping [points at the *GeoGebra* tool] So, one conjecture is… can be that …	It seems like **C** has convinced himself of his “transformation theory”.
(1:29:37)	**E**: So, let’s try to produce it…if you take this one [points to the original ellipse]	**E** suggested that they could make the transformation by using the *GeoGebra* tool.
(1:29:50)	**C**: Click on it and it will say what we are supposed to do [points at the text in *GeoGebra*]. Object to enlarge [reads the text laud] so it is the ellipse, yes. Then click on the point *D*. [a small window appears with the text: “Number”]	They utilized an affordance provided by the software, namely the help text: “Select object to enlarge, then centre point and enter factor”
(1:30:30)	**C**: Number. Ok, let’s try…the number is *DE* over *DF’*…one half in this case. **E**: Yes	They calculated the scale factor.

(1:30:43)	**C**: And we got it! [the new object coincide with the trace in figure 12]	Once again, they have convinced themselves of a further generalization.

A discussion followed about whether this transformation has a special name. The observer informed the students about the term "homothety" and the students went to Google to get a description of it. The students also discussed how they should prove this even if they never wrote it down. At this stage, the participants were rather sure that there were nothing more to change (in the original description), or as **E** expressed it: "I think our imagination is exhausted". But only a few seconds after this, **C** made the following utterances: "Maybe you can change the curve *completely*." (1:34:11) and "Yes, it is so. It is like a definition of this transformation."(1:34:43). Finally, they wrote this most general conjecture on the worksheet and their work with the task-situation was finished.

6.2 Summary of the results

When we compared our predictions about student performances with the results from the case study we found both agreements and differences. The main agreements are that the students

- added construction elements to explore and explain geometrical properties and relations. This was made even more frequently than expected.
- exploited the simplicity to check different examples and test ideas provided by the DGE. For example, they used the following dragging modalities: *bounded dragging*, *guided dragging* and *dragging with trace activated.*
- made further reflections when they formulated their conclusions in writing.
- found the most general case by changing the premises in the initial description. However, they did this in a more systematic way than expected.

The main unexpected results are that the students

- performed task (a) and task (b) at the same time.
- found it hard to distinguish between task (c) and task (d).

Some further observations are that the students

- expressed their written answers rather shortly. For example, they did not express the premises in their conjectures, they only wrote the conclusions.
- frequently opened a new window (and saved the old constructions) when they started with a new construction. This made it possible for them to switch between different windows to get ideas and support for their mathematical reasoning.
- used special cases to explore and confirm conjectures.
- spent almost 80% of the time on the last task, i.e. *further generalizations.*

7 Conclusion and Discussion

In this section we first introduce the revised model and discuss each task in the model. Then, we discuss how the model could be used.

7.1 The Revised Model

Results from the case study have been considered in the revision of the initial model (table 1). A suggestion of a revised model is introduced in table 4.

Table 4 The revised model with the modifications highlighted in bold.

Description of the mathematical situation	
(a)	Make an appropriate construction in [the DGE, e.g. *GeoGebra*] and study the position of [a dependent object, e.g. a point], for different positions of [an independent object]. Make a conjecture.
(b)	**Are you convinced of the truth of your conjecture? If not, try to use [the DGE] to support your conjecture. When you are convinced, go to the next task.**[7]
(c)	Explain in your own words why your conjecture is true.
(d)	Construct a **formal** proof.
(e)	**Investigate if your conjecture can be generalized. Perform the tasks above with new premises.**

7.1.1 Task a: Make an appropriate construction in [the DGE, e.g. GeoGebra] and study the position of [a dependent object, e.g. a point], for different positions of [an independent object]. Make a conjecture.

The first task is the same as in the initial model. This task presupposes that students have been introduced to the software in use. The students in the case study were introduced to *GeoGebra* very shortly, and different dragging modalities were demonstrated some days before the study. Furthermore, one of them spent some hours "playing" with the software. These preparations were sufficient for them to be able to work with this task and the subsequent tasks. During their work with the tasks, the students got more and more acquainted with the software. This was a good preparation for their work with generalizations in the last task.

An alternative to the formulation "Make an appropriate construction" would have been to give detailed descriptions of the different steps in the construction as in the examples discussed by Leung (2011). We argue that it might be instructive for students to reflect on what an "appropriate construction" should include and how it should be made. However, the level of guidance should be adapted to both how difficult the construction is and the students' abilities.

One important principle in our model is that students should make their own conjectures. This is consistent with the notion *open problem* (e.g. Mogetta et al. 1999). To make the problem even more open, the instruction "study the position of…" could be replaced with "search for mathematical properties". This formulation had probably worked equally well for the students in our case study since they are high achieving students. The level of openness in the instruction

[7] We take for granted that students will make a new conjecture in case their conjecture turns out to be false. We have omitted the words "or refute" in the initial model.

should be adapted to the mathematical situation under consideration and the students' abilities.

In our model the request "make a conjecture" is formulated in the first task. In Leung's model, on the other hand, this request is explicitly stated in the last mode, the Situated Discourse Mode (SDM). One reason for this difference between our model and Leung's model could be that the models are based on different types of mathematical problems. In geometrical locus problems, DGE often provide a quick visualization of a conjecture as in the example (see table 2).

We argue that it might be important for students to formulate their conjectures in writing early in the process. In this way their conjectures could serve as a base for further reflections (Kieran & Saldanha 2008). These reflections could lead to further explorations and improved conjectures (Lin et al. 2012). That the request for written answers could make students to reflect further was also shown in our case study. For example, when the students were to write "inside the ellipse" they hesitated and said "actually what about outside?" (minute 1:12:40).

7.1.2 Task b: Are you convinced of the truth of your conjecture? If not, try to use [the DGE] to support your conjecture. When you are convinced, go to the next task.

The formulation of this task is changed completely. When the students in the case study read the instruction for task (b) they felt that they already had performed it. The reason for this was that they supported their conjecture by investigating a further example before they formulated the conjecture in writing. To avoid this kind of confusion, we chose to reformulate task (b) so that students can proceed with the next task in case they already are convinced of the truth of their conjecture.

7.1.3 Task c: Explain in your own words why your conjecture is true.

The formulation of this task is unchanged. This task is challenging for many students (Hölzl 2001). However, for the students in our case study it was not too hard to explain their conjecture. The strategies they used were that they

- referred to mathematical theory, in this case the definition of an ellipse
- added construction elements frequently to find and investigate different ideas
- used special cases to find and investigate different ideas

It might be useful for teachers to have these strategies in mind when guiding and supporting students.

7.1.4 Task d: Construct a formal proof.

In this task we have added the word "formal" to elucidate the difference between this task and the preceding one. For the students in the case study there were no difference between task (c) and task (d). For them, the conjecture was already proved when they had performed the explanation in task (c). To make a formal proof construction was not a challenge for them. However, for many students this is a challenge.

This task could be omitted without affecting the rest of the task-situation. The purpose with the task-situation and the students' abilities should guide the decision whether to include this task or not. This aligns with Leung's model

(Leung 2011) where the last instruction "to explain or prove" (p. 328) indicates that the proof construction might be excluded.

7.1.5 Task e: Investigate if your conjecture can be generalized. Perform the tasks above with new premises.

We decided to change the formulation of this task because we wanted to focus even more on the process of generalization. The students in the case study worked in a systematic way when they generalized their initial conjecture. They went back to the original description and looked for premises to vary. To direct students' focus on their conjecture we decided to replace the original formulation "make new related investigations" with the formulation "Investigate if your conjecture can be generalized." It might be easier for students to perform this task if they have formulated their conjecture as a conditional statement, i.e. with both premises and conclusion. If so, they can look for premises to vary in their conjecture instead of the original description.

We realize that this reformulation implies a restriction of the model since there might appear situations when the initial conjecture not can be generalized. However, we want a model that fosters students to search for further generalizations. To find further generalizations students should be able to pose their own questions. Research (e.g. Sinclair 2003) indicates that this is a challenge for many students. The last sentence in this task could give students a clue of how to search for further generalizations, i.e. by defining and varying the *premises* in the conjecture. This general directive might not be sufficient for all students. To give further guidance, teachers could help students to pose questions like "What if…". The importance of posing this kind of questions is highlighted by several authors (e.g. Chazan 1990a; Lin et al. 2012; Mason et al. 2010; Yerushalmy 1993).

7.2 How to use the model

One way to use the model to design a task-situation is to emanate from a traditional proof task, as we do in section 5.2. However, it is possible to emanate from problems without the traditional formulation "prove that...". For example, the following locus problem elaborated by Guven (2008) could be used.

> What is the locus of points determined by the intersection of the medians of a triangle as one of the vertices of the triangle traverses a circle? (p. 252). (See figure 14)

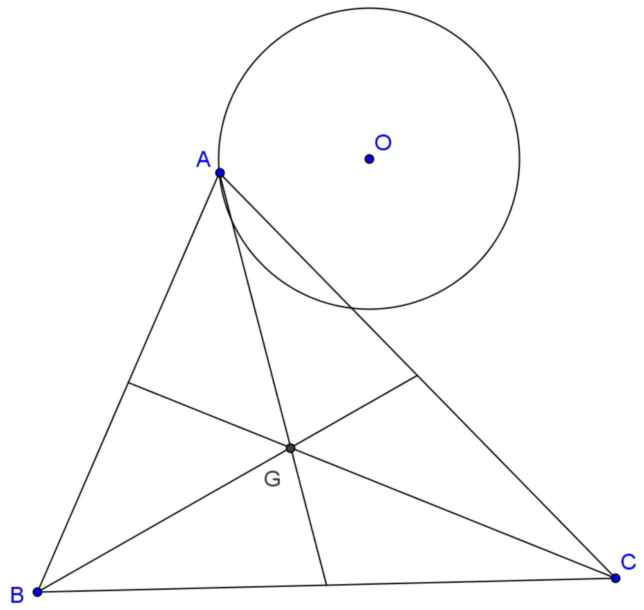

Fig. 14 Illustration of the problem introduced by Guven

In table 5, we give a suggestion of how this problem can be reformulated by using the model.

Table 5 An example of how the revised model can be used

Let G be the intersection of the medians of triangle ABC and let vertex A traverse a circle.

(a) Make an appropriate construction in GeoGebra and study the position of point G, for different positions of vertex A. Make a conjecture.
(b) Are you convinced of the truth of your conjecture? If not, try to use GeoGebra to support your conjecture. When you are convinced, go to the next task.
(c) Explain in your own words why your conjecture is true.
(d) Construct a formal proof.
(e) Investigate if your conjecture can be generalized. Perform the tasks above with new premises.

This problem provides several opportunities for students to make further generalizations. There are several premises in the initial description that could be varied and investigated. Some interesting "what if…" questions that could be posed are

- What if *G* is an ar*bitrary point* on one of the medians?
- What if *G* is an arbitrary point on *an arbitrary segment* between a vertex and its opposite side in the triangle?
- What if *A* is a point on an *arbitrary figure* instead of a circle?

The more generalizations made the better the chance to discover that the findings can be explained by homothety, also in this example. Though, even if this problem differs a lot from the example used in the case study, the problems can both be explained by the same mathematical property, i.e. homothety. Most likely it is possible to find other relations that can be discovered and generalized in a DGE and explained through homothety. We suggest that starting with a basic principle and trying to find special cases founded on this principle can be an alternative way to find examples where the model can be used to design task-situations that promote generalizations.

In this paper we have restricted our model to *geometrical* locus problems. However, with a minor revision of task (a) the model could be used to reformulate other types of locus problems. For example, consider the following problem from calculus:

Investigate the locus of the extreme point of the function $f(x) = x^2 + bx$ *when the parameter b changes.*

This problem could be reformulated in the following way by using the model

Let E be the extreme point of the function $f(x) = x^2 + bx$*, where b is a real number.*

(a) Make an appropriate construction in GeoGebra and study the position of point E, for different values of the parameter b. Make a conjecture.

The subsequent tasks (b) – (e) could be the same as before. In this problem, the *slider tool* could be used to vary the value of the parameter. This tool is not discussed in the literature referred to in this paper. However, Zehavi and Mann (2011) discuss this tool in connection to dynamical explorations in a CAS environment.

It would be interesting to investigate the generality of our model. Foremost, it is the formulation in task (a) that restricts the model to geometrical locus problems. We suppose that the model could be used to design task-situations based on other kinds of problems if task (a) is reformulated.

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