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ON HALF-SPACE PROBLEMS FOR THE WEAKLY NON-LINEAR DISCRETE BOLTZMANN EQUATION

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Abstract. Existence of solutions of weakly non-linear half-space problems for the general discrete velocity (with arbitrarily finite number of velocities) model of the Boltzmann equation are studied. The solutions are assumed to tend to an assigned Maxwellian at infinity, and the data for the outgoing particles at the boundary are assigned, possibly linearly depending on the data for the incoming particles. The conditions, on the data at the boundary, needed for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution of the problem are investigated. In the non-degenerate case (corresponding, in the continuous case, to the case when the Mach number at infinity is different of -1, 0 and 1) implicit conditions are found. Furthermore, under certain assumptions explicit conditions are found, both in the non-degenerate and degenerate cases. Applications to axially symmetric models are studied in more detail.

1. **Introduction.** The planar stationary Boltzmann equation, see Ref. [16] and [17], with inflow boundary condition reads

$$\begin{cases} \xi^1 \dfrac{\partial F}{\partial x} = Q\left(F,F\right),\ F = F\left(x,\xi\right) \\ F\left(0,\xi\right) = F_0\left(\xi\right) \text{ for } \xi^1 > 0 \\ F \to M_\infty \text{ as } x \to \infty, \end{cases} \tag{1}$$

where $x \in \mathbb{R}_+$, $\xi = \left(\xi^1,\xi^2,\xi^3\right) \in \mathbb{R}^3$, $M_\infty = \dfrac{\rho_\infty}{\left(2\pi T_\infty\right)^{3/2}} e^{-\left|\xi - \mathbf{u}_\infty\right|^2/(2T_\infty)}$, ρ_∞, $\mathbf{u}_\infty = \left(u^1,u^2,u^3\right)$, and T_∞ are constant, and the collision integral $Q\left(F,F\right)$ is quadratic in F (for more details see Refs. [16] and [17]). After the transformation $F = M_\infty + M_\infty^{1/2} f$, the non-linear equation (1) reads

$$\begin{cases} \xi^1 \dfrac{\partial f}{\partial x} + Lf = S(f,f),\ f = f\left(x,\xi\right) \\ f\left(0,\xi\right) = f_0\left(\xi\right) \text{ for } \xi^1 > 0, \\ f \to 0 \text{ as } x \to \infty, \end{cases} \tag{2}$$

where $Lf = -2M_\infty^{-1/2} Q\left(M_\infty, M_\infty^{1/2} f\right)$ and $S(f,f) = M_\infty^{-1/2} Q\left(M_\infty^{1/2} f, M_\infty^{1/2} f\right)$. The problem (2) can by a shift in the velocity space be rewritten as

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$$\begin{cases} \left(\xi^1 + u^1\right) \dfrac{\partial f}{\partial x} + L_0 f = S_0(f,f) \\ f\left(0,\xi\right) = f_0\left(\xi\right) \text{ for } \xi^1 + u^1 > 0, \\ f \to 0 \text{ as } x \to \infty, \end{cases} \tag{3}$$

where $L_0 f = -2M_0^{-1/2} Q\left(M_0, M_0^{1/2} f\right)$, $S_0(f,f) = M_0^{-1/2} Q\left(M_0^{1/2} f, M_0^{1/2} f\right)$ and $M_0 = \dfrac{\rho_\infty}{\left(2\pi T_\infty\right)^{3/2}} e^{-|\xi|^2/(2T_\infty)}$ (cf. Ref. [21]).

The general boundary condition at $x = 0$ (at the wall) in Eq. (1) reads:

$$F\left(0,\xi\right) = g_0\left(\xi\right) + \int\limits_{\xi_*^1 < 0} K(\xi,\xi_*) F\left(0,\xi_*\right)\, d\xi_* \text{ for } \xi^1 > 0, \tag{4}$$

where (i) $g_0\left(\xi\right) \geq 0$ for $\xi^1 > 0$; (ii) the kernel $K(\xi,\xi_*)$, fulfills $K(\xi,\xi_*) \geq 0$ for $\xi^1 > 0$ and $\xi_*^1 < 0$; and (iii)

$$M_w(\xi) = g_0\left(\xi\right) + \int\limits_{\xi_*^1 < 0} K(\xi,\xi_*) M_w\left(\xi_*\right)\, d\xi_* \text{ for } \xi^1 > 0,$$

where $M_w = \dfrac{\rho_w}{\left(2\pi T_w\right)^{3/2}} e^{-|\xi|^2/(2T_w)}$, T_w is the temperature of the wall and ρ_w is the saturated gas density at temperature T_w, if the boundary is at rest, see Refs. [34] and [35].

For a non-condensable gas (i.e. with no mass flux of the gas across the wall) we can put $g_0\left(\xi\right) \equiv 0$. A particular case is the boundary conditions introduced by Maxwell in Ref. [30, Appendix],

$$F\left(0,\xi\right) = (1-\alpha) F(0,\xi_-) + \frac{\alpha \sigma_w}{\left(2\pi T_w\right)^{3/2}} e^{-|\xi|^2/(2T_w)} \text{ for } \xi^1 > 0,$$

$$\text{with } \sigma_w = -\sqrt{\frac{2\pi}{T_w}} \int\limits_{\xi^1 < 0} \xi^1 F\left(0,\xi\right)\, d\xi \text{ and } \xi_- = \left(-\xi^1, \xi^2, \xi^3\right),$$

where T_w is the temperature of the wall and α, with $0 \leq \alpha \leq 1$, is the accommodation coefficient. The case $\alpha = 1$ is called diffuse reflection, and the case $\alpha = 0$ specular reflection. The Maxwell boundary conditions can be obtained by taking

$$K(\xi,\xi_*) = (1-\alpha)\delta(\xi_* - \xi + 2\xi^1 \mathbf{e}_1) - \frac{\alpha}{2\pi T_w^2} \xi_*^1 e^{-|\xi|^2/(2T_w)},$$

with $\mathbf{e}_1 = (1,0,0)$, in Eq. (4).

In this paper we study the corresponding problem

$$\begin{cases} \xi_i^1 \dfrac{dF_i}{dx} = Q_i\left(F,F\right), \, x \in \mathbb{R}_+, \; i = 1, ..., n, \\ F_i\left(0\right) = F_{0i} \text{ for } \xi_i^1 > 0 \\ F_i(x) \to M_{\infty i} \text{ as } x \to \infty, \end{cases}$$

for the general discrete velocity model in Refs. [15] and [23]. More general boundary conditions (see Eq. (30) below), corresponding to boundary condition (4) in the continuous case, are also considered. Discrete velocity models (DVMs) of the Boltzmann equation are models, where the velocity is discretized, i.e. the velocity is assumed to be able to take only a finite (or in general a discrete) number of different values. It is a well-known fact that the Boltzmann equation can be approximated by DVMs, see Refs. [12], [22], [31] and [32], and that these approximations can be used for numerical methods. The study of DVMs can also give a better conceptual understanding and new ideas, which can be applied to the Boltzmann equation. In the planar stationary case, the general DVM reduces to a system of ordinary differential equations. We continue here the study of DVMs in the directions formulated in Refs. [9], [10] and [7]. Important tools in these studies are the results in Ref. [10] (see Section 2.1 below) on the dimensions of the stable, unstable and center manifolds of the singular points (Maxwellians for DVMs).

Half-space problems for the Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers, see Refs. [16] and [17]. For a comprehensive and detailed description of the asymptotic theory see Refs. [34] and [35]. The half-space problems provide the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. Mathematical results on the half-space problem for the Boltzmann equation for a single-component gas are reviewed in Ref. [6]. Sone and Aoki with coworkers have under a long time considered problems related to these questions, both from a theoretical and numerical point of view, see Refs. [34] and [35] and references therein.

The half-space problems for the linearized Boltzmann equation are well investigated, see Refs. [5], [21] and [27]. A classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete Boltzmann equation has been made in Ref. [7], based on results obtained in Ref. [10]. The results in Ref. [5] have been extended to yield also in the case of binary mixtures, for the homogeneous, as well as the inhomogeneous, linearized Boltzmann equation, by Aoki, Bardos and Takata in Ref. [1].

In Ref. [38] Ukai, Yang and Yu studied the non-linear problem with inflow boundary conditions for a hard sphere gas, assuming that the solutions tend to an assigned Maxwellian at infinity. The conditions on the data at the boundary needed for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution of the problem are investigated. In the cases when the Mach number at infinity is different of -1, 0 and 1 the number of conditions needed is found. Similar existence results have followed for cut-off hard potentials in Ref. [19], and cut-off soft potentials in Ref. [41], and for boundary conditions of diffuse and specular reflection type in Ref. [36]. In Ref. [25] Golse studied the case when the Mach number is 0. Also the non-linear stability of boundary layer solutions have been investigated in Refs. [39], [40] and [42].

Ukai considered in Ref. [37] the same problem for the discrete Boltzmann equation, in the case corresponding to the case when the Mach number is less than -1 for the full Boltzmann equation. This result was generalized by Kawashima and Nishibata in Ref. [28], where they still considered inflow boundary conditions, and in Ref. [29], for different boundary conditions. However, Kawashima and Nishibata in Refs. [28] and [29] still assumed some quite restrictive conditions. In Ref. [2] Babovsky studied a degenerate case for the non-linear (and linearized) DVM, with slightly perturbed specular reflection (cf. Ref. [26]), but with a quite restrictive condition on the non-linear part of the collision operator.

In Ref. [18] Cercignani et al. have shown that the solutions of the half-space problem for the general non-linear DVM with inflow boundary conditions tend to Maxwellians at infinity (without specifying the Maxwellians). In the present paper, the singular point (Maxwellian for DVMs) approached at infinity is fixed and small deviations of the solutions from the singular point is studied. The data for the outgoing particles at the boundary are assigned, possibly linearly depending on the data for the incoming particles. The conditions on the data at the boundary needed for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution of the problem are investigated. In the non-degenerate case (corresponding, in the continuous case, to the case when the Mach number at infinity is different of -1, 0 and 1) implicit conditions have been found by using arguments by Ukai, Yang and Yu in Ref. [38] for the continuous Boltzmann equation. Furthermore, under certain assumptions explicit conditions are found, both in the non-degenerate and degenerate cases. The results extend, not only by more general boundary conditions, but also by more general assumptions, previous results for the discrete Boltzmann equation by Ukai in Ref. [37], and Kawashima and Nishibata in Refs. [28] and [29], and include also (for DVMs) the results obtained by Ukai, Yang and Yu in Ref. [38] for the continuous Boltzmann equation. Applications to axially symmetric models have also been studied, generalizing the results by Babovsky in Ref. [2].

All results are obtained for an arbitrary finite number of velocities. Similar results as in this paper can also be obtained for DVMs for mixtures. Existence of weak shock wave solutions for the discrete Boltzmann equation has also been proved based on the same ideas in Ref. [8].

This paper is organized as follows: In Section 2, we introduce the planar stationary discrete Boltzmann equation and review some of its properties. We make an expansion around an equilibrium Maxwellian, and review, Theorem 2.1 in Subsection 2.1, the results in Ref. [10] on the dimensions of the stable, unstable and center manifolds of the system of ODEs. The problem and the main results on existence and uniqueness are stated in Section 3 (Theorem 3.1 and Theorem 3.2). The boundary conditions at the "wall" are discussed in more detail in Section 4. In particular, inflow boundary conditions and Maxwell-type boundary conditions (Subsection 4.1) are considered. The results of [10] (stated in Theorem 2.1) are used to investigate the number of additional conditions needed to obtain well-posedness of the weakly non-linear problem in Section 5 and Section 6 respectively, and thereby to prove Theorem 3.1 (Section 5) and Theorem 3.2 (Section 6) in Section 3. Implicit conditions for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution in the non-degenerate case and also for the degenerate case, but then with some restrictions on the non-linear part of the collision operator, are obtained (Section 5). The results are in accordance with corresponding results for the continuous Boltzmann equation obtained in the non-degenerate case, with inflow boundary conditions in Ref. [38]. We also obtain explicit conditions for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution (Section 6), but with more restrictions, at least in the non-degenerate case, on the non-linear part. However, in some degenerate cases we obtain weaker restrictions on the non-linear part than in Theorem 3.1. The more general case when we allow velocities inducing a singular "velocity-matrix" (that is, if we allow velocities that have zero as first component) is discussed in Section 7. Applications to axially symmetric models is studied in Section 8. The degenerate cases for axially symmetric DVMs (in the "shock wave context"), if we have expanded around a non-drifting Maxwellian in Section 2, are discussed in Subsection 8.1. The results are in accordance with the results for the continuous Boltzmann equation in Ref. [21]. We also apply our results (Theorem 3.2) in Section 3 to a boundary layer problem of the type studied by Golse, Perthame and Sulem in Ref. [26] for the Boltzmann equation, and by Babovsky in Ref. [2] for DVMs (with quite restrictive conditions on the non-linear part of the collision operator). We first consider a plane 12-velocity DVM in Subsection 8.2, but also a more general axially symmetric DVM (cf. Ref. [2]) in Subsection 8.3.

2. **Discrete Boltzmann equation.** The planar stationary system for the discrete Boltzmann equation (DBE) reads

$$\xi_i^1 \frac{dF_i}{dx} = Q_i(F,F), \; x \in \mathbb{R}_+, \; i = 1, ..., n, \tag{5}$$

where $\mathrm{V} = \{\xi_1, ..., \xi_n\}$, $\xi_i \in \mathbb{R}^d$, is a finite set of velocities, $F_i = F_i(x) = F(x, \xi_i)$, and $F = F(x, \xi)$ represents the microscopic density of particles with velocity $\xi = (\xi^1, ..., \xi^d)$ at position $\mathbf{x} = (x, x^2, ..., x^d) \in \mathbb{R}^d$. We also assume (except in Section 7) that

$$\xi_i^1 \neq 0, \text{ for } i = 1, ..., n.$$

For a function $g = g(\xi)$ (possibly depending on more variables than ξ), we will identify g with its restriction to the set V, but also when suitable consider it like a vector function

$$g = (g_1, ..., g_n), \text{ with } g_i = g(\xi_i).$$

Consistently, we say that g is non-negative (positive), $g \geq 0$ ($g > 0$), if and only if $g_i \geq 0$ ($g_i > 0$) for all $1 \leq i \leq n$.

Then Eq. (5) can be rewritten as

$$B \frac{dF}{dx} = Q(F,F), \text{ with } x \in \mathbb{R}_+ \text{ and } B = \mathrm{diag}(\xi_1^1, ..., \xi_n^1). \tag{6}$$

Below we review some properties of the discrete Boltzmann equation.

The collision operators $Q_i(F,F)$ in Eq. (5) are given by the bilinear expressions

$$Q_i(F,G) = \frac{1}{2} \sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} \left(F_k G_l + G_k F_l - F_i G_j - G_i F_j \right), \tag{7}$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0,$$

with equality unless the conservation laws

$$\xi_i + \xi_j = \xi_k + \xi_l \text{ and } |\xi_i|^2 + |\xi_j|^2 = |\xi_k|^2 + |\xi_l|^2 \tag{8}$$

are satisfied (preservation of momentum and energy).

Remark 1. Our main results, presented in Section 3, do not depend on the preservation of energy (even if we indeed use it in some of our applications), i.e., Eqs. (8) could be replaced by

$$\xi_i + \xi_j = \xi_k + \xi_l,$$

without affecting our main results. In fact, our main results do not depend on what set of collision invariants (cf. Eq. (9)) we have.

A function $\phi = \phi(\xi)$ is a collision invariant if and only if

$$\phi_i + \phi_j = \phi_k + \phi_l, \tag{9}$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$, or, equivalently, if and only if

$$\langle \phi, Q(F,F) \rangle = 0, \tag{10}$$

for all non-negative functions F. We have the trivial collision invariants (also called the physical collision invariants) $\phi_0 = 1$, $\phi_1 = \xi^1, ..., \phi_d = \xi^d$, $\phi_{d+1} = |\xi|^2$ (including all linear combinations of these). Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$.

We consider below (even if this restriction is not necessary in our general context) only normal DVMs. That is, DVMs without spurious (or non-physical) collision invariants, i.e. any collision invariant is of the form

$$\phi = a + \mathbf{b} \cdot \xi + c\, |\xi|^2 \tag{11}$$

for some constant $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$ (methods of their construction are described in Refs. [11], [13] and [14]). In this case the equation (10) has the general solution (11).

A Maxwellian distribution (or just a Maxwellian) is a function $M = M(\xi)$, such that

$$Q(M, M) = 0 \text{ and } M > 0.$$

All Maxwellian distributions are of the form

$$M = e^{\phi} = Ae^{\mathbf{b}\cdot\xi + c|\xi|^2}, \text{ with } A = e^a > 0 \text{ and } c < 0, \tag{12}$$

where ϕ is a collision invariant (11) (the latter equality is due to the assumption of normal DVMs). In general a, $\mathbf{b}$ and c can be functions of x, but since we assume that our solutions tend to a global, i.e. with absolute constant a, $\mathbf{b}$ and c, Maxwellian at infinity, our interest is in global Maxwellians, and so when we below refer to a Maxwellian, we will mean a global Maxwellian.

Given a Maxwellian M we denote

$$F = M + M^{1/2} f, \tag{13}$$

in Eq. (5), and obtain

$$\xi_i^1 \frac{df_i}{dx} = -\left(Lf\right)_i + S_i\left(f, f\right), \, i = 1, ..., n,$$

where L is the linearized collision operator ($n \times n$ matrix) given by

$$Lf = -2M^{-1/2} Q(M, M^{1/2} f), \tag{14}$$

and S is the quadratic part given by

$$S(f, g) = M^{-1/2} Q(M^{1/2} f, M^{1/2} g). \tag{15}$$

In more explicit forms, the operators (14) and (15) read

$$\left(Lf\right)_i = -\sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} M_j^{1/2} \left(M_k^{1/2} f_l + M_l^{1/2} f_k - M_i^{1/2} f_j - M_j^{1/2} f_i \right),$$

and

$$S_i\left(f, f\right) = \sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} M_j^{1/2} \left(f_k f_l - f_i f_j\right).$$

The matrix L is symmetric and semi-positive, and the null-space $N(L)$ of L is (for normal DVMs) given by

$$N(L) = \operatorname{span}(M^{1/2}, M^{1/2}\xi^1, ..., M^{1/2}\xi^d, M^{1/2}\, |\xi|^2).$$

Furthermore, $S\left(f, f\right)$ belong to the orthogonal complement of $N(L)$, i.e.

$$S\left(f, f\right) \in N(L)^{\perp}.$$

Then the system (6) transforms into

$$B\frac{df}{dx}+Lf=S(f,f). \tag{16}$$

The diagonal matrix B (6) (under our assumptions) has no zero diagonal elements and is non-singular. If we denote $f|_{x=0}=f_0$ (the boundary conditions imposed by all ξ_i), then we can rewrite Eq. (16) as

$$f(x)=e^{-xB^{-1}L}f_0+\int_0^x e^{(\sigma-x)B^{-1}L}\left[S\left(f,f\right)\right](\sigma)\,d\sigma.$$

2.1. **Characteristic numbers.** We denote by $n^{\pm}$, where $n^{+}+n^{-}=n$, and $m^{\pm}$, with $m^{+}+m^{-}=q$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices B and $B^{-1}L$ respectively, and by m^{0} the number of zero eigenvalues of $B^{-1}L$. Moreover, we denote by k^{+}, k^{-} and l the numbers of positive, negative and zero eigenvalues of the $p\times p$ matrix K ($p=d+2$ for normal DVMs), with entries $k_{ij}=\left\langle y_i,y_j\right\rangle_B=\left\langle y_i,By_j\right\rangle$, such that $\{y_1,...,y_p\}$ is a basis of the null-space of L, i.e. in our case $\operatorname{span}(y_1,...,y_p)=N(L)=\operatorname{span}\left(M^{1/2},M^{1/2}\xi^{1},...,M^{1/2}\xi^{d},M^{1/2}\left|\xi\right|^{2}\right)$. Here and below, we denote $\left\langle\cdot,\cdot\right\rangle_B=\left\langle\cdot,B\cdot\right\rangle$. We also recall the notation $N(L)$ for the null-space of L.

In applications, the number p of collision invariants is usually relatively small compared to n (note that formally $n=\infty$ for the continuous Boltzmann equation whenas $p\leq 5$). Also, the matrix B is diagonal and therefore all its eigenvalues are known. This explains the importance of the following result by Bobylev and Bernhoff in Ref. [10] (see also Ref. [7]).

Theorem 2.1. *The numbers of positive, negative and zero eigenvalues of $B^{-1}L$ are given by*

$$\begin{cases} m^{+}=n^{+}-k^{+}-l\\ m^{-}=n^{-}-k^{-}-l\\ m^{0}=p+l.\end{cases}$$

Remark 2. In Ref. [10] Theorem 2.1 is proved for any real symmetric matrices L and B, such that L is semi-positive and B is invertible.

In the proof of Theorem 2.1 a basis

$$u_1,...,u_q,y_1,...,y_k,z_1,...,z_l,w_1,...,w_l \tag{17}$$

of $\mathbb{R}^n$, such that

$$y_i,z_r\in N(L),\ B^{-1}Lw_r=z_r \text{ and } B^{-1}Lu_\alpha=\lambda_\alpha u_\alpha, \tag{18}$$

and

$$\begin{aligned}
&\left\langle u_\alpha,u_\beta\right\rangle_B=\lambda_\alpha\delta_{\alpha\beta},\text{ with }\lambda_1,...,\lambda_{m^+}>0\text{ and }\lambda_{m^++1},...,\lambda_q<0,\\
&\left\langle y_i,y_j\right\rangle_B=\gamma_i\delta_{ij},\text{ with }\gamma_1,...,\gamma_{k^+}>0\text{ and }\gamma_{k^++1},...,\gamma_k<0,\\
&\left\langle u_\alpha,z_r\right\rangle_B=\left\langle u_\alpha,w_r\right\rangle_B=\left\langle u_\alpha,y_i\right\rangle_B=\left\langle w_r,y_i\right\rangle_B=\left\langle z_r,y_i\right\rangle_B=0,\\
&\left\langle w_r,w_s\right\rangle_B=\left\langle z_r,z_s\right\rangle_B=0\text{ and }\left\langle w_r,z_s\right\rangle_B=\delta_{rs},
\end{aligned} \tag{19}$$

is constructed.

The Jordan normal form of $B^{-1}L$ (with respect to the basis (17)-(19)) is

$$\begin{pmatrix} \lambda_1 & & & & & & & & & & & & \\ & \ddots & & & & & & & & & & & \\ & & \lambda_q & & & & & & & & & & \\ & & & 0 & & & & & & & & & \\ & & & & \ddots & & & & & & & & \\ & & & & & 0 & & & & & & & \\ & & & & & & 0 & 1 & & & & & \\ & & & & & & 0 & 0 & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & 0 & 1 & \\ & & & & & & & & & 0 & 0 \end{pmatrix},$$

where there are l blocks of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For any $h \in \mathbb{R}^n$, we obtain

$$e^{-xB^{-1}L}h = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left(\left(\eta_j - x\alpha_j \right) z_j + \alpha_j w_j \right) + \sum_{r=1}^{q} \beta_r e^{-\lambda_r x} u_r,$$

where

$$\mu_i = \frac{\langle h, y_i \rangle_B}{\langle y_i, y_i \rangle_B},\ \beta_r = \frac{\langle h, u_r \rangle_B}{\lambda_r},\ \alpha_j = \langle h, z_j \rangle_B \text{ and } \eta_j = \langle h, w_j \rangle_B .$$

3. **Statement of the problem and main results.** We consider the non-linear system

$$B\frac{df}{dx} + Lf = S(f,f), \tag{20}$$

where the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty, \tag{21}$$

and

$$S\left(f,f\right) \in N(L)^{\perp}. \tag{22}$$

The boundary conditions (21) correspond to the case when we have made the expansion (13) around a Maxwellian $M = M_\infty$, such that $F \to M_\infty$ as $x \to \infty$.

We can (without loss of generality) assume that

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix},$$

where

$$\begin{aligned} &B_+ = \operatorname{diag}\left(\xi_1^1, ..., \xi_{n^+}^1\right) \text{ and } B_- = -\operatorname{diag}\left(\xi_{n^++1}^1, ..., \xi_n^1\right), \\ &\text{with } \xi_1^1, ..., \xi_{n^+}^1 > 0 \text{ and } \xi_{n^++1}^1, ..., \xi_n^1 < 0. \end{aligned} \tag{23}$$

We also define the projections $R_+ : \mathbb{R}^n \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^n \to \mathbb{R}^{n^-}$, $n^- = n - n^+$, by

$$R_+ s = s^+ = (s_1, ..., s_{n^+}) \text{ and } R_- s = s^- = (s_{n^++1}, ..., s_n)$$

for $s = (s_1, ..., s_n)$.

At $x = 0$ we assume the general boundary conditions (cf. Eqs. (31) below)

$$f^+(0) = Cf^-(0) + h_0, \tag{24}$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$. We introduce the operator $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^{n^+}$, given by

$$\mathcal{C} = R_+ - CR_-.$$

We will also assume that the matrix C fulfills one of the conditions

$$\begin{aligned} &\dim \mathcal{C}U_+ = m^+, \\ &\text{with } U_+ = \operatorname{span}\left(u \middle| \; Lu = \lambda Bu, \text{ with } \lambda > 0\right) = \operatorname{span}\left(u_1, ..., u_{m^+}\right), \end{aligned} \tag{25}$$

and

$$\dim \mathcal{C}X_+ = n^+, \text{ with } X_+ = \operatorname{span}\left(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1, ..., z_l\right). \tag{26}$$

Remark 3. For the continuous Boltzmann equation (with $d = 3$), if we have made the expansion (13) around a non-drifting Maxwellian

$$M = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} e^{-|\xi|^2/2T_\infty},$$

$k^+ = 1$, $l = 3$ and the collision invariants y_1, y_2, z_1, z_2 and z_3 can be chosen as, cf. Ref. [21],

$$\begin{aligned} y_1 &= \left(\frac{\xi^1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30}T_\infty}\right) M^{1/2}, \; y_2 = \left(-\frac{\xi^1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30}T_\infty}\right) M^{1/2}, \\ z_1 &= \left(\sqrt{\frac{5}{2}} - \frac{|\xi|^2}{\sqrt{10}T_\infty}\right) M^{1/2}, \; z_2 = \frac{\xi^2}{\sqrt{T_\infty}} M^{1/2} \text{ and } z_3 = \frac{\xi^3}{\sqrt{T_\infty}} M^{1/2}. \end{aligned}$$

Moreover,

$$w_j = L^{-1}\xi^1 z_j,$$

in the continuous case, and the continuous analogue of equation $Lu = \lambda Bu$ is

$$Lh = \lambda \xi^1 h, \; h = h(\xi), \tag{27}$$

(see Ref. [16] for a discussion on the eigenvalue problem (27)). We also want to point out that, in the continuous case, the boundary conditions (before the expansion (13)), that correspond to conditions (24), are given by Eqs. (4).

We now state our main results.

Theorem 3.1. *Let condition (26) be fulfilled and suppose that* $\left\langle S\left(f(x), f(x)\right), w_j\right\rangle = 0$ *for* $j = 1, ..., l$, *and that* $\left\langle h_0, h_0\right\rangle_{B_+}$ *is sufficiently small. Then with* $k^+ + l$ *conditions on* h_0, *the system (20) with the boundary conditions (21) and (24) has a locally unique solution.*

Theorem 3.1 is proved in Section 5.

Theorem 3.2. *Let condition (25) be fulfilled and assume that*

$$\begin{aligned} &h_0, \mathcal{C}e^{xB^{-1}L}B^{-1}S(f(x), f(x)) \in \mathcal{C}U_+ \textit{ for all } x \in \mathbb{R}_+, \\ &\textit{with } U_+ = \textit{span}(u : Lu = \lambda Bu, \; \lambda > 0) = \textit{span}\left(u_1, ..., u_{m^+}\right). \end{aligned} \tag{28}$$

Then there is a positive number δ_0, *such that if*

$$|h_0| \leq \delta_0,$$

then the system (20) with the boundary conditions (21) and (24) has a locally unique solution.

The proof of Theorem 3.2 is outlined in Section 6.

Remark 4. If condition (25) is fulfilled, then the condition

$$h_0 \in \mathcal{C}U_+.$$

implies that we have $k^+ + l$ conditions on h_0.

Remark 5. If the conditions

$$\begin{aligned} \mathcal{C}U_- &\subseteq \mathcal{C}U_+, \\ \text{with } U_- &= \operatorname{span}\left(\{u|\ Lu = \lambda Bu, \text{ with } \lambda < 0\} \cup \{z_1, ..., z_l\}\right) \\ &= \operatorname{span}\left(u_{m^++1}, ..., u_q, z_1, ..., z_l\right), \end{aligned} \tag{29}$$

and (22) are fulfilled, then

$$\mathcal{C}e^{xB^{-1}L}B^{-1}S(f,f) \in \mathcal{C}U_+ \text{ for all } x \in \mathbb{R}_+.$$

Lemma 3.3. *(see Ref. [7]) Let B_+ and B_- be the matrices defined in Eqs. (23). Then*
i) condition (26) is fulfilled, if

$$C^T B_+ C < B_- \text{ on } R_- X_+;$$

ii) condition (25) is fulfilled, if

$$C^T B_+ C \leq B_- \text{ on } R_- U_+.$$

Proof. ii) Let $u \in U_+$ and $C^T B_+ C \leq B_-$ on $R_- U_+$. Then

$$\langle u, u \rangle_B > 0.$$

Furthermore, if $u \neq 0$ and $\mathcal{C}u = 0$, then

$$\langle u, u \rangle_B = \left\langle Cu^-, Cu^- \right\rangle_{B_+} - \left\langle u^-, u^- \right\rangle_{B_-} = \left\langle (C^T B_+ C - B_-)u^-, u^- \right\rangle \leq 0.$$

Hence, if $\mathcal{C}u = 0$, then $u = 0$. That is, $\dim \mathcal{C}U_+ = \dim U_+ = m^+$, and part ii) of the lemma is proved.

Part i) of the lemma is proved in a similar way (see also Ref. [7]). □

Corollary 1. *(see Ref. [7]) If $C = 0$, then the conditions (25) and (26) are fulfilled. In particular, $\left\{u_1^+, ..., u_{m^+}^+, y_1^+, ..., y_{k^+}^+, z_1^+, ..., z_l^+\right\}$ is a basis of $\mathbb{R}^{n^+}$.*

Ukai considered the case with $m^+ = n^+$ and $C = 0$ in Ref. [37]. Then conditions (25), (26) and (28) are trivially fulfilled, since $R_+ U_+ = \mathbb{R}^{n^+}$. This is the discrete correspondence of the case when the Mach number of the Maxwellian M_∞ is less than -1, i.e. $\mathcal{M}_\infty = u_{\infty,1}/\sqrt{\frac{5}{3}T_\infty} < -1$, for the full Boltzmann equation.

This result was generalized by Kawashima and Nishibata in Ref. [28], where they still considered the case $C = 0$ (but allowed zero first components of the velocities, which was ruled out in Ref. [37]). Then conditions (25) and (26) are fulfilled by Corollary 1. They assumed that $\dim R_+ \left(BN(L)\right)^\perp = m_+$ (equivalent to assumption [A] in Ref. [28]), which implies that $R_+ \left(BN(L)\right)^\perp = R_+ U_+$, and hence, that $l = 0$ and $R_+ U_- \subseteq R_+ U_+$ (Eq. (29) with $C = 0$). Therefore, condition (28) is fulfilled if the boundary data satisfies the consistency condition, equivalent to the condition $h_0 \in R_+ \left(BN(L)\right)^\perp$, in Ref. [28]. In their subsequent paper [29], Kawashima and Nishibata assumed that $\dim R_+ \left(BN(L)\right)^\perp = m^+$, $CR_- \left(BN(L)\right)^\perp \subseteq R_+ \left(BN(L)\right)^\perp$, and that $\langle u, u \rangle_B < 0$ if $\mathcal{C}u = 0$ and $u \neq 0$ (cf. Lemma 2.1 in Ref. [29]). Conditions (25) and (26) are fulfilled by the latter assumption and Lemma 3.3. By the first assumption, $l = 0$ and $R_+ U_- \subseteq R_+ U_+$. Therefore, the second assumption implies that condition (28) is fulfilled if the boundary data satisfies the consistency condition, equivalent to condition $h_0 \in \mathcal{C}\left(BN(L)\right)^{\perp}$, in Ref. [29]. Note that a_0 in Eq. (30) is assumed to be zero, $a_0 = 0$, in Ref. [29].

Remark 6. All our results can be extended in a natural way, to yield also for singular matrices B (see Section 7), if

$$N(L) \cap N(B) = \{0\} .$$

4. **Boundary conditions.** If $M = M_\infty = Ae^{\mathbf{b}\cdot\xi + c|\xi|^2} \in \mathbb{R}^n$ is the Maxwellian we have made the expansion (13) around, i.e.,

$$F(x) = M + M^{1/2} f(x) ,$$

then the general boundary conditions (cf. boundary condition (4) in the continuous case)

$$F^{+}(0) = C_0 F^{-}(0) + a_0, \tag{30}$$

where C_0 is a given $n^+ \times n^-$ matrix and $a_0 \in \mathbb{R}^{n^+}$, at $x = 0$, lead to the following C and h_0 in Eq. (24),

$$C = M_{+}^{-1/2} C_0 M_{-}^{1/2} \text{ and } h_0 = M_{+}^{-1/2}(C_0 M^{-} - M^{+} + a_0), \tag{31}$$

with

$$M_{+}^{-1/2} = \mathrm{diag}(M_1^{-1/2}, ..., M_{n^+}^{-1/2}) \text{ and } M_{-}^{1/2} = \mathrm{diag}(M_{n^+ +1}^{1/2}, ..., M_n^{1/2}).$$

Example 1. If we assume inflow boundary conditions, i.e. $C_0 = 0$, as is the case when we have complete condensation, then $C = 0$ and $h_0 = M_{+}^{-1/2}\left(a_0 - M^{+}\right)$.

Example 2. Let $n^- = n^+$. The discrete version of the Maxwell-type boundary conditions reads

$$F^{+}(0) = C_0 F^{-}(0), \text{ with } C_0 = (1-\alpha)I + \alpha C_{0d},\ 0 \le \alpha \le 1,$$

where I is the identity matrix and C_{0d} is the $n^+ \times n^+$ matrix, with the elements $c_{0d,ij} = \dfrac{\xi^1_{n^+ + j} M_{0i}}{\left\langle B_{-} M_0^{-}, 1\right\rangle}$ for some Maxwellian M_0, cf. Ref. [24]. The cases $\alpha = 0$ and $\alpha = 1$ correspond to specular and diffuse reflection, respectively.

After the expansion (13), the Maxwell-type boundary conditions reads, cf. Ref. [7],

$$\begin{aligned} f^{+}(0) &= C_M f^{-}(0) + h_0, \text{ with } C_M = (1-\alpha) M_{+}^{-1/2} M_{-}^{1/2} + \alpha C_d,\ 0 \le \alpha \le 1, \\ h_0 &= M_{+}^{-1/2}\left((1-\alpha)M^{-} + \alpha \frac{\left\langle B_{-} M^{-}, 1\right\rangle}{\left\langle B_{-} M_0^{-}, 1\right\rangle} M_0^{+} - M^{+}\right), \end{aligned} \tag{32}$$

where C_d is the $n^+ \times n^+$ matrix, with the elements

$$c_{d,ij} = \frac{\xi^1_{n^+ + j} M_i^{-1/2} M_{n^+ + j}^{1/2} M_{0i}}{\left\langle B_{-} M_0^{-}, 1\right\rangle}.$$

We obtain that

$$\begin{aligned}\left\langle C_d^T B_+ C_d u, u\right\rangle &= \frac{\left\langle B_- \sqrt{M^-}, u\right\rangle^2 \left\langle B_+ M_0^+, M_+^{-1} M_0^+\right\rangle}{\left\langle B_- M_0^-, 1\right\rangle^2} \\ &\leq \frac{\left\langle B_- M^-, 1\right\rangle \left\langle B_+ M_0^+, \left(M^+\right)^{-1} M_0^+\right\rangle}{\left\langle B_- M_0^-, 1\right\rangle^2} \left\langle B_- u, u\right\rangle,\end{aligned}$$

with equality if and only if $u \in \operatorname{span}(\sqrt{M^-})$.

Remark 7. In the general case of Remark 2 we fix an orthonormal basis

$$\left\{e_1, ..., e_n\right\}, \text{ with } \left\langle e_i, e_j\right\rangle = \delta_{ij},$$

of $\mathbb{R}^n$, such that

$$Be_i = b_i e_i,$$

where

$$b_1, ..., b_{n^+} > 0 \text{ and } b_{n^++1}, ..., b_n < 0,$$

and define $R_+ : \mathbb{R}^n \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^n \to \mathbb{R}^{n^-}$, by

$$R_+ s = s^+ = (s_1, ..., s_{n^+}) \text{ and } R_- s = s^- = (s_{n^++1}, ..., s_n)$$

for $s = \sum\limits_{i=1}^{n} s_i e_i$. We also introduce the matrices B_+ and B_-, defined by

$$B_+ = \operatorname{diag}\left(b_1, ..., b_{n^+}\right) \text{ and } B_- = -\operatorname{diag}\left(b_{n^++1}, ..., b_n\right).$$

4.1. **Maxwell-type boundary conditions.** We now consider the Maxwell-type boundary conditions (32).

If we assume that

$$\xi_{i+n^+} = (-\xi_i^1, \xi_i^2 ..., \xi_i^d),\ \xi_i^1 > 0, \text{ for } i = 1, ..., n^+, \tag{33}$$

then $M^- = e^{-2b\xi^1} M^+$ and therefore, $M_+^{-1/2} M_-^{1/2} = \operatorname{diag}(e^{-b\xi_1^1}, ..., e^{-b\xi_{n^+}^1})$, where b is the first component of $\mathbf{b}$ in Eqs. (12). Note also that $B_- = B_+$. Furthermore, we assume that

$$M_0 = A_0 e^{c_0 |\xi|^2} \text{ and } \mathbf{b} = (b, 0, ..., 0).$$

Then

$$\begin{aligned}\left\langle C_M^T B_+ C_M u, u\right\rangle &= (1-\alpha)^2 \left\langle B_+ \left(M^+\right)^{-1} M^- u, u\right\rangle \\ &\quad + 2(\alpha - \alpha^2) \left\langle M_+^{-1/2} M_-^{1/2} B_+ u, C_d u\right\rangle + \alpha^2 \left\langle C_d^T B_+ C_d u, u\right\rangle,\end{aligned}$$

where

$$\left\langle C_d^T B_+ C_d u, u\right\rangle \leq \frac{\left\langle B_+ M^-, 1\right\rangle \left\langle B_+ M_0^+, \left(M^+\right)^{-1} M_0^+\right\rangle}{\left\langle B_+ M_0^+, 1\right\rangle^2} \left\langle B_- u, u\right\rangle,$$

and

$$\begin{aligned}&\left\langle M_+^{-1/2}M_-^{1/2}B_+C_d u,u\right\rangle\\ =&\frac{\left\langle B_-M_-^{1/2},u\right\rangle\left\langle B_+M_-^{1/2}\left(M^+\right)^{-1}M_0^+,u\right\rangle}{\left\langle B_-M_0^+,1\right\rangle}\\ \leq&\frac{\left\langle B_+M^-,1\right\rangle^{1/2}\left\langle B_+\left(M^+\right)^{-2},M^-\left(M_0^+\right)^2\right\rangle^{1/2}}{\left\langle B_-M_0^+,1\right\rangle}\left\langle B_-u,u\right\rangle.\end{aligned}$$

Hence, if $b\geq 0$ we obtain the following rough estimate

$$\left\langle C_M^T B_+C_M u,u\right\rangle\leq e^{-2b\xi_{\min}^1}(1+\alpha(e^{|c-c_0|(|\xi|_{\max}^2-|\xi|_{\min}^2)/2}-1))^2\left\langle B_-u,u\right\rangle,\qquad(34)$$

where $\xi_{\min}^1=\min\left(\xi_1^1,...,\xi_{n^+}^1\right)$, $|\xi|_{\min}^2=\min(|\xi_1|^2,...,|\xi_{n^+}|^2)$ and $|\xi|_{\max}^2=\max(|\xi_1|^2,...,|\xi_{n^+}|^2)$. If $b<0$, then we can replace $\xi_{\min}^1$ with $\xi_{\max}^1$, where $\xi_{\max}^1=\max\left(\xi_1^1,...,\xi_{n^+}^1\right)$, in Eq. (34).

Lemma 4.1. *Let $M_0=A_0e^{c_0|\xi|^2}$ and $M=M_\infty=Ae^{b\xi^1+c|\xi|^2}$, and assume that assumption (33) is fulfilled. Then $C=sC_M$ fulfills conditions (25) and (26), if*

$$s<\left\{\begin{array}{l}e^{b\xi_{\min}^1}(1+\alpha(e^{|c-c_0|(|\xi|_{\max}^2-|\xi|_{\min}^2)/2}-1))^{-1}\text{ if }b\geq 0\\ e^{b\xi_{\max}^1}(1+\alpha(e^{|c-c_0|(|\xi|_{\max}^2-|\xi|_{\min}^2)/2}-1))^{-1}\text{ if }b<0\end{array}\right..\qquad(35)$$

Proof. If s satisfies inequality (35), then it follows from inequality (34) that

$$\left\langle\left(sC_M\right)^T B_+sC_M u,u\right\rangle=s^2\left\langle C_M^T B_+C_M u,u\right\rangle<\left\langle B_-u,u\right\rangle$$

and the lemma follows by Lemma 3.3. □

5. **With damping term.** We add (following the structure in Ref. [38] for the full Boltzmann equation) a damping term $-\gamma P_0^+f$ to the right-hand side of the system (20) and obtain

$$B\frac{df}{dx}+Lf=S(f,f)-\gamma P_0^+f,\qquad(36)$$

where $\gamma>0$ and

$$P_0^+f=\sum_{i=1}^{k^+}\frac{\left\langle f(x),y_i\right\rangle_B}{\left\langle y_i,y_i\right\rangle_B}y_i+\sum_{j=1}^{l}\left\langle f(x),w_j\right\rangle_B z_j.$$

First we consider the corresponding linearized inhomogeneous system

$$B\frac{df}{dx}+Lf=g-\gamma P_0^+f,\qquad(37)$$

where $g=g(x):\mathbb{R}_+\to\mathbb{R}^n$ is a given function such that

$$g\left(x\right)\in N(L)^\perp\text{ for all }x\in\mathbb{R}_+.\qquad(38)$$

The system (37) with the boundary conditions (21) has (under the assumption that all necessary integrals exist) the general solution, using the notations in Eqs. (17)-(19),

$$f(x)=\sum_{i=1}^{k^+}\mu_i\left(x\right)y_i+\sum_{j=1}^{l}\eta_j\left(x\right)z_j+\sum_{r=1}^{q}\beta_r\left(x\right)u_r,\qquad(39)$$

where

$$\begin{cases} \mu_i(x) = \mu_i(0)\, e^{-\gamma x},\ i = 1, ..., k^+, \\ \eta_j(x) = \eta_j(0)\, e^{-\gamma x} + \int\limits_0^x e^{(\tau - x)\gamma} \widetilde{\eta}_j(\tau)\ d\tau,\ j = 1, ..., l, \\ \beta_r(x) = \beta_r(0)\, e^{-\lambda_r x} + \int\limits_0^x e^{(\tau - x)\lambda_r} \widetilde{\beta}_r(\tau)\ d\tau,\ r = 1, ..., m^+, \\ \beta_r(x) = -\int\limits_x^\infty e^{(\tau - x)\lambda_r} \widetilde{\beta}_r(\tau)\ d\tau,\ r = m^+ + 1, ..., q, \end{cases} \tag{40}$$

with

$$\widetilde{\eta}_j(x) = \langle g(x), w_j \rangle \text{ and } \widetilde{\beta}_r(x) = \langle g(x), u_r \rangle. \tag{41}$$

From the boundary conditions (24), we obtain the system

$$\begin{aligned} &\sum_{r=1}^{m^+} \beta_r(0)\, \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_i(0)\, \mathcal{C} y_i + \sum_{j=1}^{l} \eta_j(0)\, \mathcal{C} z_j \\ =& h_0 + \sum_{r=m^++1}^{q} \int_0^\infty e^{\tau \lambda_r} \widetilde{\beta}_r(\tau)\ d\tau\, \mathcal{C} u_r, \text{ with } \mathcal{C} = R_+ - C R_-. \end{aligned} \tag{42}$$

For $h_0 = 0$ in (24), we have the trivial solution $f(x) \equiv 0$. Therefore, we consider only non-zero h_0, $h_0 \neq 0$, below. The system (42) has (under the assumption that all necessary integrals exist) a unique solution if we assume that the condition (26) is fulfilled.

Theorem 5.1. *Assume that the conditions (26) and (38) are fulfilled and that all necessary integrals exist. Then the system (37) with the boundary conditions (21) and (24), has a unique solution given by Eqs. (39)-(42).*

We fix a number σ, such that

$$0 < \sigma \leq \min\{|\lambda| \neq 0;\ \det(\lambda B - L) = 0\} \text{ and } \sigma \leq \gamma$$

and introduce the norm (cf. Ref. [29])

$$|h|_\sigma = \sup_{x \geq 0} e^{\sigma x} |h(x)|,$$

the Banach space

$$\mathcal{X} = \left\{ h \in \mathcal{B}^0[0, \infty) \,|\, |h|_\sigma < \infty \right\}$$

and its closed convex subset

$$\mathcal{S}_R = \left\{ h \in \mathcal{B}^0[0, \infty) \,|\, |h|_\sigma \leq R\, |h_0| \right\},$$

where R is a, so far, undetermined positive constant.

We assume that the condition (26) is fulfilled and introduce the operator $\Theta(f)$ on $\mathcal{X}$, defined by

$$\Theta(f) = \sum_{i=1}^{k^+} \mu_i(f(x))\, y_i + \sum_{j=1}^{l} \eta_j(f(x))\, z_j + \sum_{r=1}^{q} \beta_r(f(x))\, u_r, \tag{43}$$

where

$$\begin{cases} \mu_i(f(x)) = \mu_i(f(0))\, e^{-\gamma x},\ i = 1, ..., k^+, \\ \eta_j(f(x)) = \eta_j(f(0))\, e^{-\gamma x} + \int\limits_0^x e^{(\tau - x)\gamma} \widetilde{\eta}_j(f(\tau))\, d\tau,\ j = 1, ..., l, \\ \beta_r(f(x)) = \beta_r(f(0))\, e^{-\lambda_r x} + \int\limits_0^x e^{(\tau - x)\lambda_r} \widetilde{\beta}_r(f(\tau))\, d\tau,\ r = 1, ..., m^+, \\ \beta_r(f(x)) = -\int\limits_x^\infty e^{(\tau - x)\lambda_r} \widetilde{\beta}_r(f(\tau))\, d\tau,\ r = m^+ + 1, ..., q, \end{cases}$$

with $\beta_1(f(0)), ..., \beta_{m^+}(f(0)), \mu_1(f(0)), ..., \mu_{k^+}(f(0))$ and $\eta_1(f(0)), ..., \eta_l(f(0))$ given by the system

$$\begin{aligned} &\sum_{r=1}^{m^+} \beta_r(f(0))\, \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_i(f(0))\, \mathcal{C} y_i + \sum_{j=1}^{l} \eta_j(f(0))\, \mathcal{C} z_j \\ =& h_0 + \sum_{r=m^+ + 1}^{q} \int_0^\infty e^{\tau \lambda_r} \widetilde{\beta}_r(f(\tau))\, d\tau\, \mathcal{C} u_r, \end{aligned}$$

and

$$\mathcal{C} = R_+ - C R_-,\ \widetilde{\eta}_j(f) = \langle S(f, f), w_j \rangle,\ \text{and}\ \widetilde{\beta}_r(f) = \langle S(f, f), u_r \rangle.$$

Lemma 5.2. *Let $f, h \in \mathcal{X}$ and assume that the condition (26) is fulfilled. Then there is a positive constant K (independent of f and h), such that*

$$\left|\Theta(0)\right|_\sigma \leq K \left|h_0\right|, \tag{44}$$

$$\left|\Theta(f) - \Theta(h)\right|_\sigma \leq K(\left|f\right|_\sigma + \left|h\right|_\sigma) \left|f - h\right|_\sigma. \tag{45}$$

Proof. Let $\mathcal{C}^{-1}$ denote the inverse map of the linear map $\mathcal{C} = R_+ - C R_-$ on $X_+ = \operatorname{span}(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1, ..., z_l)$. The map $\mathcal{C}^{-1}$ is also linear and therefore bounded. We denote

$$P = (u_1 ... u_q y_1 ... y_k z_1 ... z_l w_1 ... w_l)$$

(cf. Eqs. (17)-(19)). By Eq. (19)

$$P^{-1} = D^{-1} \widetilde{P}^t B, \text{ where}$$

$$\widetilde{P} = (u_1 ... u_q y_1 ... y_k z_1 ... z_l w_1 ... w_l) \text{ and } D = \operatorname{diag}(\lambda_1 ... \lambda_q \gamma_1 ... \gamma_k 1 ... 1).$$

Then

$$\begin{aligned} \left|\Theta(0)\right|_\sigma &= \left|P P^{-1} \Theta(0)\right|_\sigma \leq \left|P\right| \left|P^{-1} \Theta(0)\right|_\sigma \\ &= \left|P\right| \left| \sum_{r=1}^{m^+} \beta_r(0)\, e^{-\lambda_r x} P^{-1} u_r + e^{-\gamma x} P^{-1} \left(\sum_{i=1}^{k^+} \mu_i(0)\, y_i + \sum_{j=1}^{l} \eta_j(0)\, z_j \right) \right|_\sigma \\ &\leq \left|P\right| \left| P^{-1} \left(\sum_{r=1}^{m^+} \beta_r(0)\, u_r + \sum_{i=1}^{k^+} \mu_i(0)\, y_i + \sum_{j=1}^{l} \eta_j(0)\, z_j \right) \right| \\ &= \left|P\right| \left|P^{-1} \mathcal{C}^{-1} h_0\right| \leq K_0 \left|h_0\right|, \text{ with } K_0 = \left|P\right| \left|P^{-1} \mathcal{C}^{-1}\right|. \end{aligned}$$

Clearly,

$$\left|f\right|_\sigma < \infty \Rightarrow \left|S(f, f)\right|_{2\sigma} < \infty,$$

and

$$\left|\mathcal{C}^{-1}\mathcal{C}\sum_{r=m^{+}+1}^{q}\int_{0}^{\infty}e^{\tau\lambda_{r}}\left(\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)-\widetilde{\beta}_{r}\left(h\left(\tau\right)\right)\right)\,d\tau\,u_{r}\right|$$
$$\leq\left|\mathcal{C}^{-1}\mathcal{C}P\right|\left|\int_{0}^{\infty}e^{-\tau\sigma}\sum_{r=m^{+}+1}^{q}\left(\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)-\widetilde{\beta}_{r}\left(h\left(\tau\right)\right)\right)P^{-1}u_{r}\,d\tau\right|$$
$$\leq\left|\mathcal{C}^{-1}\mathcal{C}P\right|\int_{0}^{\infty}e^{-3\tau\sigma}\,d\tau\,\left|P^{-1}B^{-1}(S(f,f)-S(h,h))\right|_{2\sigma}.$$

Hence, we obtain

$$\begin{aligned}
&\left|\Theta(f)-\Theta(h)\right|_{\sigma}\\
=&\left|PP^{-1}\left(\Theta(f)-\Theta(h)\right)\right|_{\sigma}\leq\left|P\right|\left|P^{-1}\left(\Theta(f)-\Theta(h)\right)\right|_{\sigma}\\
\leq&\left|P\right|\sup_{x\geq0}\left(\int_{x}^{\infty}e^{(2x-\tau)\sigma}\left|P^{-1}\sum_{r=m^{+}+1}^{q}(\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)-\widetilde{\beta}_{r}\left(h\left(\tau\right)\right))u_{r}\right|d\tau\right.\\
&\quad+\int_{0}^{x}e^{\tau\sigma}\left|P^{-1}\left(\sum_{r=1}^{m^{+}}(\widetilde{\beta}_{r}\left(f\left(\tau\right)\right)-\widetilde{\beta}_{r}\left(h\left(\tau\right)\right))u_{r}\right.\right.\\
&\quad\left.\left.\left.+\sum_{j=1}^{l}(\widetilde{\eta}_{j}\left(f\left(\tau\right)\right)-\widetilde{\eta}_{j}\left(h\left(\tau\right)\right))z_{j}\right)\right|d\tau\right)\\
&\quad+\left|P^{-1}\left(\sum_{r=1}^{m^{+}}(\beta_{r}\left(f(0)\right)-\beta_{r}\left(h(0)\right))u_{r}+\sum_{j=1}^{l}(\eta_{j}\left(f\left(0\right)\right)-\eta_{j}\left(h\left(0\right)\right))z_{j}\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^{k^{+}}(\mu_{i}\left(f\left(0\right)\right)-\mu_{i}\left(h\left(0\right)\right))y_{i})\right)\right|\\
\leq&\left|P\right|(\sup_{x\geq0}\int_{x}^{\infty}e^{(2x-3\tau)\sigma}\,d\tau+\int_{0}^{\infty}e^{-\tau\sigma}\,d\tau+\left|P^{-1}\right|\left|\mathcal{C}^{-1}\mathcal{C}P\right|\int_{0}^{\infty}e^{-3\tau\sigma}\,d\tau)\\
&\quad*\left|P^{-1}B^{-1}(S(f,f)-S(h,h))\right|_{2\sigma}\\
\leq&K_{1}\left|S(f,f)-S(h,h)\right|_{2\sigma},\text{ with }K_{1}=\frac{1}{3\sigma}\left|P\right|\left|D^{-1}\widetilde{P}^{t}\right|\left(4+\left|P^{-1}\right|\left|\mathcal{C}^{-1}\mathcal{C}P\right|\right).
\end{aligned}$$

The quadratic function $S(f,f)$ is bilinear in its arguments and bounded, and hence, there is a positive constant K_2 (independent of f and h), such that

$$\left|S(f,f)-S(h,h)\right|=\left|S(f+h,f-h)\right|\leq K_{2}(\left|f\right|+\left|h\right|)\left|f-h\right|.$$

Therefore,

$$\left|S(f,f)-S(h,h)\right|_{2\sigma}\leq K_{2}(\left|f\right|_{\sigma}+\left|h\right|_{\sigma})\left|f-h\right|_{\sigma}.$$

Let $K=\min(K_0,K_1K_2)$. □

Theorem 5.3. *Let condition* (26) *be fulfilled. Then there is a positive number* δ_0*, such that if*

$$\left|h_{0}\right|\leq\delta_{0},$$

then the system (36) with the boundary conditions (21) and (24) has a unique solution $f = f(x)$ in $\mathcal{S}_R$ for a suitable chosen R.

Proof. By estimates (44) and (45), there is a positive number K such that

$$\left|\Theta(f)\right|_\sigma = \left|\Theta(f) - \Theta(0) + \Theta(0)\right|_\sigma \leq K(\left|h_0\right| + \left|f\right|_\sigma^2) \tag{46}$$

if $f \in \mathcal{X}$.

Let $R = 2K$ and let δ_0 be a positive number, such that $\delta_0 < \dfrac{1}{R^2}$. By estimates (45) and (46)

$$\left|\Theta(f)\right|_\sigma \leq (\frac{1}{2} + 2K^2 \left|h_0\right|) R \left|h_0\right| \leq R \left|h_0\right|$$

and

$$\left|\Theta(f) - \Theta(h)\right|_\sigma \leq 2KR \left|h_0\right| \left|f - h\right|_\sigma \leq R^2 \delta_0 \left|f - h\right|_\sigma, \; R^2\delta_0 < 1,$$

if $f, h \in \mathcal{S}_R$ and $\left|h_0\right| \leq \delta_0$.

The theorem follows by the contraction mapping theorem (see Ref. [33, p.2]). $\square$

Theorem 5.4. *Suppose that $\left\langle S\left(f, f\right), w_j\right\rangle = 0$ for $j = 1, ..., l$. Then the solution of Theorem 5.3 is a solution of the problem (20), (21) and (24) if and only if $P_0^+ f(0) = 0$.*

Proof. The relations

$$\left\{\begin{array}{l} \mu_i(f(x)) = \mu_i(f(0)) e^{-\gamma x}, \; i = 1, ..., k^+, \\ \eta_j(f(x)) = \eta_j(f(0)) e^{-\gamma x}, \; j = 1, ..., l, \end{array}\right.$$

are fulfilled if $f(x)$ is a solution of Theorem 5.3 and $\left\langle S\left(f, f\right), w_j\right\rangle = 0$. Hence, $P_0^+ f(0) = 0$ if and only if $P_0^+ f(x) \equiv 0$. $\square$

We denote by $\mathbb{I}^\gamma$ the linear solution operator

$$\mathbb{I}^\gamma(h_0) = f(0),$$

where $f(x)$ is given by

$$\left\{\begin{array}{l} B\dfrac{df}{dx} + Lf + \gamma P_0^+ f = 0 \\ \mathcal{C} f(0) = h_0 \\ f \rightarrow 0, \text{ as } x \rightarrow \infty \end{array}\right. .$$

Similarly, we denote by $\mathcal{I}^\gamma$ the nonlinear solution operator

$$\mathcal{I}^\gamma(h_0) = f(0),$$

where $f(x)$ is given by

$$\left\{\begin{array}{l} B\dfrac{df}{dx} + Lf = S(f, f) - \gamma P_0^+ f \\ \mathcal{C} f(0) = h_0 \\ f \rightarrow 0, \text{ as } x \rightarrow \infty \end{array}\right. .$$

We assume that $\left\langle S\left(f, f\right), w_j\right\rangle = 0$ for $j = 1, ..., l$. By Theorem 5.4, the solution of Theorem 5.3 is a solution of the problem (20), (21) and (24) if and only if $P_0^+ \mathcal{I}^\gamma(h_0) \equiv 0$.

Let

$$r_i = \frac{r_i'}{\sqrt{\langle r_i', r_i' \rangle_{B_+}}},$$

$$\text{with } r_i' = \mathcal{C}y_i - \sum_{r=1}^{m^+} \frac{\langle \mathcal{C}y_i, \mathcal{C}u_r \rangle_{B_+}}{\langle \mathcal{C}u_r, \mathcal{C}u_r \rangle_{B_+}} \mathcal{C}u_r - \sum_{j=1}^{i-1} \langle \mathcal{C}y_i, r_j \rangle_{B_+} r_j \neq 0, \, i = 1, ..., k^+,$$

and

$$r_{i+k^+} = \frac{r_{i+k^+}'}{\sqrt{\left\langle r_{i+k^+}', r_{i+k^+}' \right\rangle_{B_+}}},$$

$$\text{with } r_{i+k^+}' = \mathcal{C}z_i - \sum_{r=1}^{m^+} \frac{\langle \mathcal{C}z_i, \mathcal{C}u_r \rangle_{B_+}}{\langle \mathcal{C}u_r, \mathcal{C}u_r \rangle_{B_+}} \mathcal{C}u_r - \sum_{j=1}^{i+k^+-1} \langle \mathcal{C}z_i, r_j \rangle_{B_+} r_j \neq 0, \, i = 1, ..., l.$$

Then

$$P_0^+ \mathbb{I}^\gamma \equiv 0 \Leftrightarrow h_0 \in \mathcal{R}^{\perp_{B_+}},$$
$$\text{with } \mathcal{R}^{\perp_{B_+}} = \left\{ u \in \mathbb{R}^{n^+} \,\middle|\, \langle u, r_i \rangle_{B_+} = 0 \text{ for } i = 1, ..., k^+ + l \right\}$$

and

$$\mathcal{I}^\gamma(h_0) \equiv \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1),$$
$$\text{with } h_0 = \sum_{i=1}^{k^++l} a_i r_i + h_1, \, h_1 \in \mathcal{R}^{\perp_{B_+}} \text{ and } a_i = \langle h_0, r_i \rangle_{B_+}.$$

Lemma 5.5. *Suppose that $P_0^+ \mathcal{I}^\gamma(h_0) \equiv 0$. Then h_0 is a function of h_1 if $\langle h_0, h_0 \rangle_{B_+}$ is sufficiently small.*

Proof. It is obvious that $\mathcal{I}^\gamma(0) = 0$ and that we for the Fréchet derivative of $\mathcal{I}^\gamma(\epsilon h_0)$ have

$$\frac{d}{d\epsilon} \mathcal{I}^\gamma(\epsilon h_0) \Big|_{\epsilon=0} = \mathbb{I}^\gamma(h_0).$$

Then if $h_0 = r_i$

$$\frac{\partial}{\partial a_i} \left\langle \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1), u \right\rangle_B \Big|_{a_i=0} = \frac{d}{d\epsilon} \langle \mathcal{I}^\gamma(\epsilon h_0), u \rangle_B \Big|_{\epsilon=0} = \langle \mathbb{I}^\gamma(h_0), u \rangle_B \neq 0,$$

where $u = y_i$ if $i = 1, ..., k^+$ and $u = w_{i-k^+}$ if $i = k^+ + 1, ..., k^+ + l$. By the implicit function theorem, $\left\langle \widetilde{\mathcal{I}}^\gamma(a_1, ..., a_{k^++l}, h_1), y_1 \right\rangle_B = 0$ defines $a_1 = a_1(a_2, ..., a_{k^++l}, h_1)$. Induction gives that

$$a_1 = a_1(h_1), ..., a_{k^++l} = a_{k^++l}(h_1).$$

$\square$

6. **Direct approach without damping term.** In this section we deal directly with the general system (20) with the boundary conditions (21) and (24).

Let $g = g(x) : \mathbb{R}_+ \to \mathbb{R}^n$ be a given function, such that $g(x) \in N(L)^\perp$ for all $x \in \mathbb{R}_+$. The linearized inhomogeneous system

$$B \frac{df}{dx} + Lf = g, \tag{47}$$

with the boundary conditions (21) have (under the assumption that all necessary integrals exist) the general solution

$$f(x)=\sum_{j=1}^{l}\eta_j\left(x\right)z_j+\sum_{r=1}^{q}\beta_r\left(x\right)u_r, \tag{48}$$

where

$$\eta_j\left(x\right)=-\int_x^{\infty}\widetilde{\eta}_j\left(\tau\right)\,d\tau,\ \widetilde{\eta}_j\left(x\right)=\left\langle g\left(x\right),w_j\right\rangle,\ j=1,...,l, \tag{49}$$

and $\beta_1\left(x\right),...,\beta_q\left(x\right)$ are given by Eq. (40). From the boundary conditions (24), we obtain the system

$$\begin{aligned}\mathcal{C}\sum_{r=1}^{m^+}\beta_r\left(0\right)u_r &= h_0+\mathcal{C}\int_0^{\infty}\sum_{r=m^++1}^{q}e^{\tau\lambda_r}\widetilde{\beta}_r\left(\tau\right)u_r+\sum_{j=1}^{l}\widetilde{\eta}_j\left(\tau\right)z_j\,d\tau, \\ \text{with } \mathcal{C} &= R_+-CR_-.\end{aligned} \tag{50}$$

The system (50) has (under the assumption that all necessary integrals exist) a solution if we assume that

$$\begin{aligned}&h_0,\mathcal{C}e^{xB^{-1}L}B^{-1}g(x)\in\mathcal{C}U_+\text{ for all }x\in\mathbb{R}_+,\\ &\text{with }U_+=\operatorname{span}(u:Lu=\lambda Bu,\,\lambda>0)=\operatorname{span}\left(u_1,...,u_{m^+}\right)\end{aligned} \tag{51}$$

and a unique solution if and only if, additionally, condition (25) is fulfilled.

Theorem 6.1. *Assume that the conditions (25), (38) and (51) are fulfilled and that all necessary integrals exist. Then the system (47) with the boundary conditions (21) and (24) has a unique solution given by Eqs. (48)-(50).*

Following the lines of the proof of Theorem 5.3 we now obtain Theorem 3.2.

7. **Extension to singular operators** B. To study the case when the operator B is singular (i.e. the case when $\xi_i^1=0$ for some i) we assume (cf. Refs. [29] and [7]) that

$$N(L)\cap N(B)=\left\{0\right\}, \tag{52}$$

and introduce the orthogonal projections

$$P_0:\mathbb{R}^n\rightarrow N(B)\text{ and }P_1:\mathbb{R}^n\rightarrow\operatorname{Im}(B).$$

The assumption (52) ensures that the operator P_0LP_0 is non-singular on $N(B)$.

The system (47) is equivalent with the system (see Ref. [7])

$$\left\{\begin{array}{l}P_0f=-\left(P_0LP_0\right)^{-1}P_0LP_1f+\left(P_0LP_0\right)^{-1}P_0g(x)\\ \widetilde{B}\dfrac{dP_1f}{dx}+\widetilde{L}P_1f=\widetilde{g}(x)\end{array}\right.,$$

where

$$\begin{aligned}&\widetilde{L}=P_1L(I-P_0\left(P_0LP_0\right)^{-1}P_0L)P_1,\ \widetilde{B}=P_1BP_1\text{ and}\\ &\widetilde{g}(x)=P_1(I-LP_0\left(P_0LP_0\right)^{-1}P_0)g(x).\end{aligned}$$

The restrictions, $\widetilde{L}_{\mathrm{Im}}$ and $\widetilde{B}_{\mathrm{Im}}$, of $\widetilde{L}$ and $\widetilde{B}$ to $\operatorname{Im}(B)$, are linear operators ($\widetilde{n}\times\widetilde{n}$ matrices, with $\widetilde{n}=n-\dim(N(B)))$ on $\operatorname{Im}(B)$.

Lemma 7.1. *Let $g(x) \in N(L)^{\perp}$ and assume that*

$$N(L) \cap N(B) = \{0\} .$$

Then the linear operators $\widetilde{L}_{\mathrm{Im}}$ and $\widetilde{B}_{\mathrm{Im}}$ on $\mathrm{Im}(B)$ have the following properties: $\widetilde{L}_{\mathrm{Im}}$ and $\widetilde{B}_{\mathrm{Im}}$ are real symmetric operators, $\widetilde{L}_{\mathrm{Im}}$ is semi-positive, $\widetilde{B}_{\mathrm{Im}}$ is non-singular, $\dim(N(\widetilde{L}_{\mathrm{Im}})) = p$, and the numbers k^+, k^- and l are the same for the system

$$\widetilde{B}_{\mathrm{Im}} \frac{dP_1 f}{dx} + \widetilde{L}_{\mathrm{Im}} P_1 f = \widetilde{g}(x), \tag{53}$$

as for the original system (47). Furthermore, $\widetilde{g}(x) \in N(\widetilde{L})^{\perp}$.

Proof. (cf. Ref. [7]) It is clear that the operators $\widetilde{L}$ and $\widetilde{B}$ are real and symmetric and that $\widetilde{B}$ is non-singular on $\mathrm{Im}(B)$. Hence, this is true also for the restrictions to $\mathrm{Im}(B)$. The linear operator $\widetilde{L}_{\mathrm{Im}}$ is semi-positive, since

$$\begin{aligned} 0 &= \left\langle (P_0 L - P_0 L P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h, \left(P_0 L P_0\right)^{-1} P_0 L P_1 h \right\rangle \\ &= \left\langle L(I - P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h, P_0 \left(P_0 L P_0\right)^{-1} P_0 L P_1 h \right\rangle \end{aligned}$$

if $h \in \mathrm{Im}(B)$, and hence,

$$\begin{aligned} \left\langle \widetilde{L}_{\mathrm{Im}} h, h \right\rangle &= \left\langle \widetilde{L} h, h \right\rangle = \left\langle L(I - P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h, P_1 h \right\rangle \\ &= \left\langle L(I - P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h, (I - P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h \right\rangle \geq 0 \end{aligned}$$

for all $h \in \mathrm{Im}(B)$. By assumption (52), $\dim(N(\widetilde{L})) = \dim(N(L)) = p$ and $N(\widetilde{L}) \subseteq P_1 N(L)$, since

$$\begin{aligned} \widetilde{L} P_1 h &= P_1 L(I - P_0 \left(P_0 L P_0\right)^{-1} P_0 L) P_1 h \\ &= P_1 L(P_1 + P_0 \left(P_0 L P_0\right)^{-1} P_0 L P_0) h = P_1 L(P_1 + P_0) h = 0 \end{aligned}$$

if $Lh = L(P_0 + P_1)h = 0$, for $h \in \mathbb{R}^n$. Hence,

$$N(\widetilde{L}_{\mathrm{Im}}) = N(\widetilde{L}) = P_1 N(L).$$

Furthermore, the numbers k^+, k^- and l are the same for the system (53) as for the original system, since

$$\left\langle \widetilde{B}_{\mathrm{Im}} P_1 h, P_1 h \right\rangle = \langle B P_1 h, P_1 h \rangle = \langle Bh, h \rangle$$

for all $h \in \mathbb{R}^n$. If $h \in N(L)$ then

$$\begin{aligned} &\left\langle P_1 L P_0 \left(P_0 L P_0\right)^{-1} P_0 g(x), P_1 h \right\rangle \\ &= \left\langle L P_0 \left(P_0 L P_0\right)^{-1} P_0 g(x), P_1 h \right\rangle \\ &= - \left\langle L P_0 \left(P_0 L P_0\right)^{-1} P_0 g(x), P_0 h \right\rangle = - \langle P_0 g(x), h \rangle \end{aligned}$$

and so

$$\langle \widetilde{g}(x), P_1 h \rangle = \langle P_1 g(x), h \rangle + \langle P_0 g(x), h \rangle = \langle g(x), h \rangle = 0.$$

Hence,

$$\widetilde{g}(x) \in (P_1 N(L))^{\perp} = N(\widetilde{L})^{\perp},$$

and the lemma is proved. □

Denote

$$\widetilde{\Theta}(f) = \Theta(P_1 f),$$

where Θ is the operator (43) when L and $S(f,f)$ are replaced with $\widetilde{L}$ and $\widetilde{S}(f,f) = P_1(I - LP_0 (P_0LP_0)^{-1} P_0)S(f,f)$ in Eq. (36). We introduce

$$\widehat{\Theta}(f) = \left(I - (P_0LP_0)^{-1} P_0L\right) \widetilde{\Theta}(f) + (P_0LP_0)^{-1} P_0S(f,f)$$

and denote

$$\lambda_{\min} = \min |\lambda_i| \text{ and } \lambda_{\max} = \max |\lambda_i|,$$

where $\lambda_1, ..., \lambda_{n-p}$ are the non-zero eigenvalues of L. Then

$$\left|\widehat{\Theta}(0)\right|_\sigma = \left|\left(I - (P_0LP_0)^{-1} P_0L\right) \widetilde{\Theta}(0)\right|_\sigma \leq \widehat{K}_0 |h_0|, \text{ with } \widehat{K}_0 = (1 + \lambda_{\min}^{-1}\lambda_{\max})\widetilde{K},$$

and

$$\begin{aligned}
&\left|\widehat{\Theta}(f) - \widehat{\Theta}(g)\right|_\sigma \\
=& \left|\left(I - (P_0LP_0)^{-1} P_0L\right) \left(\widetilde{\Theta}(f) - \widetilde{\Theta}(g)\right) + (P_0LP_0)^{-1} P_0 \left(S(f,f) - S(g,g)\right)\right|_\sigma \\
\leq& (1 + \lambda_{\min}^{-1}\lambda_{\max})\widetilde{K}(|f|_\sigma + |h|_\sigma) |f - h|_\sigma + \lambda_{\min}^{-1} K_2(|f|_\sigma + |h|_\sigma) |f - h|_\sigma \\
=& \widehat{K}_1(|f|_\sigma + |h|_\sigma) |f - h|_\sigma, \text{ with } \widehat{K}_1 = \widetilde{K} + \lambda_{\min}^{-1}(\widetilde{K}\lambda_{\max} + K_2).
\end{aligned}$$

We can now extend our main results in Section 3 to yield also for singular operators B.

8. **Axially symmetric DVMs.** In this section we consider only such symmetric sets of velocities V, such that

$$\text{if } \xi_i = (\xi_i^1, ..., \xi_i^d) \in \mathrm{V}, \text{ then } (\pm\xi_i^1, ..., \pm\xi_i^d) \in \mathrm{V} \tag{54}$$

for any combinations of signs (see also Ref. [7]). We can, without loss of generality, assume that

$$(\xi_{i+N}^1, \xi_{i+N}^2, ..., \xi_{i+N}^d) = (-\xi_i^1, \xi_i^2, ..., \xi_i^d) \text{ and } \xi_i^1 > 0,$$

for $i = 1, ..., N$, with $n = 2N$.

Example 3. The plane 12-velocity model in Ref. [11], with velocities

$$(\pm 1, \pm 1), \ (\pm 1, \pm 3) \text{ and } (\pm 3, \pm 1),$$

and the infinitely many (obvious, from the constructions in Ref. [11] - "with three corners of a square in the model, add the fourth") symmetric normal extensions of this model are examples of (normal) such DVMs. These extensions include the plane square models, with (all combinations of) coordinates from the set of all odd integers with absolute values less or equal than a maximal odd integer (these models are called Nicodin p-th squares in Ref. [20], but are also, at least implicitly, constructed in Ref. [11]).

Example 4. In three dimensions the 32-velocity model, with velocities

$$(\pm 1, \pm 1, \pm 1), \ (\pm 1, \pm 1, \pm 3), \ (\pm 1, \pm 3, \pm 1) \text{ and } (\pm 3, \pm 1, \pm 1),$$

and the infinitely many (obvious) symmetric normal extensions of this model are examples of such (normal) DVMs. These extensions include the cubic models, with (all combinations of) coordinates from the set of all odd integers with absolute values less or equal than a maximal odd integer. The 32-velocity model can be obtained by normal extensions, with the starting point in the 9-velocity asymmetric normal model with velocities $(\pm 1, \pm 1, \pm 1)$ and $(3, -1, 1)$. The 24-velocity models, with velocities $(\pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 3, \pm 1)$ and $(\pm 3, \pm 1, \pm 1)$, and $(\pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 3)$ and $(\pm 3, \pm 1, \pm 1)$, respectively, are DVMs with fewer velocities (earlier in the "evolution"), that can be constructed from the same asymmetric model.

8.1. **Explicit calculation of the characteristic numbers.** We now assume that (i) we have a symmetric set (54) of velocities; (ii) our DVM is normal; (iii) we have made the expansion (13) around a non-drifting Maxwellian M, i.e. with $\mathbf{b} = \mathbf{0}$ in Eq. (12); and (iv)

$$B = \mathrm{diag}(\xi_1^1, ..., \xi_N^1, -\xi_1^1, ..., -\xi_N^1), \text{ with } \xi_1^1, ..., \xi_N^1 > 0.$$

In this section we study, instead of Eq. (20), the equation

$$(B + uI)\frac{df}{dx} + Lf = S(f, f), \tag{55}$$

(cf. Eq. (3)). Note, however, that Eqs. (20) and (55) are never equivalent for non-zero u, as Eqs. (2) and (3) are in the continuous case, for DVMs with a finite number of velocities.

The linearized collision operator L has the null-space

$$N(L) = \mathrm{span}\,(\phi_1, ..., \phi_{d+2}),$$

where

$$\begin{cases} \phi_1 = M^{1/2} \cdot (1, ..., 1) \\ \phi_2 = M^{1/2} \cdot (\xi_1^1, ..., \xi_N^1, -\xi_1^1, ..., -\xi_N^1) \\ \phi_3 = M^{1/2} \cdot (|\xi_1|^2, ..., |\xi_N|^2, |\xi_1|^2, ..., |\xi_N|^2) \\ \phi_{i+2} = M^{1/2} \cdot (\xi_1^i, ..., \xi_N^i, -\xi_1^i, ..., -\xi_N^i),\ i = 2, ..., d, \end{cases} . \tag{56}$$

Then the degenerate values of u, i.e. the values of u for which $l \geq 1$, are

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5 - \chi_3^2)}}, \tag{57}$$

where $K = (\langle\phi_i, \phi_j\rangle_{B+uI})$, $\chi_1 = \langle\phi_1, \phi_1\rangle$, $\chi_2 = \langle\phi_2, \phi_2\rangle$, $\chi_3 = \langle\phi_1, \phi_3\rangle$, $\chi_4 = \langle\phi_2, \phi_3\rangle_B$, $\chi_5 = \langle\phi_3, \phi_3\rangle$, see Ref. [7]. Moreover, we have the following table for the values of k^+, k^- and l (see Ref. [7]):

	$u < u_-$	$u = u_-$	$u_- < u < 0$	$u = 0$	$0 < u < u_+$	$u = u_+$	$u_+ < u$
k^+	0	0	1	1	$d+1$	$d+1$	$d+2$
k^-	$d+2$	$d+1$	$d+1$	1	1	0	0
l	0	1	0	d	0	1	0

Example 5. The degenerate values of u for the 12-velocity model in Example 3 are

$$u = 0 \text{ and } u = \pm\sqrt{\frac{1 + 50s^2}{1 + 10s^2}},$$

and for the 32-velocity model in Example 4 the degenerate values of u are

$$u = 0 \text{ and } u = \pm\sqrt{\frac{3 + 121s^2}{3 + 33s^2}},$$

where, in both cases, $s = e^{4c}$ and c is a negative constant given by the Maxwellian M, cf. Eq. (12).

Remark 8. For the continuous Boltzmann equation (with $d = 3$) the numbers $\chi_1, ..., \chi_5$ are given by

$$\chi_1 = \rho,\ \chi_2 = \rho T,\ \chi_3 = 3\rho T,\ \chi_4 = 5\rho T^2 \text{ and } \chi_5 = 15\rho T^2,$$

(where ρ and T denote the density and the temperature respectively), if we have made the expansion (13) around a non-drifting Maxwellian

$$M = \frac{\rho}{(2\pi T)^{3/2}} e^{-|\xi|^2/2T}.$$

Therefore, for the Boltzmann equation (with $d = 3$) the degenerate values (57) are (cf. Ref. [21])

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{5T}{3}}.$$

Below we return to study Eq. (20).

8.2. **Plane 12-velocity model.** For $d = 2$ the equations (5) admit a class of solutions satisfying

$$F_i = F_{i'} \text{ if } \xi_i^1 = \xi_{i'}^1 \text{ and } |\xi_i|^2 = |\xi_{i'}|^2 . \tag{58}$$

This reduces the number n of equations (5) to the number $2N < n$ of different combinations $(\xi_i^1, |\xi_i|^2)$ in the velocity set. However, the structure of the collision terms (7) (in slightly different notations) remains unchanged. We can, without loss of generality, assume that

$$(\xi_{i+N}^1, |\xi_{i+N}|^2) = (-\xi_i^1, |\xi_i|^2) \text{ and } \xi_i^1 > 0$$

for $i = 1, ..., N$. Then, the Maxwellians are of the form

$$M_i = Ae^{b\xi_i^1 + c|\xi_i|^2} = M_{i+N}e^{2b\xi_i^1},\ i = 1, ..., N,$$

for some constant $A, b, c \in \mathbb{R}$, with $A > 0$.

For the 12-velocity model in Example 3 (see Ref. [7]), the system (5) reduces by reduction (58) to a system of the form

$$\begin{cases} \dfrac{dF_1}{dx} = \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3 \\ \dfrac{dF_2}{dx} = \sigma_1 q_1 - \sigma_2 q_2 + \sigma_4 q_4 \\ 3\dfrac{dF_3}{dx} = -(\sigma_1 q_1 + \sigma_4 q_4) \\ -\dfrac{dF_4}{dx} = -(\sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3) \\ -\dfrac{dF_5}{dx} = \sigma_2 q_2 - \sigma_3 q_3 + \sigma_4 q_4 \\ -3\dfrac{dF_6}{dx} = \sigma_3 q_3 - \sigma_4 q_4 \end{cases},$$

where

$$q_1 = F_3F_4 - F_1F_2, q_2 = F_2F_4 - F_1F_5, q_3 = F_4F_5 - F_1F_6, q_4 = F_3F_6 - F_2F_5$$

and $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \geq 0$.

We assume below that we have linearized around a non-drifting Maxwellian

$$M = K(1, s^2, s^2, 1, s^2, s^2),$$

where $s = e^{4c}$ and $K = Ae^{2c}$, c and A are constant, with $A > 0$. The null-space of L is given by

$$N(L) = \operatorname{span}(\phi_1, \phi_2, \phi_3),$$

where

$$\begin{cases} \phi_1 = K^{1/2}(1, s, s, 1, s, s) \\ \phi_2 = K^{1/2}(1, s, 3s, -1, -s, -3s) \\ \phi_3 = 2K^{1/2}(1, 5s, 5s, 1, 5s, 5s) \end{cases}$$

and

$$B = \operatorname{diag}(1, 1, 3, -1, -1, -3).$$

A typical choice of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ (cf. Refs. [15] and [23]) is

$$\begin{cases} \sigma_1 = \sigma_3 = 2S \\ \sigma_2 = S(\sqrt{2} + \sqrt{5}) \\ \sigma_4 = S\sqrt{10}, \end{cases}.$$

Therefore, we assume below that $\sigma_1 = \sigma_3, \sigma_2, \sigma_4 > 0$. Then

$$L = \left(\begin{array}{cccccc} (2\sigma_1 + \sigma_2)s^2 & (\sigma_1 - \sigma_2)s & -\sigma_1 s & -(2\sigma_1 + \sigma_2)s^2 & (\sigma_2 - \sigma_1)s & \sigma_1 s \\ (\sigma_1 - \sigma_2)s & \sigma_1 + \sigma_2 + \sigma_4 s^2 & -(\sigma_1 + \sigma_4 s^2) & (\sigma_2 - \sigma_1)s & -\sigma_2 + \sigma_4 s^2 & -\sigma_4 s^2 \\ -\sigma_1 s & -(\sigma_1 + \sigma_4 s^2) & \sigma_1 + \sigma_4 s^2 & \sigma_1 s & -\sigma_4 s^2 & \sigma_4 s^2 \\ -(2\sigma_1 + \sigma_2)s^2 & (\sigma_2 - \sigma_1)s & \sigma_1 s & (2\sigma_1 + \sigma_2)s^2 & (\sigma_1 - \sigma_2)s & -\sigma_1 s \\ (\sigma_2 - \sigma_1)s & -\sigma_2 + \sigma_4 s^2 & -\sigma_4 s^2 & (\sigma_1 - \sigma_2)s & \sigma_1 + \sigma_2 + \sigma_4 s^2 & -(\sigma_1 + \sigma_4 s^2) \\ \sigma_1 s & -\sigma_4 s^2 & \sigma_4 s^2 & -\sigma_1 s & -(\sigma_1 + \sigma_4 s^2) & \sigma_1 + \sigma_4 s^2 \end{array} \right)$$

and

$$S(f, f) = \begin{pmatrix} s\sigma_1(q_1 + q_3) + s\sigma_2 q_2 \\ \sigma_1 q_1 - \sigma_2 q_2 + s\sigma_4 q_4 \\ -\sigma_1 q_1 - s\sigma_4 q_4 \\ -s\sigma_1(q_1 + q_3) - s\sigma_2 q_2 \\ \sigma_2 q_2 - \sigma_1 q_3 + s\sigma_4 q_4 \\ \sigma_1 q_3 - s\sigma_4 q_4 \end{pmatrix},$$

and if we denote

$$\begin{cases} y_1 = (5\phi_1 + \phi_2 - \dfrac{\phi_3}{2})K^{-1/2} = (5, s, 3s, 3, -s, -3s) \\ y_2 = (5\phi_1 - \phi_2 - \dfrac{\phi_3}{2})K^{-1/2} = (3, -s, -3s, 5, s, 3s) \\ z = (\dfrac{\chi_3\phi_1 - \chi_2\phi_3}{8s})K^{-1/2} = (10s, -1, -1, 10s, -1, -1) \\ w = \left(0, -\dfrac{2}{\sigma_2}, -\dfrac{3}{\sigma_1} - \dfrac{2}{\sigma_2}, 0, \dfrac{2}{\sigma_2}, \dfrac{3}{\sigma_1} + \dfrac{2}{\sigma_2}\right), \end{cases}$$

then

$$\begin{aligned}
&y_1, y_2, z \in N(L),\ Lw = Bz,\\
&\langle y_1, y_2\rangle_B = \langle z, z\rangle_B = \langle w, w\rangle_B = \langle z, y_i\rangle_B = \langle w, y_i\rangle_B = 0 \text{ for } i = 1, 2,\\
&\langle y_1, y_1\rangle_B = -\langle y_2, y_2\rangle_B = 16 \text{ and } \langle z, w\rangle_B = \frac{18}{\sigma_1} + \frac{16}{\sigma_2}.
\end{aligned}$$

Furthermore, if

$$\left\{\begin{array}{l} u_1 = \sqrt{2}\widetilde{u}_1 + [2\sqrt{2}\sigma_1 - \sqrt{(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}]\widetilde{u}_2 \\ u_2 = \sqrt{2}\widetilde{u}_1 + [2\sqrt{2}\sigma_1 + \sqrt{(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}]\widetilde{u}_2 \end{array}\right. , \text{ with}$$

$$\begin{aligned}
&\widetilde{u}_1 = (3s(3\sigma_2 - 4\sigma_1), -3(4\sigma_1 + 3\sigma_2), 4\sigma_1, 3s(3\sigma_2 - 4\sigma_1), -9\sigma_2, 0) \text{ and}\\
&\widetilde{u}_2 = (0, 3, -1, 0, -3, 1),
\end{aligned}$$

then

$$Lu_i = \lambda_i Bu_i \text{ for } i = 1, 2, \text{ with } \lambda_1 = -\lambda_2 = \frac{\sqrt{2(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}}{3},$$

$$\begin{aligned}
&\langle u_1, u_2\rangle_B = \langle u_i, z\rangle_B = \langle u_i, w\rangle_B = \langle u_i, y_j\rangle_B = 0 \text{ for } i, j = 1, 2, \text{ and}\\
&\langle u_1, u_1\rangle_B = -\langle u_2, u_2\rangle_B = 12(8\sigma_1 + 9\sigma_2)\sqrt{2(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}.
\end{aligned}$$

We have that

$$\begin{aligned}
&(R_+ - R_-)u_1 = -2\sqrt{(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}R_+\widetilde{u}_2 = -(R_+ - R_-)u_2 \text{ and}\\
&(R_+ - R_-)z = 0
\end{aligned}$$

and so Theorem 3.2 is applicable for $C = \operatorname{diag}(1,1,1)$ ($C =$ the identity operator) and $h_0 \in \operatorname{span}((0,3,-1))$ sufficiently small (cf. Refs. [2] and [26]).

8.3. **More general axially symmetric DVMs.** Now, additionally to assumptions (i)-(iv) above, we assume that (v) the coefficients Γ_{ij}^{kl} in Eq. (7) satisfy the additional symmetric conditions

$$\Gamma_{ij}^{kl} = \Gamma_{\pi(i)\pi(j)}^{\pi(k)\pi(l)},$$

where $\pi(i) = \left\{\begin{array}{l} i + N, \text{ if } 1 \leq i \leq N, \\ i - N, \text{ if } N + 1 \leq i \leq 2N, \end{array}\right.$.

Then $L = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix}$, where L_1 and L_2 are two $N \times N$ matrices (cf. Refs. [2], [3], [4] and [7]). We choose

$$\left\{\begin{array}{l} \varphi_1 = \phi_2 + \phi_3 \\ \varphi_2 = \phi_2 - \phi_3 \\ \varphi_3 = \chi_4\phi_1 - \chi_2\phi_3 \end{array}\right. .$$

where $\chi_2 = \langle \phi_2, \phi_2\rangle$, $\chi_4 = \langle \phi_2, \phi_3\rangle_B$ and ϕ_1, ϕ_2, ϕ_3 are given in Eq. (56). Then

$$K = 2\chi_2\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $K = (\langle \varphi_i, \varphi_j\rangle_B)$. Hence, $k^+ = k^- = 1$ and $l = d$, since $\phi_4, ..., \phi_{d+2}$ are all orthogonal, with respect to the scalar product $\langle\cdot,\cdot\rangle_B$, to ϕ_1, ϕ_2 and ϕ_3. Since

$B^{-1}L\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \lambda\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$, with $u^+, u^- \in \mathbb{R}^N$, implies that $B^{-1}L\begin{pmatrix} u^- \\ u^+ \end{pmatrix} = -\lambda\begin{pmatrix} u^- \\ u^+ \end{pmatrix}$, we obtain (cf. Refs. [2], [3], [4] and [7]) that the non-negative eigenvalues of $B^{-1}L$ are $\pm\lambda_1, ..., \pm\lambda_{N-l-1}$, where $\lambda_i > 0$, with corresponding eigenvectors $u_1^{\pm}, ..., u_{N-l-1}^{\pm}$, where $R_+u_i^- = R_-u_i^+$ and $R_-u_i^- = R_+u_i^+$.

Therefore, the Jordan normal form of $B^{-1}L$ for $d = 3$ is (the number of blocks $\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$ is equal to the dimension d, that is, in this case 3)

$$\begin{pmatrix}
\lambda_1 &&&&&&&&&&&&&& \\
& \ddots &&&&&&&&&&&&& \\
&& \lambda_{m^+} &&&&&&&&&&&& \\
&&& -\lambda_1 &&&&&&&&&&& \\
&&&& \ddots &&&&&&&&&& \\
&&&&& -\lambda_{m^+} &&&&&&&&& \\
&&&&&& 0 &&&&&&&& \\
&&&&&&& 0 &&&&&&& \\
&&&&&&&& 0 & 1 &&&&& \\
&&&&&&&& 0 & 0 &&&&& \\
&&&&&&&&&& 0 & 1 &&& \\
&&&&&&&&&& 0 & 0 &&& \\
&&&&&&&&&&&& 0 & 1 \\
&&&&&&&&&&&& 0 & 0
\end{pmatrix}.$$

If $C = \mathrm{diag}(1, ..., 1)$, i.e. if C is the identity operator, and $h_0 \in (R_+ - R_-)\mathrm{span}\left(u_1^+, ..., u_{N-l-1}^+\right)$, then the conditions (25) and (28) are fulfilled (see Remark 5).

Under the assumptions (i)-(v) given above, the following theorem (see Ref. [26] for the case of the continuous Boltzmann Equation) follows by Theorem 3.2.

Theorem 8.1. *Let $h_0 \in (R_+ - R_-)U_+$, where $U_+ = span\left(u_1^+, ..., u_{N-l-1}^+\right)$. Then there is a positive number δ_0, such that if*

$$|h_0| \leq \delta_0,$$

then the system (20) with the boundary conditions

$$f(x) \to 0, \text{ as } x \to \infty, \text{ and } (R_+ - R_-)f(0) = h_0,$$

has a locally unique solution $f = f(x)$.

Remark 9. The same problem, for $d = 2$, is also studied by Babovsky in Ref. [2], but then under the quite restrictive condition $\langle S(f, f), w_i\rangle = 0$ for $i = 1, 2$ (in our notations).

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