Regularity for Diffuse Reflection Boundary Problem to the Stationary Linearized Boltzmann Equation in a Convex Domain

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Stationary linearized Boltzmann equation in a convex domain in $\mathbb{R}^3$

We consider

$$\begin{cases} \zeta \cdot \nabla f(x,\zeta) = L(f), \\ \qquad x \in \Omega, \\ \qquad \zeta \in \mathbb{R}^3, \end{cases} \tag{1}$$

where Ω is a C^2 strictly convex bounded domain in $\mathbb{R}^3$.
Here, we consider hard sphere, cutoff hard potential, or cutoff Maxwellian molecular gases, i.e., $0 \leq \gamma \leq 1$.

Diffuse Reflection Boundary Condition

Diffuse Reflection Boundary Condition:

- Velocity distribution function leaving the boundary is in thermal equilibrium with the boundary temperature.
- There is no net flux on boundary.

Existence of solutions:

- Convex domain: Guiraud (1970 J. de Mc.)
- General domain: Esposito, Guo, Kim, and Marra (2013 CMP)

Regularity:

- Continuous alway from the grazing set: Esposito, Guo, Kim, and Marra (2013 CMP)

Regularity for the time evolutional problem (weakly nonlinear):

$$\frac{\partial}{\partial t} f(x,\zeta,t) + \zeta \cdot \bigtriangledown_x f(x,\zeta,t) = L(f) + \Gamma(f,f), \tag{2}$$

- Kim (2011 CMP): Discontinuity from boundary in a nonconvex domain.
- Guo, Kim, Tonon, and Trescases (2016 ARMA): BV estimate in a nonconvex domain.
- Guo, Kim, Tonon, and Trescases (2016 Inv. Math.) : Regularity in a convex domain.

All these results are NOT uniform in time.

In (2016 Inv. Math.), they establish weighted C^1 estimate, which grows severely with time. This motivates us to look at the regularity to the stationary solution directly.

Cut-off hard potential and cut-off Maxwellian gas

We consider
Cross-section:

$$B(|\zeta - \zeta_*|, \theta) = |\zeta - \zeta_*|^\gamma \beta(\theta), \qquad (3)$$

where $0 \le \gamma \le 1$ and $0 \le \beta(\theta) \le C \cos\theta \sin\theta$.

Here, the cross section is for the binary collision operator

$$J(F, F) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (F' F'_* - FF_*) B(|\zeta_* - \zeta|, \theta) d\theta d\epsilon d\xi_*. \qquad (4)$$

Properties of the collision operator

$$L(f) = -\nu(|\zeta|)f + K(f),$$

$$K(f)(x,\zeta) = \int_{\mathbb{R}^3} k(\zeta,\zeta_*)f(x,\zeta_*)d\zeta_*,$$

$$\nu(|\zeta|) = \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2}|\eta-\zeta|^\gamma d\eta$$

$$\nu_0(1+|\zeta|)^\gamma \leq \nu(|\zeta|) \leq \nu_1(1+|\zeta|)^\gamma.$$

Estimates for Kernel

Let $0 < \delta < 1$.

$$|k(\zeta, \zeta_*)| \leq$$

$$C_1 |\zeta - \zeta_*|^{-1} (1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)} e^{-\frac{1}{4}(1-\delta)\left(|\zeta-\zeta_*|^2 + (\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)},$$

$$| \nabla_\zeta k(\zeta, \zeta_*)| \leq$$

$$C_2 \frac{1 + |\zeta|}{|\zeta - \zeta_*|^2} (1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)} e^{-\frac{1}{4}(1-\delta)\left(|\zeta-\zeta_*|^2 + (\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}.$$

Definition of solution

We define

$$p(x,\zeta) : \text{backward trajectory} \cap \partial\Omega,$$
$$\tau_-(x,\zeta) : \text{traveling time.}$$

We write

$$\zeta \cdot \nabla f(x,\zeta) + \nu(|\zeta|) f(x,\zeta) = K(f). \tag{5}$$

Integral equation:

$$f(x,\zeta) = f(p(x,\zeta),\zeta) e^{-\nu\tau_-(x,\zeta)} + \int_0^{\tau_-(x,\zeta)} e^{-\nu s} K(f)(x-\zeta s,\zeta)\, ds. \tag{6}$$

We say $f(x,\zeta)$ is a solution to the stationary linearized Boltzmann equation if the integral equation is satisfied a. e.

Holder Continutiy

- (C. 2016) Local Hölder continuity for incoming boundary condition.

  Key idea: Velocity averaging for stationary equation.

Main Theorem

Theorem (C., Hsia, Kawagoe, 2017)

Let Ω be a C^2 strictly convex domain in $\mathbb{R}^3$ and $f \in L^\infty_{x,\zeta}$ be a stationary solution to the diffuse reflection boundary problem within Ω, (1). *Suppose the derivative of the boundary temperature is bounded. Then, for $\epsilon > 0$,*

$$\sum_{i=1}^{3} \left|\frac{\partial}{\partial_{x_i}} f(x,\zeta)\right| + \sum_{i=1}^{3} \left|\frac{\partial}{\partial_{\zeta_i}} f(x,\zeta)\right| \leq C(1 + d_x^{-1})^{\frac{4}{3}+\epsilon}, \qquad (7)$$

where d_x is the distance between x and $\partial\Omega$.

Sketch of proof

Diffuse reflection boundary condition for linearized Boltzmann equation:
Let $T(x)$ be the temperature on the boundary.
For $x \in \partial\Omega$ and $\zeta \cdot n(x) < 0$,

$$f(x,\zeta) = \sigma(x) M^{\frac{1}{2}} + T(x)(|\zeta|^2 - \frac{3}{2}) M^{\frac{1}{2}}, \tag{8}$$

where

$$M = M(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}.$$

$$\sigma(x) = -\frac{1}{2} T(x) + 2\sqrt{\pi} \int_{\zeta \cdot n > 0} f(x,\zeta) |\zeta \cdot n| M^{\frac{1}{2}} d\zeta. \tag{9}$$

Let

$$\psi(x) := 2\sqrt{\pi}\int_{\zeta\cdot n>0} f(x,\zeta)|\zeta\cdot n|M^{\frac{1}{2}}d\zeta. \tag{10}$$

Substitute f above by the integral equation (6) and boundary condition (8) and (9).

$$\begin{aligned}\psi(x) =& 2\sqrt{\pi}\int_{\zeta\cdot n>0} T(p(x,\zeta))(|\zeta|^2-2)M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta\\ &+2\sqrt{\pi}\int_{\zeta\cdot n>0}\psi(p(x,\zeta))M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta\\ &+2\sqrt{\pi}\int_{\zeta\cdot n>0}\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}K(f)(x-s\zeta,\zeta)M^{\frac{1}{2}}(\zeta)|\zeta\cdot n|dsd\zeta\\ =:& B_T+B_\psi+D_f.\end{aligned} \tag{11}$$

Sketch of proof:

- ψ is bounded provided $f \in L^\infty_{x,\zeta}$
- First derivatives of B_T, B_ψ are bounded provided T, ψ are bounded.
- D_f is Hölder continuous provided $f \in L^\infty_{x,\zeta}$.

Now, we can conclude f is locally Hölder continuous by using analysis in (C. 2016). However, we shall further improve the regularity to differentiability.

- D_f is bounded differentiable provided f is locally Hölder up to boundary.
- We have the desired estimate for first derivatives of f provided derivatives of ψ, T are bounded and f is locally Hölder.

Recall

$$D_f(x) = 2\sqrt{\pi} \int_{\zeta \cdot n > 0} \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x - s\zeta, \zeta) M^{\frac{1}{2}}(\zeta) |\zeta \cdot n| ds d\zeta. \tag{12}$$

Proposition (C. Hsia 2015)

Let $1 \leq p \leq \infty$*.*

$$\| \bigtriangledown_\zeta K(f)(x,\zeta)\|_{L^p_\zeta} \leq C\|f(x,\zeta)\|_{L^p_\zeta}. \tag{13}$$

Notice that $\| \bigtriangledown_\zeta k(\zeta,\zeta_*)\|_{L^\infty_\zeta L^1_{\zeta_*}}$ and $\| \bigtriangledown_\zeta k(\zeta,\zeta_*)\|_{L^\infty_{\zeta_*} L^1_\zeta}$ are bounded. By an argument similar to the proof of Young's inequality, we have the proposition.

Transfer regularity from velocity to space

Idea: Combination of averaging or collision and transport can transfer regularity in velocity to space. For time evaluational problem in whole space,

- Velocity averaging lemma (Golse, Perthame, Sentis 1985)
- Mixture lemma (Liu, Yu ARMA 2004)

In present research, we realize this effect for stationary problem in a convex domain by interplaying between velocity and space.

Let $-n(x),\ e_2,\ e_3$ be an orthonormal basis. Let

$$\begin{aligned} \zeta' &= -\rho\cos\theta n(x) + \rho\sin\theta\cos\phi e_2 + \rho\sin\theta\sin\phi e_3, \\ r &= \rho s, \\ \hat{\zeta}' &= \frac{\zeta'}{|\zeta'|}. \end{aligned} \tag{14}$$

Then,

$$\begin{aligned} D_f =& 2\pi^{-\frac{1}{4}} \int_0^{\infty} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|} \\ & e^{-\frac{\nu(\rho)}{\rho} r} K(f)(x - r\hat{\zeta}, \zeta) |\hat{\zeta} \cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta\, dr d\phi d\theta d\rho. \end{aligned} \tag{15}$$

$$D_f = 2\pi^{-\frac{1}{4}} \underbrace{\int_0^{\infty}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}}_{\zeta\cdot n(x)<0} \int_0^{|x-p(x,\zeta)|} e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta}\cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta dr d\phi d\theta d\rho. \tag{16}$$

$$D_f = 2\pi^{-\frac{1}{4}} \int_0^{\infty} \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{|x-p(x,\zeta)|}}_{\Omega} e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta}\cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta dr d\phi d\theta d\rho. \tag{17}$$

Let $y = x - r\hat{\zeta}$

$$D_f = 2\pi^{-\frac{1}{4}} \int_0^{\infty} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|} e^{-\frac{\nu(\rho)}{\rho} r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta} \cdot n(x)| e^{-\frac{\rho^2}{2}} \rho^2 \sin\theta \, dr d\phi d\theta d\rho$$

$$= 2\pi^{-\frac{1}{4}} \int_0^{\infty} \int_{\Omega} e^{-\frac{\nu(\rho)}{\rho}|x-y|} K(f)(y, \rho\frac{(x-y)}{|x-y|}) \frac{(x-y) \cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 dy d\rho$$

$$= 2\pi^{-\frac{1}{4}} \int_0^{\infty} \int_{\Omega} \int_{\mathbb{R}^3} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho\frac{x-y}{|x-y|}, \zeta') f(y, \zeta') \frac{(x-y) \cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 d\zeta' dy d\rho. \quad (18)$$

Suppose $g(t)$ is on $\partial\Omega$ passing x and

$$g(0) = x,$$
$$g'(0) = v,$$

where $v \in T_x\partial\Omega$. We define

$$\nabla_v^x F(x, \zeta) = \frac{d}{dt} F(g(t), \zeta)\Big|_{t=0}. \tag{19}$$

$$
\begin{aligned}
\nabla_v^x D_f(x) &= \int_0^\infty \int_\Omega \int_{\mathbb{R}^3} \nabla_v^x \left(e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} \right) \\
&\cdot \left[f(y,\zeta') - f(x,\zeta') \right] e^{-\frac{\rho^2}{2}} \rho^2 d\zeta' dy d\rho \\
&- \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega \mathrm{div}_y \\
&\left(e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') v \right) dy d\zeta' d\rho \\
&+ \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega \\
&e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot \nabla_v^x(n(x))}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') dy d\zeta' d\rho \\
&=: \nabla^x D_f^1 + \nabla^x D_f^2 + \nabla^x D_f^3.
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\nabla^x D_f^{2,\epsilon} =& -\int_0^\infty \int_{\mathbb{R}^3} \int_{\partial\Omega\setminus B(x,\epsilon)} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho\frac{x-y}{|x-y|},\zeta')\frac{(x-y)\cdot n(x)}{|x-y|^3} \\
& e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')[v\cdot n(y)]dA(y)d\zeta' d\rho \\
& -\int_0^\infty \int_{\mathbb{R}^3} \int_{\partial B(x,\epsilon)\cap\Omega} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho\frac{x-y}{|x-y|},\zeta')\frac{(x-y)}{|x-y|}\cdot n(x) \\
& e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')[v\cdot \frac{x-y}{|x-y|}]\frac{1}{\epsilon^2} dA(y)d\zeta' d\rho \\
& =: S^\epsilon + B^\epsilon.
\end{aligned}
\tag{21}
$$

Thank you!