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BOUNDARY LAYERS AND SHOCK PROFILES FOR THE DISCRETE BOLTZMANN EQUATION FOR MIXTURES

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Abstract. We consider the discrete Boltzmann equation for binary gas mixtures. Some known results for half-space problems and shock profile solutions of the discrete Boltzmann for single-component gases are extended to the case of two-component gases. These results include well-posedness results for half-space problems for the linearized discrete Boltzmann equation, existence results for half-space problems for the weakly non-linear discrete Boltzmann equation, and existence results for shock profile solutions of the discrete Boltzmann equation. A characteristic number, corresponding to the speed of sound in the continuous case, is calculated for axially symmetric models. Some explicit calculations are also made for a simplified 6 + 4 -velocity model.

1. **Introduction.** We consider the discrete Boltzmann equation (DBE) for two-component gases and extend some known results for single-component gases to this case. In the planar stationary case systems of ODEs with the same structure as the systems obtained for single species, cf. [10],[4],[5], and [6], are obtained. It is then possible to extend the well-posedness results for the half-space problems for the linearized DBE in [4], the existence results for the half-space problems for the weakly non-linear DBE in [5], and the existence results for shock profile solutions of the DBE in [9], to the case of binary mixtures. We exemplify some of our results, by explicit calculations for a simplified plane 6 + 4 -velocity model. However, we want to stress that our general results are valid for any finite number of velocities.

We give below a brief review of related publications. Half-space problems for the Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers [28],[29]. Mathematical results for half-space problems for the Boltzmann equation for a single-component gas is reviewed in [3]. The Kyoto group of Y. Sone, K. Aoki, and coworkers, has under a long time considered problems related to these questions, both from a theoretical and numerical point of view [28],[29]. Some of these results have also been extended to the case of mixtures [31].

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In the planar stationary case, the DBE reduces to a system of ODEs. It is well-known that the Boltzmann equation can be approximated up to any order by the DBE [13],[22],[27].

Half-space problems for the linearized Boltzmann equation are well investigated [2], and in the case of binary mixtures by Aoki, Bardos and Takata in [1]. For the linearized DBE a classification of well-posed half-space problems has been made in [4], based on results in [10] on the dimensions of the corresponding stable, unstable and center manifolds for singular points (Maxwellians for the DBE) to general systems of ODEs of the same type.

In [33] Ukai, Yang and Yu studied the non-linear case with inflow boundary conditions, assuming that the solutions tend to an assigned Maxwellian at infinity. The conditions on the data at the boundary needed for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution of the problem are investigated. Similar problems have also been studied for the DBE in [32],[23],[24], and [5]. The quite general results in [5] include (for DVMs) the results obtained by Ukai, Yang and Yu in [33] for the continuous Boltzmann equation. In this connection, we also mention the recent paper by Yang [34], and the recent work by Liu and Yu in [26] on the center manifold theory of the half-space problem for the full Boltzmann equation, where also more references for the continuous case can be found.

The existence of shock profile solutions, c.f. [18] and [25], have been studied for the DBE in [16] and [9]. For the shock wave problem the DBE also becomes a system of ODEs. In [9] existence of shock profile solutions for the DBE is proved. The results concern weak shocks, i.e., when the shock speeds are close to a typical speed, corresponding to the sound speed in the continuous case. The shock-wave problem have also been studied for several explicit discrete velocity models for mixtures, see e.g. [19].

The case when one of the gases is a non-condensable gas (cf. [30]) is not included in this paper, but will be treated in a future paper [7].

The paper is organized as follows. In Section 2 we present the DBE for mixtures and some of its properties. We make an expansion around a bi-Maxwellian and obtain the linearized collision operator and the quadratic part and conclude that we actually obtain a system with the same structure as in the case of one species. We also remind a result in [10] on the dimensions of the corresponding stable, unstable and center manifolds for singular points (bi-Maxwellians for DVMs for binary mixtures) to general systems of ODEs of the same type. Then we present the extension of our results for boundary layers in [4] and [5], in Section 3, and for shock profiles in [9], in Section 4, to the case of binary mixtures. In Section 5, we calculate a number, corresponding to the speed of sound in the continuous case, for axially symmetric models. Finally, in Sections 6 and 7 we exemplify our theory for an explicit simplified model. We find exact shock profile solutions in Section 6 and consider non-linear boundary layers in the case of a moving wall with constant speed in Section 7, for a plane $6+4$ -velocity model, where we have assumed that our flow is symmetric with respect to the x -axis.

2. **Discrete velocity models for binary mixtures.** The general discrete velocity model (DVM), or the discrete Boltzmann equation, for a binary mixture of the

gases A and B reads

$$\begin{cases} \dfrac{\partial f_i^A}{\partial t} + \xi_i^A \cdot \nabla_{\mathbf{x}} f_i^A = Q_i^{AA}(f^A, f^A) + Q_i^{BA}(f^B, f^A),\ i = 1, ..., n_A, \\ \dfrac{\partial f_j^B}{\partial t} + \xi_j^B \cdot \nabla_{\mathbf{x}} f_j^B = Q_j^{AB}(f^A, f^B) + Q_j^{BB}(f^B, f^B),\ j = 1, ..., n_B \end{cases} \tag{1}$$

where $V_\alpha = \left\{\xi_1^\alpha, ..., \xi_{n_\alpha}^\alpha\right\} \subset \mathbb{R}^d$, $\alpha, \beta \in \{A, B\}$ are finite sets of velocities, $f_i^\alpha = f_i^\alpha(\mathbf{x}, t) = f^\alpha(\mathbf{x}, t, \xi_i^\alpha)$ for $i = 1, ..., n_\alpha$, and $f^\alpha = f^\alpha(\mathbf{x}, t, \xi)$ represents the microscopic density of particles (of the gas α) with velocity ξ at time $t \in \mathbb{R}_+$ and position $\mathbf{x} \in \mathbb{R}^d$. We denote by m_α the mass of a molecule of the gas α. Here and below, $\alpha, \beta, \gamma \in \{A, B\}$.

For a function $g^\alpha = g^\alpha(\xi)$ (possibly depending on more variables than ξ), we will identify g^α with its restriction to the set V_α, but also when suitable consider it like a vector function

$$g^\alpha = (g_1^\alpha, ..., g_{n^\alpha}^\alpha), \text{ with } g_i^\alpha = g^\alpha(\xi_i^\alpha).$$

Then $f^\alpha = (f_1^\alpha, ..., f_{n^\alpha}^\alpha)$ in Eq.(1).

The collision operators $Q_i^{\beta\alpha}(f^\beta, f^\alpha)$ in (1) are given by

$$Q_i^{\beta\alpha}(f^\beta, f^\alpha) = \sum_{k=1}^{n_\alpha} \sum_{j,l=1}^{n_\beta} \Gamma_{ij}^{kl}(\beta, \alpha) \left(f_k^\alpha f_l^\beta - f_i^\alpha f_j^\beta\right) \text{ for } i = 1, ..., n_\alpha,$$

where it is assumed that the collision coefficients $\Gamma_{ij}^{kl}(\beta, \alpha)$, with $1 \leq i, k \leq n_\alpha$ and $1 \leq j, l \leq n_\beta$, satisfy the relations

$$\Gamma_{ij}^{kl}(\alpha, \alpha) = \Gamma_{ji}^{kl}(\alpha, \alpha) \text{ and } \Gamma_{ij}^{kl}(\beta, \alpha) = \Gamma_{kl}^{ij}(\beta, \alpha) = \Gamma_{ji}^{lk}(\alpha, \beta) \geq 0, \tag{2}$$

with equality unless the conservation laws

$$m_\alpha \xi_i^\alpha + m_\beta \xi_j^\beta = m_\alpha \xi_k^\alpha + m_\beta \xi_l^\beta \text{ and } m_\alpha \left|\xi_i^\alpha\right|^2 + m_\beta \left|\xi_j^\beta\right|^2 = m_\alpha \left|\xi_k^\alpha\right|^2 + m_\beta \left|\xi_l^\beta\right|^2$$

are satisfied. We denote

$$f = \left(f^A, f^B\right) = \left(f^A(\xi), f^B(\xi)\right), \ g = \left(g^A, g^B\right) = \left(g^A(\xi), g^B(\xi)\right),$$
$$\text{and } Q(f, f) = \left(Q^{AA}(f^A, f^A) + Q^{BA}(f^B, f^A), Q^{AB}(f^A, f^B) + Q^{BB}(f^B, f^B)\right).$$

Then the collision operator $Q(f, f)$ can be obtained from the bilinear expressions

$$Q_i(f, g) = \frac{1}{2} \sum_{j,k,l=1}^{n_A} \Gamma_{ij}^{kl}(A, A) \left(f_k^A g_l^A + g_k^A f_l^A - f_i^A g_j^A - g_i^A f_j^A\right)$$
$$+ \frac{1}{2} \sum_{k=1}^{n_A} \sum_{j,l=1}^{n_B} \Gamma_{ij}^{kl}(B, A) \left(f_k^A g_l^B + g_k^A f_l^B - f_i^A g_j^B - g_i^A f_j^B\right),\ i = 1, ..., n_A, \text{ and}$$
$$Q_{n^A+i}(f, g) = \frac{1}{2} \sum_{k=1}^{n_B} \sum_{j,l=1}^{n_A} \Gamma_{ij}^{kl}(A, B) \left(f_k^B g_l^A + g_k^B f_l^A - f_i^B g_j^A - g_i^B f_j^A\right)$$
$$+ \frac{1}{2} \sum_{j,k,l=1}^{n_B} \Gamma_{ij}^{kl}(B, B) \left(f_k^B g_l^B + g_k^B f_l^B - f_i^B g_j^B - g_i^B f_j^B\right),\ i = 1, ..., n_B,$$

Denoting $Q(f, g) = (Q_1(f, g), ..., Q_n(f, g))$, with $n = n_A + n_B$, we see that, for arbitrary f and g

$$Q(f, g) = Q(g, f),$$

and by the relations (2), with $h=\left(h^A,h^B\right)$,

$$\begin{aligned}
&\left\langle h,Q\left(f,g\right)\right\rangle = \\
&=\frac{1}{8}\sum_{i,j,k,l=1}^{n_A}\Gamma_{ij}^{kl}(A,A)\left(h_i^A+h_j^A-h_k^A-h_l^A\right)\left(f_k^Ag_l^A+g_k^Af_l^A-f_i^Ag_j^A-g_i^Af_j^A\right)\\
&+\frac{1}{4}\sum_{i,k=1}^{n_A}\sum_{j,l=1}^{n_B}\Gamma_{ij}^{kl}(B,A)\left(h_i^A+h_j^B-h_k^A-h_l^B\right)\left(f_k^Ag_l^B+g_k^Af_l^B-f_i^Ag_j^B-g_i^Af_j^B\right)\\
&+\frac{1}{8}\sum_{i,j,k,l=1}^{n_B}\Gamma_{ij}^{kl}(B,B)\left(h_i^B+h_j^B-h_k^B-h_l^B\right)\left(f_k^Bg_l^B+g_k^Bf_l^B-f_i^Bg_j^B-g_i^Bf_j^B\right).
\end{aligned} \tag{3}$$

A vector $\phi=\left(\phi^A,\phi^B\right)$ is a collision invariant if and only if

$$\phi_i^\alpha+\phi_j^\beta=\phi_k^\alpha+\phi_l^\beta, \tag{4}$$

for all indices $1\leq i,k\leq n_\alpha$, $1\leq j,l\leq n_\beta$ and $\alpha,\beta\in\{A,B\}$, such that $\Gamma_{ij}^{kl}(\beta,\alpha)\neq 0$. By the relation (3)

$$\begin{aligned}
\left\langle \phi,Q\left(f,f\right)\right\rangle &=\frac{1}{4}\sum_{i,j,k,l=1}^{n_A}\Gamma_{ij}^{kl}(A,A)\left(\phi_i^A+\phi_j^A-\phi_k^A-\phi_l^A\right)\left(f_k^Af_l^A-f_i^Af_j^A\right)\\
&+\frac{1}{2}\sum_{i,k=1}^{n_A}\sum_{j,l=1}^{n_B}\Gamma_{ij}^{kl}(B,A)\left(\phi_i^A+\phi_j^B-\phi_k^A-\phi_l^B\right)\left(f_k^Af_l^B-f_i^Af_j^B\right)\\
&+\frac{1}{4}\sum_{i,j,k,l=1}^{n_B}\Gamma_{ij}^{kl}(B,B)\left(\phi_i^B+\phi_j^B-\phi_k^B-\phi_l^B\right)\left(f_k^Bf_l^B-f_i^Bf_j^B\right).
\end{aligned} \tag{5}$$

which is zero, independently of our choice of non-negative vector f ($f_i^\alpha\geq 0$ for all $1\leq i\leq n_\alpha$), if and only if ϕ is a collision invariant.

We consider below (even if this restriction is not necessary in our general reasoning) only DVMs, such that any collision invariant is of the form

$$\phi=\left(\phi^A,\phi^B\right),\text{ with }\phi^\alpha=\phi^\alpha(\xi)=a_\alpha+m_\alpha\mathbf{b}\cdot\xi+cm_\alpha\left|\xi\right|^2, \tag{6}$$

for some constant $a_A,a_B,c\in\mathbb{R}$ and $\mathbf{b}\in\mathbb{R}^d$. In this case the equation

$$\left\langle \phi,Q\left(f,f\right)\right\rangle =0$$

has the general solution (6). Discussions on constructions of DVMs for binary mixtures can be found in e.g. [11],[12],[20],[21],[14] and [15].

A binary Maxwellian distribution (or just a bi-Maxwellian) is a function $M=\left(M^A,M^B\right)$, such that

$$Q(M,M)=0\text{ and }M_i^\alpha\geq 0\text{ for all }1\leq i\leq n_\alpha.$$

All bi-Maxwellians are of the form

$$M=e^\phi,\text{ i.e. }M=\left(M^A,M^B\right),\text{ with }M^\alpha=e^{\phi^\alpha}=e^{a_\alpha+m_\alpha\mathbf{b}\cdot\xi+cm_\alpha\left|\xi\right|^2}, \tag{7}$$

where $\phi=\left(\phi^A,\phi^B\right)$ is given by Eq.(6). Assuming that f is non-negative, we let

$\phi = \log f$ in Eq.(5) and obtain that

$$\left\langle \log f, Q\left(f,f\right)\right\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{n_A}\Gamma_{ij}^{kl}(A,A)\left(f_k^A f_l^A - f_i^A f_j^A\right)\log\frac{f_i^A f_j^A}{f_k^A f_l^A}$$
$$+\frac{1}{2}\sum_{i,k=1}^{n_A}\sum_{j,l=1}^{n_B}\Gamma_{ij}^{kl}(B,A)\left(f_k^A f_l^B - f_i^A f_j^B\right)\log\frac{f_i^A f_j^B}{f_k^A f_l^B}$$
$$+\frac{1}{4}\sum_{i,j,k,l=1}^{n_B}\Gamma_{ij}^{kl}(B,B)\left(f_k^B f_l^B - f_i^B f_j^B\right)\log\frac{f_i^B f_j^B}{f_k^B f_l^B} \leq 0,$$

with equality if and only if

$$f_k^\alpha f_l^\beta = f_i^\alpha f_j^\beta$$

for all indices $1 \leq i,k \leq n_\alpha$, $1 \leq j,l \leq n_\beta$ and $\alpha, \beta \in \{A,B\}$, such that $\Gamma_{ij}^{kl}\left(\beta,\alpha\right) \neq 0$, or equivalently, if and only if f is a bi-Maxwellian. Hence, f is a bi-Maxwellian if and only if $\log f$ is a collision invariant.

For a bi-Maxwellian $M = \left(M^A, M^B\right)$, we obtain, by denoting

$$f = M + \sqrt{M}h, \tag{8}$$

in Eq.(1), the system

$$\frac{\partial h}{\partial t} + \xi\cdot\nabla_{\mathbf{x}}h = -Lh + S\left(h\right),$$

where $\xi\cdot\nabla_{\mathbf{x}}h = (\xi_1^A\cdot\nabla_{\mathbf{x}}h_1^A, ..., \xi_{n_A}^A\cdot\nabla_{\mathbf{x}}h_{n_A}^A, \xi_1^B\cdot\nabla_{\mathbf{x}}h_1^B, ..., \xi_{n_B}^B\cdot\nabla_{\mathbf{x}}h_{n_B}^B)$. Furthermore, L is the linearized collision operator ($n\times n$ matrix, with $n = n_A + n_B$) given by

$$Lh = -\frac{2}{\sqrt{M}}Q(M, \sqrt{M}h), \tag{9}$$

and the quadratic part S is given by

$$S\left(h,h\right) = \frac{1}{\sqrt{M}}Q(\sqrt{M}h, \sqrt{M}h). \tag{10}$$

By Eq.(3) and the relations $M_i^\alpha M_j^\beta = M_k^\alpha M_l^\beta \neq 0$, we obtain the equality

$$\left\langle g, Lh\right\rangle = -2\left\langle \frac{g}{\sqrt{M}}, Q(M,\sqrt{M}h)\right\rangle$$
$$= \frac{1}{4}\sum_{i,j,k,l=1}^{n_A}\Gamma_{ij}^{kl}(A,A)\left(\sqrt{M_l^A}g_k^A + \sqrt{M_k^A}g_l^A - \sqrt{M_j^A}g_i^A - \sqrt{M_i^A}g_j^A\right)$$
$$\times\left(\sqrt{M_l^A}h_k^A + \sqrt{M_k^A}h_l^A - \sqrt{M_j^A}h_i^A - \sqrt{M_i^A}h_j^A\right)$$
$$+\frac{1}{2}\sum_{i,k=1}^{n_A}\sum_{j,l=1}^{n_B}\Gamma_{ij}^{kl}(B,A)\left(\sqrt{M_l^B}g_k^A + \sqrt{M_k^A}g_l^B - \sqrt{M_j^B}g_i^A - \sqrt{M_i^A}g_j^B\right)$$
$$\times\left(\sqrt{M_l^B}h_k^A + \sqrt{M_k^A}h_l^B - \sqrt{M_j^B}h_i^A - \sqrt{M_i^A}h_j^B\right)$$
$$+\frac{1}{4}\sum_{i,j,k,l=1}^{n_B}\Gamma_{ij}^{kl}(B,B)\left(\sqrt{M_l^B}g_k^B + \sqrt{M_k^B}g_l^B - \sqrt{M_j^B}g_i^B - \sqrt{M_i^B}g_j^B\right)$$
$$\times\left(\sqrt{M_l^B}h_k^B + \sqrt{M_k^B}h_l^B - \sqrt{M_j^B}h_i^B - \sqrt{M_i^B}h_j^B\right).$$

Hence, the matrix L is symmetric, i.e.

$$\langle g, Lh \rangle = \langle Lg, h \rangle$$

for all g and h, and semi-positive, i.e.

$$\langle h, Lh \rangle \geq 0$$

for all h. Also $\langle h, Lh \rangle = 0$ if and only if

$$\sqrt{M_k^\alpha} h_l^\beta + \sqrt{M_l^\beta} h_k^\alpha = \sqrt{M_i^\alpha} h_j^\beta + \sqrt{M_j^\beta} h_i^\alpha \tag{11}$$

for all indices $1 \leq i, k \leq n_\alpha$, $1 \leq j, l \leq n_\beta$, and $\alpha, \beta \in \{A, B\}$, satisfying $\Gamma_{ij}^{kl}(\beta, \alpha) \neq 0$. We let $h = \sqrt{M}\phi$ in Eq.(11), and obtain Eq.(4), by the relations $M_i^\alpha M_j^\beta = M_k^\alpha M_l^\beta \neq 0$. Hence,

$$Lh = 0 \text{ if and only if } h = \sqrt{M}\phi,$$

where ϕ is a collision invariant. In consequence,

$$\left\langle S\left(h, h\right), \sqrt{M}\phi \right\rangle = \left\langle Q\left(f, f\right), \phi \right\rangle + \left\langle h, L\sqrt{M}\phi \right\rangle = 0$$

for all collision invariants ϕ.

In the planar stationary case our system for mixtures reads

$$D\frac{dh}{dx} + Lh = S(h, h),\ x \in \mathbb{R},$$

where $D = \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}$, with $D_\alpha = \operatorname{diag}(\xi_1^{\alpha,1}, ..., \xi_{n_\alpha}^{\alpha,1})$, and the operators L and S are given by Eqs.(9)-(10).

We consider below the case when D is non-singular, i.e. when all $\xi_i^{\alpha,1} \neq 0$ are non-zero. For the case of singular matrices D, see Remark 5 below.

We denote by $n^\pm$, where $n^+ + n^- = n$, and $m^\pm$, with $m^+ + m^- = q$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices D and $D^{-1}L$ respectively, and by m^0 the number of zero eigenvalues of $D^{-1}L$. Moreover, we denote by k^+, k^-, and l, with $k^+ + k^- = k$, where $k + l = p$, the numbers of positive, negative, and zero eigenvalues of the $p \times p$ matrix K ($p = d + 3$ for normal DVMs for binary mixtures), with entries $k_{ij} = \langle y_i, y_j \rangle_D = \langle y_i, Dy_j \rangle$, such that $\{y_1, ..., y_p\}$ is a basis of the null-space of L, $N(L)$. In our case, $\operatorname{span}(y_1, ..., y_p) = N(L) = \operatorname{span}\left(R_A M^{1/2}, R_B M^{1/2}, M^{1/2}\xi^1, ..., M^{1/2}\xi^d, M^{1/2}\left|\xi\right|^2\right)$, where $R_A h = (h_1, ..., h_{n_A}, 0, ..., 0)$ and $R_B h = (1 - R_A)h$ for $h \in \mathbb{R}^n$. Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$ and denote $\langle \cdot, \cdot \rangle_D = \langle \cdot, D\cdot \rangle$.

In applications, the number p of collision invariants is usually relatively small compared to n (note that formally $n = \infty$ for the continuous Boltzmann equation whenas $p \leq 6$). Also, the matrix D is diagonal and therefore all its eigenvalues are known. This explains the importance of the following result by Bobylev and Bernhoff [10] (see also [4]).

Theorem 2.1. *The numbers of positive, negative and zero eigenvalues of $D^{-1}L$ are given by*

$$\begin{cases} m^+ = n^+ - k^+ - l \\ m^- = n^- - k^- - l \\ m^0 = p + l. \end{cases}$$

In the proof of Theorem 2.1 a basis

$$u_1, ..., u_q, y_1, ..., y_k, z_1, ..., z_l, w_1, ..., w_l$$

of $\mathbb{R}^n$, such that

$$y_i, z_r \in N(L),\ D^{-1}Lw_r = z_r \text{ and } D^{-1}Lu_\tau = \lambda_\tau u_\tau,$$

and

$$\begin{gathered}\langle u_\tau, u_\nu \rangle_D = \lambda_\tau \delta_{\tau\nu}, \text{ with } \lambda_1, ..., \lambda_{m^+} > 0 \text{ and } \lambda_{m^++1}, ..., \lambda_q < 0,\\ \langle y_i, y_j \rangle_D = \gamma_i \delta_{ij}, \text{ with } \gamma_1, ..., \gamma_{k^+} > 0 \text{ and } \gamma_{k^++1}, ..., \gamma_k < 0,\\ \langle u_\tau, z_r \rangle_D = \langle u_\tau, w_r \rangle_D = \langle u_\tau, y_i \rangle_D = \langle w_r, y_i \rangle_D = \langle z_r, y_i \rangle_D = 0,\\ \langle w_r, w_s \rangle_D = \langle z_r, z_s \rangle_D = 0 \text{ and } \langle w_r, z_s \rangle_D = \delta_{rs},\end{gathered}$$

is constructed.

3. **Applications to boundary layers.** The main results for half-space problems for single species in [4] and [5] can now be applied in the case of binary mixtures. For the sake of completeness we present the results here. All proofs are similar to the ones for single species found in [4] and [5].

We consider the inhomogeneous (or homogeneous if $g = 0$) linearized problem

$$D\frac{df}{dx} + Lf = g, \tag{12}$$

where $g = g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$, with one of the boundary conditions

(O) the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty;$$

(P) the solution is bounded, i.e.

$$|f(x)| < \infty \text{ for all } x \in \mathbb{R}_+;$$

(Q) the solution can be slowly increasing, i.e.

$$|f(x)|\ e^{-\epsilon x} \to 0 \text{ as } x \to \infty, \text{ for all } \epsilon > 0;$$

at infinity.

In the case of boundary condition (O) at infinity we additionally assume that

$$g(x) \in N(L)^\perp \text{ for all } x \in \mathbb{R}_+. \tag{13}$$

Remark 1. The boundary condition (O) corresponds to the case when we have made the expansion (8) around a Maxwellian M, such that $F \to M$ as $x \to \infty$. The boundary conditions (P) and (Q) are the boundary conditions in the Milne and Kramers problem respectively.

We can (without loss of generality) assume that

$$D_\alpha = \begin{pmatrix} D_\alpha^+ & 0 \\ 0 & -D_\alpha^- \end{pmatrix}, \tag{14}$$

where

$$\begin{gathered}D_\alpha^+ = \operatorname{diag}\left(\xi_1^{\alpha,1}, ..., \xi_{n_\alpha^+}^{\alpha,1}\right) \text{ and } D_\alpha^- = -\operatorname{diag}\left(\xi_{n_\alpha^++1}^{\alpha,1}, ..., \xi_{n_\alpha}^{\alpha,1}\right), \text{ with}\\ \xi_1^{\alpha,1}, ..., \xi_{n_\alpha^+}^{\alpha,1} > 0 \text{ and } \xi_{n_\alpha^++1}^{\alpha,1}, ..., \xi_{n_\alpha}^{\alpha,1} < 0.\end{gathered} \tag{15}$$

We also define the projections $R_+ : \mathbb{R}^n \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^n \to \mathbb{R}^{n^-}$, $n^- = n - n^+$, by

$$\begin{aligned} R_+ s &= s^+ = \left(s_1, ..., s_{n_A^+}, s_{n_A+1}, ..., s_{n_A+n_B^+}\right) \text{ and} \\ R_- s &= s^- = \left(s_{n_A^+ +1}, ..., s_{n_A}, s_{n_A+n_B^+ +1}, ..., s_n\right) \end{aligned}$$

for $s = (s_1, ..., s_n)$.

At $x = 0$ we assume the general boundary condition

$$f^+(0) = Cf^-(0) + h_0, \tag{16}$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$. In applications,

$$C = \begin{pmatrix} C_A & 0 \\ 0 & C_B \end{pmatrix},$$

where C_α are given $n_\alpha^+ \times n_\alpha^-$ matrices.

We introduce the operator $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^{n^+}$, given by

$$\mathcal{C} = R_+ - CR_-.$$

In order to be able to obtain existence and uniqueness of solutions of the linearized half-space problems we will assume that the matrix C fulfills the condition

$$\dim \mathcal{C}U_+ = m^+, \text{ with } U_+ = \operatorname{span}\left(u_1, ..., u_{m^+}\right), \tag{17}$$

as we consider boundary condition (O) at infinity, the condition

$$\dim \mathcal{C}X_+ = n^+, \text{ with } X_+ = \operatorname{span}\left(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1, ..., z_l\right), \tag{18}$$

as we consider boundary condition (P) at infinity, and the condition (18) or the condition

$$\dim \mathcal{C}\widetilde{X}_+ = n^+, \text{ with } \widetilde{X}_+ = \operatorname{span}\left(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1 + w_1, ..., z_l + w_l\right), \tag{19}$$

as we consider boundary condition (Q) at infinity.

Theorem 3.1. *(i) Assume that the conditions (13) and (17) are fulfilled and that*

$$h_0, \mathcal{C}e^{xD^{-1}L}D^{-1}f(x) \in \mathcal{C}U_+ \text{ for all } x \in \mathbb{R}_+. \tag{20}$$

Then the system (12) with the boundary conditions (O) and (16) has a unique solution.

(ii) Assume that the condition (18) is fulfilled. Then the system (12) with the boundary conditions (P) and (16) has a unique solution with the asymptotic flow

$$f_{as} = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \eta_j z_j,$$

if the k^- parameters $\mu_{k^+ +1}, ..., \mu_k$ are prescribed.

(iii) Assume that the condition (18) is fulfilled. Then the system (12) with the boundary conditions (Q) and (16) has a unique solution with the asymptotic flow

$$f_{as}(x) = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left(\left(\eta_j - x\alpha_j\right) z_j + \alpha_j w_j\right), \tag{21}$$

if the $k^- + l$ parameters $\mu_{k^+ +1}, ..., \mu_k$ and $\alpha_1, ..., \alpha_l$ are prescribed.

(iv) Assume that the condition (19) is fulfilled. Then the system (12) with the boundary conditions (Q) and (16) has a unique solution with the asymptotic flow (21) if the k^-+l parameters $\mu_{k^++1},...,\mu_k$ and $\vartheta_1,...,\vartheta_l$, $\vartheta_i=\eta_i+\alpha_i$, are prescribed.

Especially, for the homogeneous system (12) with $g=0$, the condition (20) is reduced to

$$h_0 \in \mathcal{C}U_+.$$

Lemma 3.2. *Let* $D^+ = \begin{pmatrix} D_A^+ & 0 \\ 0 & D_B^+ \end{pmatrix}$ *and* $D^- = \begin{pmatrix} D_A^- & 0 \\ 0 & D_B^- \end{pmatrix}$, *cf. Eq.(15). Then*

i) the condition (18) is fulfilled, if

$$C^T D^+ C < D^- \text{ on } R_- X_+;$$

ii) the conditions (17) and (19) are fulfilled, if

$$C^T D^+ C \leq D^- \text{ on } R_- U_+ \text{ and } R_- \widetilde{X}_+, \text{ respectively.}$$

Corollary 1. *If $C=0$, then the conditions (17)-(19) are fulfilled.*

In particular, $\left\{u_1^+,...,u_{m^+}^+,y_1^+,...,y_{k^+}^+,z_1^+,...,z_l^+\right\}$ is a basis of $\mathbb{R}^{n^+}$.

We consider the non-linear system

$$D\frac{df}{dx} + Lf = S(f,f), \tag{22}$$

where the solution tends to zero at infinity. Furthermore, we fix a number σ, such that

$$0 < \sigma \leq \min\left\{|\lambda| \neq 0;\ \det(\lambda D - L) = 0\right\}$$

and introduce the norm

$$|h|_\sigma = \sup_{x\geq 0} e^{\sigma x} \left| h\left(x\right)\right|,$$

on $\mathcal{X} = \left\{h \in \mathcal{B}^0[0,\infty) \mid |h|_\sigma < \infty\right\}$.

We have the following existence result.

Theorem 3.3. *Let condition (18) be fulfilled and suppose that $\left\langle S\left(f(x),f(x)\right),w_j\right\rangle = 0$ for $j=1,...,l$, and that $\left\langle h_0,h_0\right\rangle_{D^+}$ is sufficiently small. Then with k^++l conditions on h_0, the system (22) with the boundary conditions (O),(16), has a locally unique solution (with respect to the norm $|\cdot|_\sigma$).*

Remark 2. It was recently proved that one can get rid of the restrictive assumptions $\left\langle S\left(f(x),f(x)\right),w_j\right\rangle = 0$ for $j=1,...,l$, on the quadratic part in the degenerate cases, in Theorem 3.3, for one-component as well as two-component gases [8], by a slight modification of the proof of Theorem 3.3, if one instead of condition (18) assume that

$$\dim \mathcal{C}\widehat{X}_+ = n^+, \text{ with } \widehat{X}_+ = \operatorname{span}\left(u_1,...,u_{m^+},y_1,...,y_{k^+},w_1,...,w_l\right). \tag{23}$$

Furthermore, the condition (23) is fulfilled, if

$$C^T D^+ C < D^- \text{ on } R_- \widehat{X}_+,$$

and, especially, $\left\{u_1^+,...,u_{m^+}^+,y_1^+,...,y_{k^+}^+,w_1^+,...,w_l^+\right\}$ is a basis of $\mathbb{R}^{n^+}$ [8].

In the following theorem we present explicit conditions on h_0, but then with restrictive conditions on the quadratic part.

Theorem 3.4. *Let condition (17) be fulfilled and assume that*

$$h_0, \mathcal{C}e^{xB^{-1}L}D^{-1}S(f(x),f(x)) \in \mathcal{C}U_+ \text{ for all } x \in \mathbb{R}_+,$$
$$\text{with } U_+ = \operatorname{span}(u : Lu = \lambda Du,\ \lambda > 0) = \operatorname{span}(u_1, ..., u_{m^+}).$$

Then there is a positive number δ_0, such that if

$$|h_0| \leq \delta_0,$$

then the system (22) with the boundary conditions (O),(16), has a locally unique solution (with respect to the norm $|\cdot|_\sigma$).

Remark 3. If condition (17) is fulfilled, then the condition

$$h_0 \in \mathcal{C}U_+$$

implies that we have $k^+ + l$ conditions on h_0.

Remark 4. If the conditions

$$\mathcal{C}U_- \subseteq \mathcal{C}U_+, \tag{24}$$

$$\begin{aligned} \text{with } U_- &= \operatorname{span}(\{u|\ Lu = \lambda Du, \text{ with } \lambda < 0\} \cup \{z_1, ..., z_l\}) \\ &= \operatorname{span}(u_{m^+ +1}, ..., u_q, z_1, ..., z_l), \end{aligned}$$

and (13) are fulfilled, then

$$\mathcal{C}e^{xD^{-1}L}D^{-1}S(f,f) \in \mathcal{C}U_+ \text{ for all } x \in \mathbb{R}_+.$$

Remark 5. All our results for half-space problems can be extended in a natural way, to yield also for singular matrices D, if

$$N(L) \cap N(D) = \{0\}.$$

4. **Applications to shock profiles.** We are interested in solutions to the problem

$$(D - cI)\frac{dF_i}{dy} = Q_i(F,F),\ i = 1, ..., n,\ c \in \mathbb{R}, \tag{25}$$

such that

$$F \to M_\pm \text{ as } y \to \pm\infty,$$

where $M_\pm$ are two bi-Maxwellians and $D = \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}$, with D_α from Eqs.(14)-(15). Here $F = (F_1, ..., F_n)$, with $F_i = F_i(y) = F(y, \xi_i)$, $i = 1, ..., n$.

Note that shifting the velocity variable in the continuous Boltzmann equation doesn't change the velocity set, while for a finite set of velocities a shift in the velocity variable changes the set of velocities. However, if we want to end up with a specific set of velocities after a given shift in the velocity variable, we can always start with a suitably shifted set of velocities. Note also that changing c in the discrete case can change the number of positive (and negative) eigenvalues of the matrix $D - cI$, and thereby the number of positive (and negative) eigenvalues of the matrix $(D - cI)^{-1}L$ can change also away from the degenerate values of c.

We denote by $\{\phi_1, ..., \phi_p\}$ ($p = d + 3$ for normal DVMs for binary mixtures) a basis for the vector space of collision invariants. If we multiply Eq.(25) scalarly by ϕ_i, $1 \leq i \leq p$, and integrate over $\mathbb{R}$, then we obtain that the bi-Maxwellians M_- and M_+ must fulfill the Rankine-Hugoniot conditions

$$\langle M_+, \phi_i \rangle_{D-cI} = \langle M_-, \phi_i \rangle_{D-cI},\ i = 1, ..., p.$$

We make the following assumptions on our DVMs.

1. There is a number c_0 ("speed of sound"), with the following properties:
   [i] $\operatorname{rank}(K) = p - 1$, where K is the $p \times p$ matrix with the elements
$$k_{ij} = \langle M_+\phi_i, \phi_j \rangle_{D-c_0 I} .$$
   The rank of K is independent of the choice of the basis $\{\phi_1, ..., \phi_p\}$. In other words, there is a unique (up to its sign) vector $\phi_\perp$ in $\operatorname{span}(\phi_1, ..., \phi_p)$, such that $\langle M_+\phi_\perp, \phi_\perp \rangle = 1$ and
$$\langle M_+\phi_\perp, \phi \rangle_{D-c_0 I} = 0 \text{ for all } \phi \in \operatorname{span}(\phi_1, ..., \phi_p). \tag{26}$$
   [ii] $c_0 \neq \xi_i^{\alpha,1}$ for $i = 1, ..., n_\alpha$, or, equivalently, $\det(D - c_0 I) \neq 0$.
2. The vector(s) $\phi_\perp$ fulfilling Eqs.(26), also satisfy $\left\langle M_+\phi_\perp, \phi_\perp^2 \right\rangle_{D-c_0 I} \neq 0$. We choose the sign of the vector $\phi_\perp$, such that $\left\langle M_+\phi_\perp, \phi_\perp^2 \right\rangle_{D-c_0 I} > 0$.

We assume that assumptions 1 and 2 are fulfilled and denote
$$\|h\| = \|h(y)\| = \sup_{y\in\mathbb{R}} |h(y)|$$
for any bounded (vector or scalar) function $h(y) : \mathbb{R} \to \mathbb{R}^k$, where k is a positive integer.

Theorem 4.1. *For any given positive Maxwellian M_+, there exists a family of Maxwellians $M_- = M_-(\varepsilon)$ and shock speeds $c = c(\varepsilon) = c_0 + \varepsilon$, such that the shock wave problem (25) has a non-negative locally unique (with respect to the norm $\|\cdot\|$ and up to a shift in the independent variable) non-trivial bounded solution for each sufficiently small $\varepsilon > 0$. Furthermore, M_- is determined by M_+ and c.*

Remark 6. We can interchange M_- and M_+ in Theorem 4.1 (with $\varepsilon < 0$).

The proof of Theorem 4.1 is similar to the proof in [9] for the case of one species.

5. **"Speed of sound" for axially symmetric DVMs for mixtures.** We assume that (i) we have two axially symmetric sets of velocities (i.e. if $\left\{\xi_1^\alpha, ..., \xi_{n_\alpha}^\alpha\right\} \in V_\alpha$, then also $\left\{\pm\xi_1^\alpha, ..., \pm\xi_{n_\alpha}^\alpha\right\} \in V_\alpha$ for all possible combinations of sound); (ii) all collision invariants are of the form (6); and (iii) we have made an expansion (cf. Eq.(8)) around a non-drifting bi-Maxwellian M (i.e. with $\mathbf{b} = 0$ in Eqs.(6)-(7)). Let
$$D = \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}, \text{ with } D_\alpha = \operatorname{diag}(\xi_1^{\alpha,1}, ..., \xi_{N_\alpha}^{\alpha,1}, -\xi_1^{\alpha,1}, ..., -\xi_{N_\alpha}^{\alpha,1}),$$
and assume that $c \notin \left\{\pm\xi_1^{A,1}, ..., \pm\xi_{N_A}^{A,1}, \pm\xi_1^{B,1}, ..., \pm\xi_{N_B}^{B,1}\right\}$. The null-space of L is given by
$$N(L) = \operatorname{span}(\phi_0, ..., \phi_{d+2}),$$
where
$$\begin{cases} \phi_0 = M^{1/2} \cdot (\underbrace{1, ..., 1}_{2N_A}, \underbrace{0, ..., 0}_{2N_B}) \\ \phi_1 = M^{1/2} \cdot (\underbrace{0, ..., 0}_{2N_A}, \underbrace{1, ..., 1}_{2N_B}) \\ \phi_2 = M^{1/2} \cdot (\phi_2^A, \phi_2^B), \text{ with } \phi_2^\alpha = (\xi_1^{\alpha,1}, ..., \xi_{N_\alpha}^{\alpha,1}, -\xi_1^{\alpha,1}, ..., -\xi_{N_\alpha}^{\alpha,1}) \\ \phi_3 = M^{1/2} \cdot (\phi_3^A, \phi_3^B), \text{ with } \phi_3^\alpha = (\left|\xi_1^\alpha\right|^2, ..., \left|\xi_{N_\alpha}^\alpha\right|^2, \left|\xi_1^\alpha\right|^2, ..., \left|\xi_{N_\alpha}^\alpha\right|^2) \\ \phi_{2+i} = M^{1/2} \cdot (\phi_{2+i}^A, \phi_{2+i}^B), \text{with } \phi_{2+i}^\alpha = (\xi_1^{\alpha,i}, ..., \xi_{2N_\alpha}^{\alpha,i}), i = 2, ..., d \end{cases}.$$

Then,

$$K=\begin{pmatrix} -c\chi_1^A & 0 & \chi_2^A & -c\chi_3^A & & & \\ 0 & -c\chi_1^B & \chi_2^B & -c\chi_3^B & & & \\ \chi_2^A & \chi_2^B & -c\chi_2 & \chi_4 & & & \\ -c\chi_3^A & -c\chi_3^B & \chi_4 & -c\chi_5 & & & \\ & & & & -c\chi_6 & & \\ & & & & & \ddots & \\ & & & & & & -c\chi_{d+4} \end{pmatrix},$$

where $K=(\langle\phi_{i+1},\phi_{j+1}\rangle_{D-cI})$, $\chi_1^A=\langle\phi_0,\phi_0\rangle$, $\chi_2^A=\langle\phi_0,\phi_2\rangle_D$, $\chi_3^A=\langle\phi_0,\phi_3\rangle$, $\chi_1^B=\langle\phi_1,\phi_1\rangle$, $\chi_2^B=\langle\phi_1,\phi_2\rangle_D$, $\chi_3^B=\langle\phi_1,\phi_3\rangle$, $\chi_2=\chi_2^A+\chi_2^B=\langle\phi_2,\phi_2\rangle$, $\chi_4=\langle\phi_2,\phi_3\rangle_D$ and $\chi_{i+2}=\langle\phi_i,\phi_i\rangle$, $i=3,...,d+2$. Hence,

$$\begin{aligned} &\det(K)\\ &=c^d\chi_6...\chi_{d+4}(c^2(\chi_1^A\chi_1^B\chi_2\chi_5-\chi_1^A\chi_2(\chi_3^B)^2-\chi_1^B\chi_2(\chi_3^A)^2)+(\chi_2^A\chi_3^B-\chi_2^B\chi_3^A)^2\\ &+2\chi_4(\chi_1^A\chi_2^B\chi_3^B+\chi_1^B\chi_2^A\chi_3^A)-(\chi_1^A(\chi_2^B)^2+\chi_1^B(\chi_2^A)^2)\chi_5-\chi_1^A\chi_1^B\chi_4^2), \end{aligned}$$

and the degenerate values of c (the values of c for which $l\geq 1$) are

$$\begin{aligned} &c_0=0 \text{ and } c_\pm=\pm\sqrt{\frac{\mathcal{X}}{\chi_2(\chi_1^A\chi_1^B\chi_5-\chi_1^A(\chi_3^B)^2-\chi_1^B(\chi_3^A)^2)}}, \text{ where}\\ &\mathcal{X}=\chi_1^A\chi_1^B\chi_4^2+(\chi_1^A(\chi_2^B)^2+\chi_1^B(\chi_2^A)^2)\chi_5-2\chi_4(\chi_1^A\chi_2^B\chi_3^B+\chi_1^B\chi_2^A\chi_3^A)\\ &\quad-(\chi_2^A\chi_3^B-\chi_2^B\chi_3^A)^2. \end{aligned}$$

Here $c_+=\sqrt{\frac{\mathcal{X}}{\chi_2(\chi_1^A\chi_1^B\chi_5-\chi_1^A(\chi_3^B)^2-\chi_1^B(\chi_3^A)^2)}}$ corresponds to the speed of sound in the continuous case.

6. **Exact shock profiles for a plane 6+4-velocity model.** We now consider the shock wave problem for a mixture, in which gas A is described by a 6-velocity model with velocities $(\pm1,0)$ and $(\pm1,\pm2m)$, and the gas B is described by the classical Broadwell model [17] in plane with velocities $(\pm m,\pm m)$.

Here $m=\dfrac{m_A}{m_B}$, where we assume that $m>1$. The case $m<1$ can be studied in a similar way. If $m=1$, then for this simplified model, the degenerate values would be $c_0=0$ and $c_\pm=\pm1$. Especially, "the speed of sound" is 1, contradicting assumption 1[ii] in Section 4. However, in general we don't have to exclude the case $m=1$.

Note that for the Broadwell model we have only two linearly independent collision invariants, as the mass vector and the energy vector are linearly dependent, even if mass, momentum, and energy all are preserved. However, the DVMs for gas A and the mixture will have the correct number of linearly independent collision invariants.

For a flow symmetric around the x^1-axis we obtain the reduced system

$$\begin{cases} (1-c)\dfrac{df_1}{dy} = \sigma_1 q_1 + \sigma_2 q_2 \\ -(1+c)\dfrac{df_2}{dy} = -\sigma_1 q_1 - \sigma_2 q_2 \\ (1-c)\dfrac{df_3}{dy} = -\sigma_1 q_1 + \sigma_3 q_3 \\ -(1+c)\dfrac{df_4}{dy} = \sigma_1 q_1 - \sigma_3 q_3 \\ (m-c)\dfrac{df_5}{dy} = -\sigma_2 q_2 - \sigma_3 q_3 \\ -(m+c)\dfrac{df_6}{dy} = \sigma_2 q_2 + \sigma_3 q_3 \end{cases},$$

where $q_1 = f_2 f_3 - f_1 f_4$, $q_2 = f_2 f_5 - f_1 f_6$, and $q_3 = f_4 f_5 - f_3 f_6$, or equivalently

$$(D - cI)\frac{df}{dy} = Q(f,f),$$

where $D = \mathrm{diag}(1,-1,1,-1,m,-m)$, $f = (f_1, f_2, f_3, f_4, f_5, f_6)$, and $Q(f,f) = \sigma_1 q_1 (1,-1,-1,1,0,0) + \sigma_2 q_2 (1,-1,0,0,-1,1) + \sigma_3 q_3 (0,0,1,-1,-1,1)$.

We assume that $D - cI$ is non-singular, i.e. that $c \notin \{\pm 1, \pm m\}$, and make the natural assumption $\sigma_2 = \sigma_3$.

The set of collision invariants are generated by the collision invariants

$$\begin{cases} \phi_0 = (1,1,1,1,0,0) \\ \phi_1 = (0,0,0,0,1,1) \\ \phi_2 = (1,-1,1,-1,1,-1) \\ \phi_3 = (1,1,1+4m^2,1+4m^2,2m,2m) \end{cases}.$$

The Maxwellians are of the form

$$M = s(1,a,b,ab,d,ad), \text{ with } a,b,d,s > 0.$$

The density, momentum density, and energy density per unit volume are obtained by

$$\begin{cases} \rho = \rho^A + \rho^B, \text{ with } \rho^A = m_A(f_1 + f_2 + 2f_3 + 2f_4) \text{ and } \rho^B = 2m_B(f_5 + f_6) \\ \rho u = m_A(f_1 - f_2 + 2f_3 - 2f_4 + 2f_5 - 2f_6) \\ \rho e = m_A(f_1 + f_2 + 2(1+4m^2)(f_3 + f_4) + 4m(f_5 + f_6)) \end{cases}.$$

We consider

$$(D - cI)\frac{df}{dy} = Q(f,f), \text{ where } f \to M_+ \text{ as } y \to \infty,$$

and denote

$$F = M_+ + M_+^{1/2} h, \text{ with } M_+ = s_+(1,1,b_+,b_+,d_+,d_+) = s(1,1,b,b,d,d), \tag{27}$$

where we have assumed that M_+ is a non-drifting Maxwellian. We obtain

$$(D - cI)\frac{dh}{dy} + Lh = S(h,h), \text{ where } h \to 0 \text{ as } y \to \infty, \tag{28}$$

with

$$L=s\begin{pmatrix} l_1 & -l_1 & -\sigma_1\sqrt{b} & \sigma_1\sqrt{b} & -\sigma_2\sqrt{d} & \sigma_2\sqrt{d} \\ -l_1 & l_1 & \sigma_1\sqrt{b} & -\sigma_1\sqrt{b} & \sigma_2\sqrt{d} & -\sigma_2\sqrt{d} \\ -\sigma_1\sqrt{b} & \sigma_1\sqrt{b} & l_2 & -l_2 & -\sigma_2\sqrt{bd} & \sigma_2\sqrt{bd} \\ \sigma_1\sqrt{b} & -\sigma_1\sqrt{b} & -l_2 & l_2 & \sigma_2\sqrt{bd} & -\sigma_2\sqrt{bd} \\ -\sigma_2\sqrt{d} & \sigma_2\sqrt{d} & -\sigma_2\sqrt{bd} & \sigma_2\sqrt{bd} & \sigma_2\left(1+b\right) & -\sigma_2\left(1+b\right) \\ \sigma_2\sqrt{d} & -\sigma_2\sqrt{d} & \sigma_2\sqrt{bd} & -\sigma_2\sqrt{bd} & -\sigma_2\left(1+b\right) & \sigma_2\left(1+b\right) \end{pmatrix},$$

where $l_1=\sigma_1 b+\sigma_2 d$ and $l_2=\sigma_1+\sigma_2 d$, and

$$\begin{aligned} S(h,h) &= \sqrt{s}(\sigma_1 q_1(\sqrt{b},-\sqrt{b},-1,1,0,0)+\sigma_2 q_2(\sqrt{d},-\sqrt{d},0,0,-1,1) \\ &\quad +\sigma_2 q_3(0,0,\sqrt{d},-\sqrt{d},-\sqrt{b},\sqrt{b})). \end{aligned}$$

The linearized operator L is symmetric and semi-positive and has the null-space

$$\begin{aligned} & N(L)=\operatorname{span}(y_1,y_2,y_3,\widetilde{y}_4)\text{, with} \\ y_1 &= (1,1,0,0,0,0), y_2=(0,0,1,1,0,0), \\ y_3 &= (0,0,0,0,1,1),\text{ and } \widetilde{y}_4=(1,-1,\sqrt{b},-\sqrt{b},\sqrt{d},-\sqrt{d}) \end{aligned}$$

where we for $c\neq 0$ can replace $\widetilde{y}_4$ with

$$y_4=\Big(1+c,1-c,\sqrt{b}\left(1+c\right),\sqrt{b}\left(1-c\right),\sqrt{d}\left(m+c\right),\sqrt{d}\left(m-c\right)\Big).$$

Then we obtain

$$K=\begin{pmatrix} -2c & 0 & 0 & 0 \\ 0 & -2c & 0 & 0 \\ 0 & 0 & -2c & 0 \\ 0 & 0 & 0 & 2c\left(\left(1-c^2\right)\left(1+b\right)+d\left(m^2-c^2\right)\right) \end{pmatrix},$$

where $K=\Big(\left\langle y_i,y_j\right\rangle_{D-cI}\Big)_{i,j=1}^{4}$.

We remind that if $n^{\pm}$, with $n^{+}+n^{-}=n$, and $m^{\pm}$, denote the numbers of the positive and negative eigenvalues of the matrices $D-cI$ and $\left(D-cI\right)^{-1}L$ respectively, and k^{+}, k^{-}, and l denote the numbers of positive, negative, and zero eigenvalues of the 4×4 matrix K, then $m^{\pm}=n^{\pm}-k^{\pm}-l$. We obtain the following table

c		$-m$		c_-		-1		0		1		c_+		m	
n^+	6		5	5	5		3		3		1	1	1		0
n^-	0		1	1	1		3		3		5	5	5		6
k^+	4		4	3	3		3		1		1	0	0		0
k^-	0		0	0	1		1		3		3	3	4		4
l	0		0	1	0		0		0		0	1	0		0
m^+	2		1	1	2		0		2		0	0	1		0
m^-	0		1	0	0		2		0		2	1	1		2

, (29)

where $c_{\pm}=\pm\sqrt{\dfrac{1+b+dm^2}{1+b+d}}$.

Explicitly, the non-zero eigenvalues of the matrix $\left(D-cI\right)^{-1}L$ are

$$\left\{\begin{array}{l} \lambda_1=\dfrac{2sc}{1-c^2}(\sigma_1(1+b)+\sigma_2 d) \\ \lambda_2=2sc\sigma_2\left(\dfrac{1+b}{m^2-c^2}+\dfrac{d}{1-c^2}\right) \end{array}\right.,$$

with the corresponding eigenvectors

$$\left\{\begin{array}{l} u_1 = (\frac{\sqrt{b}}{1-c}, \frac{\sqrt{b}}{1+c}, -\frac{1}{1-c}, -\frac{1}{1+c}, 0, 0) \\ u_2 = (\frac{\sqrt{d}}{1-c}, \frac{\sqrt{d}}{1+c}, \frac{\sqrt{bd}}{1-c}, \frac{\sqrt{bd}}{1+c}, -\frac{1+b}{m-c}, -\frac{1+b}{m+c}) \end{array}\right. .$$

Plugging $h = \nu u_1 + \eta u_2$ in Eq.(28) and multiplying scalarly by $(D - cI)\, u_i$, we obtain the two equations

$$\left\{\begin{array}{l} \frac{d\nu}{dy} + \lambda_1 \nu = k\eta\nu \\ \frac{d\eta}{dy} + \lambda_2 \eta = k\eta^2 \end{array}\right. ,$$

with $k = \dfrac{2\sigma_2 \sqrt{sd} c\,(1+b)\,(m-1)}{(1-c^2)\,(m^2-c^2)}$. The solutions are

$$\left\{\begin{array}{l} \eta = \dfrac{\lambda_2}{k + C_1 e^{\lambda_2 y}} \\ \nu = \dfrac{C_2 e^{-\lambda_1 y}}{k e^{-\lambda_2 y} + C_1} = \dfrac{C_2 e^{(\lambda_2 - \lambda_1) y}}{k + C_1 e^{\lambda_2 y}} \end{array}\right. \text{ or } \left\{\begin{array}{l} \eta = 0 \\ \nu = C_2 e^{-\lambda_1 y} \end{array}\right. .$$

The parameter $C_1 \neq 0$ reflects the invariance of our equation under shifts in the invariant variable y. The sign of C_1 is, however, defined uniquely. It must be the same as the sign of k.

Let $c_+ < c < m$ or $c_- < c < -\dfrac{1+b+dm}{1+b+d}$. If $c_+ < c < m$, then $\lambda_2 > 0$ and $\lambda_1 < 0$, and hence, $\lim\limits_{y\to\infty} \nu = 0$ implies that $C_2 = 0$. If $c_- < c < -\dfrac{1+b+dm}{1+b+d}$, then $\lambda_1 > \lambda_2 > 0$, and hence, $\lim\limits_{y\to-\infty} \nu < \infty$ implies that $C_2 = 0$. Therefore, if $\eta \neq 0$,

$$h(y) = \frac{\lambda_2}{k + C_1 e^{\lambda_2 y}} u_2.$$

Moreover,

$$f(y) = M_+ + \frac{\lambda_2}{k + C_1 e^{\lambda_2 y}} M_+^{1/2} u_2$$

and the other Maxwellian is

$$M_- = M_+ + \frac{\lambda_2}{k} M_+^{1/2} u_2 = \frac{s}{m-1} (a_1, a_2, a_1 b_-, a_2 b_-, a_1 d_-, a_2 d_-),$$

where $a_1 = m + c + \dfrac{d\left(m^2-c^2\right)}{(1+b)\,(1-c)}$, $a_2 = m - c + \dfrac{d\left(m^2-c^2\right)}{(1+b)\,(1+c)}$, $b_- = b$, and $d_- = \dfrac{\left(c^2-1\right)(1+b)}{m^2-c^2}$.

If $c_- < c < -\dfrac{1+b+dm}{1+b+d}$, then $\lambda_1 > \lambda_2 > 0$, and hence, $\lim\limits_{y\to-\infty} \nu < \infty$ implies that $C_2 = 0$.

Remark 7. We can instead of Eq.(6) consider

$$(D - cI)\, \frac{df}{dy} = Q\,(f, f)\,, \text{ where } f \to M_- \text{ as } y \to -\infty,$$

with $\dfrac{1+b+dm^2}{1+b+d} < c < c_+$ or $-m < c < c_-$, and in a similar way as above, we obtain

$$f(y) = M_- + \frac{\lambda_2}{k + C_1 e^{\lambda_2 y}} M_-^{1/2} u_2$$

and

$$M_+ = M_- + \frac{\lambda_2}{k} M_-^{1/2} u_2.$$

7. **Boundary layers for the 6+4-velocity model with a moving wall.** Let c be a real number such that $c \notin \{-m, -1, 0, 1, m\}$. We assume that $m > 1$. The cases $m < 1$ and $m = 1$ can be studied in a similar way. We define the projections $R_+ : \mathbb{R}^6 \to \mathbb{R}^{n^+}$, $n^+ = n_A^+ + n_B^+$, and $R_- : \mathbb{R}^6 \to \mathbb{R}^{n^-}$, $n^- = 6 - n^+$, by

$$\begin{aligned}
R_+ s &= s^+ \text{ and } R_- s = s^-, \text{ where} \\
s^+ &= \begin{cases} - \\ s_5 \text{ if } 1 < c < m \\ (s_1, s_3, s_5) \text{ if } -1 < c < 1 \\ (s_1, s_2, s_3, s_4, s_5) \text{ if } -m < c < -1 \\ s \text{ if } c < -m \end{cases} \quad \text{and} \\
s^- &= \begin{cases} s \text{ if } c > m \\ (s_1, s_2, s_3, s_4, s_6) \text{ if } 1 < c < m \\ (s_2, s_4, s_6) \text{ if } -1 < c < 1 \\ s_6 \text{ if } -m < c < -1 \\ - \end{cases}
\end{aligned}$$

for $s = (s_1, s_2, s_3, s_4, s_5, s_6)$.

We now consider the problem

$$\begin{cases} \dfrac{\partial f}{\partial t} + D \dfrac{\partial f}{\partial x} = Q(f,f), \; x > ct, \; t > 0, \\ f^+(ct, t) = \widetilde{C} f^-(ct, t) + \widetilde{h}_0 \\ f(x, 0) = f_0(x) \\ f_0(x) \to M \text{ as } x \to \infty \end{cases},$$

where $D = (1, -1, 1, -1, m, -m)$, $M = s(1, 1, b, b, d, d)$, $\widetilde{C}$ is a given $n^+ \times n^-$ matrix, and $\widetilde{h}_0 \in \mathbb{R}^{n^+}$.

After the change of variables $y = x - ct$ and the transformation (27) we obtain

$$\begin{cases} \dfrac{\partial f}{\partial t} + (D - cI) \dfrac{\partial f}{\partial y} + Lf = S(f,f), \; y, t > 0, \\ f^+(0, t) = C f^-(0, t) + h_0 \\ f(y, 0) = f_0(y) \\ f_0(y) \to 0 \text{ as } y \to \infty \end{cases},$$

where $D = D_0 - cI$, C is an $n^+ \times n^-$ matrix, and $h_0 \in \mathbb{R}^{n^+}$.

We assume that $\dfrac{\partial f(y,t)}{\partial t} = 0$ and consider the non-linear system

$$\begin{cases} (D - cI) \dfrac{df}{dy} + Lf = S(f,f), \; f = f(y), \; y, t > 0, \\ f^+(0) = C f^-(0) + h_0 \\ f(y) \to 0 \text{ as } y \to \infty \end{cases}.$$

If $h_0 = 0$, then we always have the trivial solution $f = 0$, and if $-1 < c < 0$, $1 < c \leq c_+$, or $m < c$, then we have no other solutions, since we have non-positive eigenvalues, $\lambda_1 < 0$ and $\lambda_2 \leq 0$. If $-1 < c < 0$, then $h_0 \in \mathbb{R}^3$, and if $1 < c \leq c_+$, then $h_0 \in \mathbb{R}$. Hence, we have 3, 1, or 0 conditions on h_0, if $-1 < c < 0$, $1 < c \leq c_+$, or $m < c$, respectively.

If $c_+ < c < m$, then $C = (0, 0, 0, 0, c_B)$, $h_0 \in \mathbb{R}$, and $f_5(0) = c_B f_6(0) + h_0$. Furthermore, $\lambda_1 < 0$ and $\lambda_2 > 0$. Therefore, $\nu = 0$. Hence, if $h_0 \neq 0$ and $c_B \neq \frac{m+c}{m-c}$, then we obtain the unique solution

$$f(y) = \frac{\lambda_2}{k + C_1 e^{\lambda_2 y}} \left(\frac{\sqrt{d}}{1-c}, \frac{\sqrt{d}}{1+c}, \frac{\sqrt{bd}}{1-c}, \frac{\sqrt{bd}}{1+c}, -\frac{1+b}{m-c}, -\frac{1+b}{m+c}\right),$$

with

$$C_1 = \frac{(1+b)\left(m(c_B - 1) - c(1 + c_B)\right)}{h_0 \left(m^2 - c^2\right)} \lambda_2 - k.$$

If $0 < c < 1$, then $C = \begin{pmatrix} c_1 & c_2 & 0 \\ c_3 & c_4 & 0 \\ 0 & 0 & c_B \end{pmatrix}$ and $h_0 \in \mathbb{R}^3$. Furthermore, $\lambda_1 > 0$ and $\lambda_2 > 0$. Assume that

$$c_B \neq \frac{m+c}{m-c} \text{ and } \dim(\operatorname{span}(v_1, v_2)) = 2, \text{ where}$$
$$\begin{cases} v_1 = \left(\frac{1+c}{1-c}\sqrt{b} - c_1\sqrt{b} + c_2, -\frac{1+c}{1-c} - c_3\sqrt{b} + c_4\right) \\ v_2 = \left(\frac{1+c}{1-c} - c_1 - c_2\sqrt{b}, \frac{1+c}{1-c}\sqrt{b} - c_3 - c_4\sqrt{b}\right) \end{cases}.$$

Then we have a unique solution if and only if

$$h_0 \in \operatorname{span}\left((v_1, 0), \left(v_2, (1+c)(1+b)\frac{m(c_B - 1) - c(1 + c_B)}{\sqrt{d}\left(m^2 - c^2\right)}\right)\right),$$

which implies 1 condition on h_0.

If $-m < c < -1$, then $C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c_B \end{pmatrix}$, $h_0 \in \mathbb{R}^5$, and we must have

$$h_0 = (f_1(0), f_2(0), f_3(0), f_4(0), f_5(0) - c_B f_6(0)).$$

Assume that $c_B \neq \frac{m+c}{m-c}$. If $c_- < c < -1$, then $\lambda_1 > 0$ and $\lambda_2 > 0$, which gives us 3 conditions on h_0. On the other hand, if $-m < c \leq c_-$, then $\lambda_1 > 0$ and $\lambda_2 \leq 0$, and hence we must have

$$h_0 = C_2\left(\frac{\sqrt{b}}{1-c}, \frac{\sqrt{b}}{1+c}, -\frac{1}{1-c}, -\frac{1}{1+c}, 0\right),$$

which implies 4 conditions on h_0. Then, with $h_0 = (h_{01}, h_{02}, h_{03}, h_{04}, 0)$, the unique solution is

$$f(y) = h_{03} e^{-\lambda_1 y}\left(-\sqrt{b}, -\frac{1-c}{1+c}\sqrt{b}, 1, \frac{1-c}{1+c}, 0, 0\right).$$

If $c < -m$, then $h_0 \in \mathbb{R}^6$, and we must have $h_0 = f(0)$. Furthermore, $\lambda_1 > 0$ and $\lambda_2 > 0$. Hence, $h_0 \in \operatorname{span}(u_1, u_2)$, which implies 4 conditions on h_0.

We see that the number of conditions on h_0, in fact, equals $k^+ + l$, where k^+ and l can be found in table 29.

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REFERENCES

[1] K. Aoki, C. Bardos, and S. Takata *Knudsen layer for gas mixtures*, J. Stat. Phys., **112** (2003), 629–655.

[2] C. Bardos, R. E. Caflisch, and B. Nicolaenko, *The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas*, Comm. Pure Appl. Math., **39** (1986), 323–352.

[3] C. Bardos, F. Golse, and Y. Sone, *Half-space problems for the Boltzmann equation: a survey*, J. Stat. Phys., **124** (2006), 275–300.

[4] N. Bernhoff *On half-space problems for the linearized discrete Boltzmann equation*, Riv. Mat. Univ. Parma, **9** (2008), 73–124.

[5] N. Bernhoff *On half-space problems for the weakly non-linear discrete Boltzmann equation*, Kinet. Relat. Models, **3** (2010), 195–222.

[6] N. Bernhoff *On half-space problems for the discrete Boltzmann equation*, Nuovo Cim. C, **33** (2010), 47–54.

[7] N. Bernhoff *On half-space problems for the non-linear discrete Boltzmann equation in the presence of a non-condensable gas*, In preparation.

[8] N. Bernhoff *Boundary layers for the non-linear discrete Boltzmann equation*, In preparation.

[9] N. Bernhoff and A. Bobylev, *Weak shock waves for the general discrete velocity model of the Boltzmann equation*, Commun. Math. Sci., **5** (2007), 815–832.

[10] A. V. Bobylev and N. Bernhoff, *Discrete velocity models and dynamical systems*, in "Lecture Notes on the Discretization of the Boltzmann Equation" (eds. N. Bellomo and R. Gatignol), World Scientific, (2003), 203–222.

[11] A. V. Bobylev and C. Cercignani, *Discrete velocity models for mixtures*, J. Stat. Phys., **91** (1998), 327–341.

[12] A. V. Bobylev and C. Cercignani, *Discrete velocity models without non-physical invariants*, J. Stat. Phys., **97** (1999), 677–686.

[13] A. V. Bobylev, A. Palczewski, and J. Schneider, *On approximation of the Boltzmann equation by discrete velocity models*, C. R. Acad. Sci. Paris Sr. I Math., **320** (1995), 639–644.

[14] A. V. Bobylev and M. C. Vinerean, *Construction and classification of discrete kinetic models without spurious invariants*, Riv. Mat. Univ. Parma, **7** (2007), 1–80.

[15] A. V. Bobylev and M. C. Vinerean, *Construction of discrete kinetic models with given invariants*, J. Stat. Phys., **132** (2008), 153–170.

[16] C. Bose, R. Illner, and S. Ukai, *On shock wave solutions for discrete velocity models of the Boltzmann equation*, Transp. Th. Stat. Phys., **27** (1998), 35–66.

[17] J. E. Broadwell, *Shock structure in a simple discrete velocity gas*, Phys. Fluids, **7** (1964), 1243–1247.

[18] R. E. Caflisch and B. Nicolaenko, *Shock profile solutions of the Boltzmann equation*, Comm. Math. Phys., **86** (1982), 161–194.

[19] C. Cercignani and H. Cornille, *Shock waves for a discrete velocity gas mixture*, J. Stat. Phys., **99** (2000), 115–140.

[20] H. Cornille and C. Cercignani, *A class of planar discrete velocity models for gas mixtures*, J. Stat. Phys., **99** (2000), 967–991.

[21] H. Cornille and C. Cercignani, *Large size planar discrete velocity models for gas mixtures*, J. Phys. A: Math. Gen., **34** (2001), 2985–2998.

[22] L. Fainsilber, P. Kurlberg, and B. Wennberg, *Lattice points on circles and discrete velocity models for the Boltzmann equation*, Siam J. Math. Anal., **37** (2006), 1903–1922.

[23] S. Kawashima and S. Nishibata, *Existence of a stationary wave for the discrete Boltzmann equation in the half space*, Comm. Math. Phys., **207** (1999), 385–409.

[24] S. Kawashima and S. Nishibata, *Stationary waves for the discrete Boltzmann equation in the half space with reflective boundaries*, Comm. Math. Phys., **211** (2000), 183–206.

[25] T-P. Liu and S-H. Yu, *Boltzmann equation: mikro-macro decompositions and positivity of shock profiles*, Comm. Math. Phys., **246** (2004), 133–179.

[26] T-P. Liu and S-H. Yu, *Invariant manifolds for steady Boltzmann flows and applications*, Preprint.

[27] A. Palczewski, J. Schneider, and A. V. Bobylev, *A consistency result for a discrete-velocity model of the Boltzmann equation*, SIAM J. Numer. Anal., **34** (1997), 1865–1883.

[28] Y. Sone, "Kinetic Theory and Fluid Dynamics" Birkhuser, 2002.

[29] Y. Sone, "Molecular Gas Dynamics" Birkhuser, 2007.

[30] S. Taguchi, K. Aoki, and V. Latocha, *Vapor flows along a plane condensed phase with weak condensation in the presence of a noncondensable gas*, J. Stat. Phys., **124** (2006), 321–369.

[31] S. Takata and F. Golse, *Half-space problem of the nonlinear Boltzmann equation for weak evaporation and condensation of a binary mixture of vapors*, Eur. J. Mech. B/Fluids., **26** (2007), 105–131.

[32] S. Ukai, *On the half-space problem for the discrete velocity model of the Boltzmann equation*, in "Advances in Nonlinear Partial Differential Equations and Stochastics" (eds. S. Kawashima and T. Yanagisawa), World Scientific, (1998), 160–174.

[33] S. Ukai, T. Yang, and S-H. Yu, *Nonlinear boundary layers of the Boltzmann equation. I. Existence*, Comm. Math. Phys., **236** (2003), 373–393.

[34] X. Yang, *The solutions for the boundary layer problem of Boltzmann equation in a half-space*, J. Stat. Phys., **143** (2011), 168–196.

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