Boundary Layers for the Nonlinear Discrete Boltzmann Equation: Condensing Vapor Flow in the Presence of a Non-Condensable Gas

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Abstract. Half-space problems for the Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers. Half-space problems provide the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. Here we consider a half-space problem of condensation for a pure vapor in the presence of a non-condensable gas by using discrete velocity models (DVMs) of the Boltzmann equation. The Boltzmann equation can be approximated by DVMs up to any order, and these DVMs can be applied for numerical methods, but also for mathematical studies to bring deeper understanding and new ideas. For one-dimensional half-space problems, the discrete Boltzmann equation (the general DVM) reduces to a system of ODEs. We obtain that the number of parameters to be specified in the boundary conditions depends on whether the condensing vapor flow is subsonic or supersonic. This behavior has earlier been found numerically. We want to stress that our results are valid for any finite number of velocities. This is an extension of known results for single-component gases (and for binary mixtures of two vapors) to the case when a non-condensable gas is present. The vapor is assumed to tend to an assigned Maxwellian, with a flow velocity towards the condensed phase, at infinity, while the non-condensable gas tends to zero at infinity. Steady condensation of the vapor takes place at the condensed phase, which is held at a constant temperature. We assume that the vapor is completely absorbed, that the non-condensable gas is diffusively reflected at the condensed phase, and that vapor molecules leaving the condensed phase are distributed according to a given distribution. The conditions, on the given distribution at the condensed phase, needed for the existence of a unique solution of the problem are investigated, assuming that the given distribution at the condensed phase is sufficiently close to the Maxwellian at infinity and that the total mass of the non-condensable gas is sufficiently small. Exact solutions and solvability conditions are found for a specific simplified discrete velocity model (with few velocities).

Keywords: Boltzmann equation, boundary layers, discrete velocity models, half-space problem, non-condensable gas
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INTRODUCTION

Half-space problems for the Boltzmann equation are important in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers [1, 2]. For single-component gases half-space problems are well-studied mathematically both for the continuous Boltzmann equation as well as the discrete Boltzmann equation, see [3, 4, 5] and references therein. In the present paper we present some of our results for the discrete Boltzmann equation for binary mixtures, recently obtained in [6] and [7]. We do consider the case of a binary mixture of two vapors, but our main objective is the case of a condensing vapor in the presence of a non-condensable gas, cf. [8] , for which the main result is presented in Theorem 2. In the latter case we also present explicit solutions and solvability conditions for a reduced 6+4-velocity model in the case of a flow symmetric around the x-axis [7]. We start by reviewing some general properties for the planar stationary discrete Boltzmann equation for binary mixtures.

The planar stationary discrete Boltzmann equation for a binary mixture of the gases A and B reads [6]

$$\begin{cases} \xi_i^{A,1}\dfrac{dF_i^A}{dx} = Q_i^{AA}(F^A,F^A)+Q_i^{BA}(F^B,F^A),\ i=1,...,n_A, \\ \xi_j^{B,1}\dfrac{dF_j^B}{dx} = Q_j^{AB}(F^A,F^B)+Q_j^{BB}(F^B,F^B),\ j=1,...,n_B, \end{cases} \tag{1}$$

where $V_\alpha = \left\{\xi_1^\alpha,...,\xi_{n_\alpha}^\alpha\right\} \subset \mathbb{R}^d$, $\alpha,\beta \in \{A,B\}$, are finite sets of velocities, $F_i^\alpha = F_i^\alpha(x) = F^\alpha(x,\xi_i^\alpha)$ for $i=1,...,n_\alpha$, and $F^\alpha = F^\alpha(x,\xi)$ represents the microscopic density of particles (of the gas α) with velocity ξ at position $x \in \mathbb{R}$.

We denote by m_α the mass of a molecule of gas α. Here and below, $\alpha, \beta \in \{A, B\}$.

For a function $g^\alpha = g^\alpha(\xi)$ (possibly depending on more variables than ξ), we will identify g^α with its restrictions to the set V^α, but also when suitable consider it like a vector function

$$g^\alpha = (g_1^\alpha, ..., g_{n_\alpha}^\alpha), \text{ with } g_i^\alpha = g^\alpha(\xi_i^\alpha).$$

The collision operators $Q_i^{\beta\alpha}(F^\beta, F^\alpha)$ in (1) are given by

$$Q_i^{\beta\alpha}(F^\beta, F^\alpha) = \sum_{k=1}^{n_\alpha} \sum_{j,l=1}^{n_\beta} \Gamma_{ij}^{kl}(\beta, \alpha)\left(F_k^\alpha F_l^\beta - F_i^\alpha F_j^\beta\right) \text{ for } i = 1, ..., n_\alpha,$$

where it is assumed that the collision coefficients $\Gamma_{ij}^{kl}(\beta, \alpha)$, with $1 \leq i, k \leq n_\alpha$ and $1 \leq j, l \leq n_\beta$, satisfy the relations

$$\Gamma_{ij}^{kl}(\alpha, \alpha) = \Gamma_{ji}^{kl}(\alpha, \alpha) \text{ and } \Gamma_{ij}^{kl}(\beta, \alpha) = \Gamma_{kl}^{ij}(\beta, \alpha) = \Gamma_{ji}^{lk}(\alpha, \beta) \geq 0.$$

It is also assumed that $\Gamma_{ij}^{kl}(\beta, \alpha) = 0$ unless we have conservation of momentum and energy (mass is trivially conserved)

$$m_\alpha \xi_i^\alpha + m_\beta \xi_j^\beta = m_\alpha \xi_k^\alpha + m_\beta \xi_l^\beta \text{ and } m_\alpha |\xi_i^\alpha|^2 + m_\beta \left|\xi_j^\beta\right|^2 = m_\alpha |\xi_k^\alpha|^2 + m_\beta \left|\xi_l^\beta\right|^2.$$

A (general) collision invariant is a vector $\phi = (\phi^A, \phi^B)$, such that

$$\phi_i^\alpha + \phi_j^\beta = \phi_k^\alpha + \phi_l^\beta,$$

for all indices $1 \leq i, k \leq n_\alpha$, $1 \leq j, l \leq n_\beta$ and $\alpha, \beta \in \{A, B\}$, such that $\Gamma_{ij}^{kl}(\beta, \alpha) \neq 0$.

The DVMs for the gases A and B are normal if the only collision invariants of the forms $\phi = (\phi^A, 0)$ and $\phi = (0, \phi^B)$, respectively, fulfills

$$\phi^\alpha = \phi^\alpha(\xi) = a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha |\xi|^2,$$

for some constant $a_\alpha, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. A DVM for a mixture is normal, if any general collision invariant of the DVM is of the form

$$\phi = (\phi^A, \phi^B), \text{ with } \phi^\alpha = \phi^\alpha(\xi) = a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha |\xi|^2,$$

for some constant $a_A, a_B, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. A DVM is called supernormal [9], if it is normal both restricted to the single-component gases as well as a mixture. This is always true in the continuous case. However, in the discrete case we can also obtain so called spurious (unphysical) collision invariants [9].

A binary Maxwellian distribution (or just a bi-Maxwellian) is a function $M = (M^A, M^B)$, such that

$$Q(M, M) = 0 \text{ and } M_i^\alpha \geq 0 \text{ for all } 1 \leq i \leq n_\alpha.$$

All bi-Maxwellians are of the form $M = e^\phi$, where ϕ is a collision invariant, i.e. for normal models we will have

$$M = (M^A, M^B), \text{ with } M^\alpha = e^{\phi^\alpha} = e^{a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha |\xi|^2}. \tag{2}$$

We assume that $n_\alpha = 2n_\alpha^+$ and that the sets V_α are symmetric, such that (after possible reordering)

$$\xi_{i+n_\alpha^+}^\alpha = (-\xi_i^{\alpha,1}, \xi_i^{\alpha,2}, ..., \xi_i^{\alpha,d}), \text{ with } \xi_i^{\alpha,1} > 0, \text{ for } i = 1, ..., n_\alpha^+, \tag{3}$$

and denote

$$F = (F^A, F^B) = (F^A(\xi), F^B(\xi)) \text{ and } Q(F, F) = \left(Q^{AA}(F^A, F^A) + Q^{BA}(F^B, F^A), Q^{AB}(F^A, F^B) + Q^{BB}(F^B, F^B)\right).$$

Then the system (1) can be rewritten as

$$D\frac{dF}{dx} = Q(F, F), \text{ with } D = \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}, \text{ where}$$

$$D_\alpha = \begin{pmatrix} D_\alpha^+ & 0 \\ 0 & -D_\alpha^+ \end{pmatrix}, \text{ and } D_\alpha^+ = \operatorname{diag}(\xi_1^{\alpha,1}, ..., \xi_{n_\alpha^+}^{\alpha,1}), \text{ with } \xi_1^{\alpha,1}, ..., \xi_{n_\alpha^+}^{\alpha,1} > 0. \tag{4}$$

We also define the projections $R_\pm^\alpha : \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha^+}$, where $n_\alpha = 2n_\alpha^+$, by

$$R_+^\alpha s^\alpha = s_+^\alpha = (s_1, ..., s_{n_\alpha^+}) \text{ and } R_-^\alpha s^\alpha = s_-^\alpha = (s_{n_\alpha^+ + 1}, ..., s_{n_\alpha})$$

for $s^\alpha = (s_1, ..., s_{n_\alpha})$.

BINARY MIXTURES OF TWO VAPORS

In this section we consider the case of a binary mixture of two vapors [6] (and as a particular case the case of a single vapor [5]), to give the possibility to compare with the results for the case of a condensing vapor with a non-condensable gas present [7], presented in the next section. We assume that our DVMs are normal considered as binary mixtures. It is also preferable that the DVMs for the gases A and B are normal, even if this doesn't affect our results.

For a bi-Maxwellian $M = \left(M^A, M^B\right)$, we obtain, by substituting $F = M + \sqrt{M} f$ in Eq.(4), the system

$$D\frac{df}{dx} + Lf = S(f,f), \tag{5}$$

where the linearized operator L is a symmetric and semi-positive matrix, with the null-space

$$\begin{aligned} N(L) &= \operatorname{span}(R_A M^{1/2}, R_B M^{1/2}, M^{1/2}\xi^1, ..., M^{1/2}\xi^d, M^{1/2}|\xi|^2), \text{ where} \\ R_A h &= (h_1, ..., h_{n_A}, 0, ..., 0) \text{ and } R_B h = (1-R_A)h \text{ if } h \in \mathbb{R}^n, \text{ with } n = n_A + n_B, \end{aligned}$$

and the quadratic part $S(f,f)$ belong to the orthogonal complement of $N(L)$ [6].

At the far end we assume that

$$f(x) \to 0 \text{ as } x \to \infty, \tag{6}$$

and at the condensed phase we assume the general boundary conditions

$$f_+^A(0) = C_A f_-^B(0) + h_0^A \text{ and } f_+^B(0) = C_B f_-^B(0) + h_0^B, \tag{7}$$

where $h_0^\alpha \in \mathbb{R}^{n_\alpha^+}$ and C_α are given $n_\alpha^+ \times n_\alpha^+$ matrices, such that

$$C_\alpha^T D_\alpha^+ C_\alpha < D_\alpha^+ \text{ on } \mathbb{R}^{n_\alpha^+}. \tag{8}$$

Note that condition (8) is fulfilled if $C_A = C_B = 0$. In fact, condition (8) can be weakened, see [6].

We denote by k^+, k^-, and l, with $k^+ + k^- + l = d+3$, the numbers of positive, negative, and zero eigenvalues of the $(d+3)\times(d+3)$ matrix K, with entries $k_{ij} = \left\langle y_i, D y_j \right\rangle$, such that $\{y_1, ..., y_{d+3}\}$ is a basis of the null-space of L, i.e. $\operatorname{span}(y_1, ..., y_{d+3}) = N(L)$. Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$. Then we have the following theorem from [6].

Theorem 1 *Let condition* (8) *be fulfilled and suppose that* $\langle h_0, h_0 \rangle_{D^+}$, *with* $h_0 = (h_0^A, h_0^B)$ *is sufficiently small. Then with* $k^+ + l$ *conditions on* h_0, *the system* (5) *with the boundary conditions* (6) *and* (7), *has a locally unique solution (with respect to a weighted supremum norm* $|\cdot|_\sigma$, *see [6]).*

The case of a single vapor can be obtained by letting V_B be the empty set, i.e. $V_B = \emptyset$. Then we will have one less collision invariant, since $R_B M^{1/2}$ will disappear. Here we assume that the DVM for gas A is normal.

Let b be the first component of $\mathbf{b}$ in Eq.(2). The typical case for a single vapor is (cf. Conjecture 1 below) that there is a critical number $b_+ > 0$ (corresponding in the continuous case to speed of sound), such that

	$b < -b_+$	$b = -b_+$	$-b_+ < b < 0$	$b = 0$	$0 < b < b_+$	$b = b_+$	$b_+ < b$
k^+	0	0	1	1	$d+1$	$d+1$	$d+2$
l	0	1	0	d	0	1	0
$k^+ + l$	0	1	1	$d+1$	$d+1$	$d+2$	$d+2$

,

and for a binary mixture of two vapors (with an extra collision invariant), for a corresponding critical number $b_+ > 0$,

	$b < -b_+$	$b = -b_+$	$-b_+ < b < 0$	$b = 0$	$0 < b < b_+$	$b = b_+$	$b_+ < b$
k^+	0	0	1	1	$d+2$	$d+2$	$d+3$
l	0	1	0	$d+1$	0	1	0
$k^+ + l$	0	1	1	$d+2$	$d+2$	$d+3$	$d+3$

.

Here $b < 0$ corresponds to condensation ($-b_+ < b < 0$ subsonic, $b = -b_+$ sonic and $b < -b_+$ supersonic) and $b > 0$ to evaporation ($0 < b < b_+$ subsonic, $b = b_+$ sonic and $b_+ < b$ supersonic).

CONDENSING VAPOR FLOW IN THE PRESENCE OF A NON-CONDENSABLE GAS

In this section we study distributions F, such that $F \to \left(M^A, 0\right)$ as $x \to \infty$, where $M^A = e^{a_A + m_A \mathbf{b}\cdot\xi + c m_A |\xi|^2}$. We consider DVMs, such that the DVMs for the gases A and B are normal. It is also preferable that the DVMs are normal considered as mixtures, however, spurious collision invariants (for the mixture) doesn't seem to affect the structure of our results.

For a bi-Maxwellian $M = \left(M^A, \varepsilon^2 M^B\right)$, where $M^\alpha = e^{a_\alpha + m_\alpha \mathbf{b}\cdot\xi + c m_\alpha |\xi|^2}$, $F^A \to M^A$ as $x \to \infty$, and ε is a so far undetermined positive constant less or equal to 1, $0 < \varepsilon \leq 1$, we obtain, by denoting

$$F^A(x) = M^A + \sqrt{M^A} f^A \text{ and } F^B(x) = \varepsilon \sqrt{M^B} f^B, \tag{9}$$

in Eq.(1), the system

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A = -\varepsilon L_{BA} f^B + S_{AA}(f^A, f^A) + \varepsilon S_{BA}(f^B, f^A) \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = \varepsilon S_{BB}(f^B, f^B) + S_{AB}(f^A, f^B) \end{cases} . \tag{10}$$

Here L_{AA} and L_{AB} are symmetric and semi-positive matrices such that

$$L_{AB} f^B = 0 \text{ if } f^B \in \operatorname{span}(\sqrt{M^B}), \text{ and } L_{AA} f^A = 0 \text{ if and only if } f^A = \sqrt{M^A}\phi^A,$$

where $\phi = \left(\phi^A, 0\right)$ is a collision invariant. The matrix L_{BA} and the quadratic parts $S_{\alpha\beta}$ fulfill the following orthogonality relations

$$L_{BA} f^B, S_{BA}(f^B, f^A) \in \operatorname{span}(\sqrt{M^A})^\perp, \; S_{AA}(f^A, f^A) \in N(L_{AA})^\perp, \text{ and } S_{\alpha B} \in N(L_{AB})^\perp.$$

Here and below, we denote by $N(L_{\alpha\beta})$ the null-space of $L_{\alpha\beta}$. Note that for the continuous Boltzmann equation $\ker(L_{AB}) = \operatorname{span}(\sqrt{M^B})$ [10]. Preferable (even if not necessary, cf. Eq.(14) below) is to have

$$N(L_{AB}) = \operatorname{span}(\sqrt{M^B}). \tag{11}$$

At the condensed phase we assume that

$$f_+^A(0) = h_0 \text{ and } f_+^B(0) = C f_-^B(0), \tag{12}$$

where C is the $n_B^+ \times n_B^+$ matrix, with the elements

$$c_{ij} = \frac{\xi_j^{B,1} \sqrt{M^B_{n_B^+ + j}} M^B_{0i}}{\left\langle D_B^+ M^B_{0-}, 1 \right\rangle \sqrt{M_i^B}},$$

which is the discrete version of the diffusive boundary conditions [11, 5, 7] after the expansion (9), and

$$h_0 = \frac{1}{\sqrt{M_+^A}} (a_0 - M_+^A) \in \mathbb{R}^{n_A^+},$$

where $M_0^B = K_0^B e^{c_0 m_B |\xi|^2}$, with $K_0^B > 0$, and $F_+^A(0) = a_0$, with $a_0 \in \mathbb{R}^{n_A^+}$, is the perfect absorption condition before the expansion (9). At the far end

$$f^A(x) \to 0 \text{ and } f^B(x) \to 0 \text{ as } x \to \infty. \tag{13}$$

We denote by k_α^+, k_α^-, and l_α, with $k_\alpha^+ + k_\alpha^- + l_\alpha = p_\alpha$, the numbers of positive, negative, and zero eigenvalues of the $p_\alpha \times p_\alpha$ matrix K_α, with entries $k_{ij}^\alpha = \left\langle y_i^\alpha, D_\alpha y_j^\alpha \right\rangle$, such that $\left\{y_1^\alpha, ..., y_{p_\alpha}^\alpha\right\}$ is a basis of the null-space of $L_{A\alpha}$, i.e. in our case $p_A = d + 2$, $\operatorname{span}\left(y_1^A, ..., y_{d+2}^A\right) = N(L_{AA}) = \operatorname{span}(\sqrt{M^A}, \sqrt{M^A}\xi^1, ..., \sqrt{M^A}\xi^d, \sqrt{M^A}\,|\xi|^2)$, and $p_B \geq 1$.

For a condensing vapor flow (i.e. with $b<0$, where b is the first component of $\mathbf{b}$ in Eq.(2)), we have $k_B^- \geq 1$. Moreover, under condition (11), $k_B^- = 1$ and $k_B^+ = l_B = 0$. However, it is enough for us that $k_B^+ = l_B = 0$, i.e. that

$$k_B^- = p_B. \tag{14}$$

Conjecture 1 *For a normal DVM (for gas A) fulfilling the symmetry relations* (3) *there is a critical number $b_+ > 0$, such that*

	$b < -b_+$	$b = -b_+$	$-b_+ < b < 0$	$b = 0$	$0 < b < b_+$	$b = b_+$	$b_+ < b$
k_A^+	0	0	1	1	$d+1$	$d+1$	$d+2$
l_A	0	1	0	d	0	1	0

(15)

Conjecture 1 is true for the continuous Boltzmann equation [12], where b_+ is the speed of sound. We assume that we have a DVM that restricted to gas A fulfills Conjecture 1, at least in the case of condensation, i.e. for $b<0$. The number b_+ has been calculated for a plane axially symmetric 12-velocity model (assuming that the solution is symmetric with respect to the x-axis) in [7].

By condition (14), $\dim(\text{span}\{u : L_{AB}u = \lambda D_B u, \lambda > 0\}) = n_B^+$, see [13, 11, 7]. We assume that

$$\dim\left(\text{span}U_B^+\right) = n_B^+ - 1, \text{ where } U_B^+ = \left\{\left(R_+^B - CR_-^B\right)u : L_{AB}u = \lambda D_B u, \lambda > 0\right\}, \tag{16}$$

but, also that

$$\dim\left(\text{span}\widetilde{U}_B^+\right) = n_B^+, \text{ where } \widetilde{U}_B^+ = U_B^+ \cup \left\{\left(R_+^B - CR_-^B\right)\sqrt{M^B}\right\}. \tag{17}$$

If we would have had $\dim\left(\text{span}U_B^+\right) = n_B^+$, then $f^B(x) = 0$, i.e. the non-condensable gas would have been absent.

For $-b_+ < b < 0$ we will also assume that

$$R_+^A\sqrt{M^A} \notin R_+^A \text{span}U_A^+, \text{ with } U_A^+ = \{u : L_{AA}u = \lambda D_A u, \lambda > 0\}, \tag{18}$$

or, equivalently, since $\dim(R_+^A \text{span}U_A^+) = n_A^+ - 1$ by Eq.(15) [13, 11, 7],

$$\dim(R_+^A \text{span}\widetilde{U}_A^+) = n_A^+, \text{ with } \widetilde{U}_A^+ = U_A^+ \cup \left\{\sqrt{M^A}\right\}.$$

In fact, we can replace $\sqrt{M^A}$ in assumption (18) by any possible vector $y \in N(L_{AA})$, such that

$$\left\langle L_{BA}f^B, y\right\rangle = \left\langle S_{BA}(f^B, f^A), y\right\rangle = 0.$$

We fix $\varepsilon = \min\{|h_0|, 1\}$ and the total mass of the gas B to be m_B^{tot}, i.e.

$$\varepsilon m_B \sum_{i=1}^{n_B} \int_0^\infty \sqrt{M^B} f_i^B(x)\, dx = m_B^{tot}, \tag{19}$$

for a given positive constant m_B^{tot}. The case $m_B^{tot} = 0$, corresponds to the case of single species considered in [5].

We have the following theorem from [7].

Theorem 2 *Let conditions* (14)-(17), *and for $-b_+ < b < 0$ also condition* (18), *be fulfilled, and suppose that $\langle h_0, h_0\rangle_{D_A^+}$ is sufficiently small and that m_B^{tot} is sufficiently small relatively $|h_0|$. Then with*

$$k_A^+ + l_A = \begin{cases} 1 \text{ if } -b_+ \leq b < 0 \\ 0 \text{ if } b < -b_+ \end{cases}$$

conditions on h_0, the system (10) *with the boundary conditions* (12) *and* (13) *under condition* (19), *has a locally unique solution (with respect to a weighted supremum norm $|\cdot|_\sigma$, see [7]).*

We note that the number of conditions on h_0 is the same as if the non-condensable gas was absent, i.e. as for a pure vapor, and also that the number of conditions depends on whether the condensing vapor flow is subsonic or supersonic. Similar behavior has been found numerically in [14] and [15], in the case when the given distribution at the condensed phase is the Maxwellian at the condensed phase. However, in our case, we can't be sure that there is any Maxwellian at rest close enough to the Maxwellian at infinity, to fulfill Theorem 2.

A REDUCED 6+4 - VELOCITY MODEL

In this section we present an exact solution and solvability condition (see [7] for a complete presentation) when the vapor, gas A, is modeled by a six-velocity model with velocities

$$\xi_1^A = (1,0), \xi_2^A = (1,1), \xi_3^A = (-1,0), \xi_4^A = (-1,1), \xi_5^A = (1,-1), \text{ and } \xi_6^A = (-1,-1), \tag{20}$$

and the non-condensable gas B is modelled by the classical Broadwell model [16] in plane with velocities

$$\xi_1^B = (m,m), \xi_2^B = (-m,m), \xi_3^B = (m,-m), \text{ and } \xi_4^B = (-m,-m).$$

Here $m = \dfrac{m_A}{m_B}$. We have the correct number of collision invariants for the two gases seen as a binary mixture, even if we for the Broadwell model have only two linearly independent collision invariants, as the mass vector and the energy vector are linearly dependent. For a flow symmetric around the x-axis we obtain the reduced system

$$\frac{dF_1^A}{dx} = \frac{dF_3^A}{dx} = \sigma_1 q_1 + \sigma_2 q_2, \; \frac{dF_2^A}{dx} = \frac{dF_4^A}{dx} = -\sigma_1 q_1 + \sigma_3 q_3, \text{ and } m\frac{dF_1^B}{dx} = m\frac{dF_2^B}{dx} = \sigma_2 q_2 + \sigma_3 q_3,$$

where $q_1 = F_2^A F_3^A - F_1^A F_4^A$, $q_2 = F_3^A F_1^B - F_1^A F_2^B$ and $q_3 = F_4^A F_1^B - F_2^A F_2^B$, or equivalently

$$\begin{cases} D_A \dfrac{dF^A}{dx} = Q^{AA}(F^A, F^A) + Q^{BA}(F^B, F^A) \\ D_B \dfrac{dF^B}{dx} = Q^{AB}(F^A, F^B) + Q^{BB}(F^B, F^B) \end{cases},$$

where $D_A = \operatorname{diag}(1,1,-1,-1)$, $D_B = \operatorname{diag}(m,-m)$, $F^A = \left(F_1^A, F_2^A, F_3^A, F_4^A\right)$, $F^B = \left(F_1^B, F_2^B\right)$, $Q^{AA}(F^A,F^A) = \sigma_1 q_1 (1,-1,-1,1)$, $Q^{BA}(F^B,F^A) = \sigma_2 q_2 (1,0,-1,0) + \sigma_3 q_3 (0,1,0,-1)$, $Q^{AB}(F^A,F^B) = (\sigma_2 q_2 + \sigma_3 q_3)(1,-1)$, and $Q^{BB}(F^B,F^B) = 0$.

We assume the boundary conditions

$$\left(F_1^A(0), F_2^A(0)\right) = s_0^A (1, q_0) \text{ and } F_1^B(0) = F_2^B(0)$$

at the condensed phase, and at the far end

$$F^A \to M^A = s^A(1,q,p,pq) \text{ and } F^B \to 0 \text{ as } x \to \infty. \tag{21}$$

Here $p > 1$ (since we consider a condensing vapor flow) and $q, s^A > 0$.

We denote (transformation (9) with $\varepsilon = 1$)

$$F^A(x) = M^A + \sqrt{M^A} f^A \text{ and } F^B(x) = \sqrt{M^B} f^B,$$

where M^A is given in Eq.(21) and $M^B = (1,p)$, and obtain the system

$$\begin{cases} \dfrac{df^A}{dx} + D_A^{-1} L_{AA} f^A = -D_A^{-1} L_{BA} f^B + D_A^{-1} S_{AA}(f^A, f^A) + D_A^{-1} S_{BA}(f^B, f^A) \\ \dfrac{df^B}{dx} + D_B^{-1} L_{AB} f^B = D_B^{-1} S_{AB}(f^A, f^B) \end{cases}.$$

The linearized collision operators L_{AA} and L_{AB} are symmetric and semi-positive and have the null-spaces

$$\begin{aligned} & N(L_{AA}) = \operatorname{span}(y_1^A, y_2^A, y_3^A) \text{ and } N(L_{AB}) = \operatorname{span}(y^B) \text{ with} \\ y_1^A \quad = \quad & (\sqrt{p}, \sqrt{pq}, 1, \sqrt{q}), y_2^A = (1,0,\sqrt{p},0), y_3^A = (0,1,0,\sqrt{p}), \text{ and } y^B = (1, \sqrt{p}). \end{aligned}$$

The non-zero eigenvalues and corresponding eigenvectors of $D_A^{-1} L_{AA}$ and $D_B^{-1} L_{AB}$ are (remind that $p > 1$)

$$\lambda^A = s^A \sigma_1 (1+q)(p-1) > 0 \text{ and } u^A = (\sqrt{pq}, -\sqrt{p}, \sqrt{q}, -1),$$

and

$$\lambda^B = \frac{s^A}{m}(\sigma_2 + \sigma_3 q)(p-1) > 0 \text{ and } u^B = (\sqrt{p}, 1),$$

respectively.

The new boundary conditions become

$$\left(f_1^A(0), f_2^A(0)\right) = \frac{1}{\sqrt{s^A q}}\left(\sqrt{q}(s_0^A - s^A), s_0^A q_0 - s^A q\right) \text{ and } f_1^B(0) = \sqrt{p} f_2^B(0) \tag{22}$$

at the condensed phase, and at the far end

$$f^A \to 0 \text{ and } f^B \to 0 \text{ as } x \to \infty. \tag{23}$$

We decompose

$$f^A = \mu_1^A y_1^A + \mu_2^A y_2^A + \mu_3^A y_3^A + \beta^A u^A \text{ and } f^B = \mu^B y^B + \beta^B u^B,$$

and obtain, since $\mu_2^A = \mu_3^A = \mu^B = 0$ and the quadratic parts vanish for all solutions under condition (23), the linearized system

$$\begin{cases} \dfrac{d\beta^A}{dx} + \lambda^A \beta^A = -\beta^B \sqrt{q}(\sigma_2 - \sigma_3) \\ \dfrac{d\mu_1^A}{dx} = -\beta^B \dfrac{p-1}{1+q}(\sigma_2 + \sigma_3 q) \\ \dfrac{d\beta^B}{dx} + \lambda^B \beta^B = 0 \end{cases}, \text{ where } \begin{cases} f^A = \mu_1^A y_1^A + \beta^A u^A \\ f^B = \beta^B u^B \end{cases}. \tag{24}$$

Solving system (24), under condition (23), ends up in

$$f^A = \beta_0^B \frac{m}{s^A(1+q)} e^{-\lambda^B x} y_1^A + \left(\beta_0^B \sqrt{q} \frac{\sigma_2 - \sigma_3}{\lambda^B - \lambda^A} e^{-\lambda^B x} + k e^{-\lambda^A x}\right) u^A \text{ and } f^B = \beta_0^B e^{-\lambda^B x} u^B,$$

with $\beta_0^B = \beta^B(0)$ and k constant. If we fix the total amount of gas B to be m_B^{tot}, we obtain

$$\beta_0^B = \frac{s^A m_B^{tot}}{2 m_A \sqrt{p}}(\sigma_2 + \sigma_3 q)(p-1).$$

Furthermore, by the boundary conditions (22) at the condensed phase we obtain

$$k = \frac{s_0^A - s^A}{\sqrt{s^A p q}} + \beta_0^B \sqrt{q} \frac{\sigma_2 - \sigma_3}{\lambda^A - \lambda^B} - \frac{m \beta_0^B}{s^A \sqrt{q}(1+q)},$$

and the solvability condition

$$s_0^A(1 + q_0) = s^A(1+q) + \frac{m_B^{tot}\sqrt{s^A}}{2 m_B}(\sigma_2 + \sigma_3 q)(p-1).$$

Note that the presence of the non-condensable gas implies that the solution for f^A contains a term from the null-space of L_{AA}. Finally we obtain,

$$F^A = s^A(1, q, p, pq) + \beta_0^B \frac{m\sqrt{p}}{\sqrt{s^A}(1+q)} e^{-\lambda^B x}(1, q, 1, q) + \sqrt{pqs^A}\left(\beta_0^B \sqrt{q} \frac{\sigma_2 - \sigma_3}{\lambda^B - \lambda^A} e^{-\lambda^B x} + k e^{-\lambda^A x}\right)(1, -1, 1, -1)$$
$$\text{and } F^B(x) = \beta_0^B e^{-\lambda^B x} \sqrt{p}(1, 1).$$

We have exactly $k_A^+ + l = k_A^+ = 1$ solvability condition. All our assumptions in the preceding section are fulfilled for this reduced model, if we allow $b_+ = \infty$ in Eq.(15). Also, the given distribution at the condensed phase corresponds to a Maxwellian, and, due to that the quadratic terms disappear, we don't need any smallness assumptions at all on the total amount of the gas B, or on the closeness of the far Maxwellian and the Maxwellian at the wall for the gas A to obtain a solution. However, smallness assumptions might be needed to obtain positivity of the solution.

For the case of a condensing vapor flow (symmetric around the x-axis) modelled by the 6-velocity model with velocities (20) (in the absence of a non-condensable gas), we just let $m_B^{tot} = 0$, and obtain the solution

$$F = F^A = s^A \left(1, q, p, pq, q, pq\right) + \frac{s_0^A - s^A}{\sqrt{s^A pq}} e^{-s^A \sigma_1 (1+q)(p-1)x} \left(\sqrt{pq}, -\sqrt{p}, \sqrt{q}, -1, -\sqrt{p}, -1\right)$$

(by adding the extra components) under the solvability condition

$$s_0^A (1 + q_0) = s^A (1 + q).$$

CONCLUSIONS

In the present paper, we have considered some problems related to the half-space problem of condensation and evaporation for the discrete Boltzmann equation for binary mixtures. The number of conditions, on the assigned distribution at the condensed phase, to obtain a unique solution have been presented (under a smallness assumption on the assigned distribution) in two different cases: (i) the case of two vapors; and (ii) the case of a vapor and a non-condensable gas (under some reasonable assumptions and a smallness assumption on the amount of the non-condensable gas). As an example exact solutions and solvability conditions have been found for a simplified discrete velocity model in case (ii). The number of conditions depends on if we have subsonic or supersonic condensation or evaporation. For an evaporating flow we obtain one extra condition in case (i) compared with the case of a pure vapor. Evaporating flows are not studied in case (ii), since then the non-condensable gas is blown away by the evaporating vapor flow and can not stay in the Knudsen layer [14, 17]. For a condensing flow we obtain, both in case (i) and (ii), the same number of conditions as in the case of a pure vapor. The structure of the solutions may, however, differ as for the simplified model in the example. To our knowledge, there is in case (ii) no rigorous analytical results of this kind for the full Boltzmann equation yet. Similar behavior has, however, been found numerically [18, 14, 15], in the case when the given distribution at the condensed phase is the Maxwellian at the condensed phase.

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