Weak Shock Wave Solutions for the Discrete Boltzmann Equation

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Abstract. The analytically difficult problem of existence of shock wave solutions is studied for the general discrete velocity model (DVM) with an arbitrary finite number of velocities (the discrete Boltzmann equation in terminology of H. Cabannes). For the shock wave problem the discrete Boltzmann equation becomes a system of ordinary differential equations (dynamical system). Then the shock waves can be seen as heteroclinic orbits connecting two singular points (Maxwellians). In this work we give a constructive proof for the existence of solutions in the case of weak shocks.

We assume that a given Maxwellian is approached at infinity, and consider shock speeds close to a typical speed c_0, corresponding to the sound speed in the continuous case. The existence of a non-negative locally unique (up to a shift in the independent variable) bounded solution is proved by using contraction mapping arguments (after a suitable decomposition of the system). This solution is then shown to tend to a Maxwellian at minus infinity.

Existence of weak shock wave solutions for DVMs was proved by Bose, Illner and Ukai in 1998. In their technical proof Bose et al. are following the lines of the pioneering work for the continuous Boltzmann equation by Caflisch and Nicolaenko. In this work, we follow a more straightforward way, suiting the discrete case. Our approach is based on results by the authors on the main characteristics (dimensions of corresponding stable, unstable and center manifolds) for singular points to general dynamical systems of the same type as in the shock wave problem for DVMs. Our proof is constructive, and it is also shown (at least implicitly) how close to the typical speed c_0, the shock speed must be for our results to be valid. All results are mathematically rigorous.

Our results are also applicable for DVMs for mixtures.

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INTRODUCTION

We are concerned with the existence of shock wave solutions $f = f(x^1,\xi,t) = F(x^1 - ct,\xi)$, of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f = Q(f,f).$$

Here $\mathbf{x} = (x^1,...,x^d) \in \mathbb{R}^d$, $\xi = \left(\xi^1,...,\xi^d\right) \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ denote position, velocity and time respectively. Furthermore, $c > c_0$ denotes the speed of the wave, where c_0 is the speed of sound. The solutions are assumed to approach two given Maxwellians $M_\pm = \dfrac{\rho_\pm}{(2\pi T_\pm)^{d/2}} e^{-\left|\xi - \mathbf{u}_\pm\right|^2/(2T_\pm)}$ (ρ, $\mathbf{u}$ and T denote density, bulk velocity and temperature respectively) as $x \to \pm\infty$, which are related through the Rankine-Hugoniot conditions.

The (shock wave) problem is to find a solution $F = F(y,\xi)$ $(y = x^1 - ct)$ of the equation

$$(\xi^1 - c)\frac{\partial F}{\partial y} = Q(F,F), \tag{1}$$

such that

$$f \to M_\pm \text{ as } y \to \pm\infty. \tag{2}$$

In this paper, we consider the shock wave problem $(1),(2)$ for the general discrete velocity model (DVM) (the discrete Boltzmann equation) [1, 2]. We allow the velocity variable to take values only from a finite subset V of $\mathbb{R}^d$, i.e. $\xi \in \mathrm{V} = \{\xi_1,...,\xi_n\} \subset \mathbb{R}^d$, where n is an arbitrary natural number.

We obtain, from Eq.(1), a system of ODEs (dynamical system)

$$(\xi_i^1 - c)\frac{dF_i}{dy} = Q_i\,(F,F)\,,\ i = 1,...,n,\ c \in \mathbb{R},$$

where $F=(F_1,...,F_n)$, with $F_i=F_i(y)=F(y,\xi_i)$, $i=1,...,n$. The collision operator $Q=(Q_1,...,Q_n)$ is given by

$$Q_i(F,G)=\frac{1}{2}\sum_{j,k,l=1}^{n}\Gamma_{ij}^{kl}\left(F_kG_l+G_kF_l-F_iG_j-G_iF_j\right),\ i=1,...,n,$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations $\Gamma_{ij}^{kl}=\Gamma_{ji}^{kl}=\Gamma_{kl}^{ij}\geq 0$, with equality unless

$$\xi_i+\xi_j=\xi_k+\xi_l \text{ and } |\xi_i|^2+|\xi_j|^2=|\xi_k|^2+|\xi_l|^2.$$

$Q(F,G)$ is a bounded bilinear operator symmetric in arguments.

For normal (only with physical collision invariants) DVMs the collision invariants (i.e. all $\phi=(\phi_1,...,\phi_n)$ such that $\phi_i+\phi_j=\phi_k+\phi_l$ if $\Gamma_{ij}^{kl}\neq 0$) are on the form

$$\phi=(\phi_1,...,\phi_n),\ \phi_i=a+\mathbf{b}\cdot\xi_i+c|\xi_i|^2,\ \ a,c\in\mathbb{R},\ \mathbf{b}\in\mathbb{R}^d,$$

and the Maxwellians (positive vectors $M=(M_1,...,M_n)$, $M_1,...,M_n>0$, such that $Q(M,M)=0$) are on the form

$$M=(M_1,...,M_n), M_i=Ke^{\mathbf{b}\cdot\xi_i+c|\xi_i|^2},\text{ with } K=e^a>0,\ \ a,c\in\mathbb{R},\ \mathbf{b}\in\mathbb{R}^d.$$

We denote by $\left\{\phi_1,...,\phi_p\right\}$ ($p=d+2$ for normal DVMs) a basis for the vector space of collision invariants (note, here and below ϕ_i denotes a collision invariant, while above ϕ_i denotes the ith component of the collision invariant ϕ). Then

$$\langle\phi_i,Q(f,f)\rangle=0 \text{ for } i=1,...,p.$$

Here and below $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product and we denote $\langle\cdot,\cdot\rangle_E=\langle\cdot,E\cdot\rangle$ for symmetric matrices E.

The shock wave problem for the discrete Boltzmann equation (DBE) reads

$$(B-cI)\frac{dF}{dy}=Q(F,F),\text{ where } F\to M_\pm \text{ as } y\to\pm\infty, \tag{3}$$

where B is the diagonal matrix

$$B=\mathrm{diag}(\xi_1^1,...,\xi_n^1).$$

Note that shock waves for the DBE can be seen as heteroclinic orbits connecting two singular points (which are Maxwellians for DVMs). If we multiply Eq.(3) scalarly by ϕ_i, $1\leq i\leq p$, and integrate over $\mathbb{R}$, then we obtain that the Maxwellians M_- and M_+ must fulfill the Rankine-Hugoniot conditions

$$\langle M_+,\phi_i\rangle_{B-cI}=\langle M_-,\phi_i\rangle_{B-cI},\ i=1,...,p.$$

The rest of this paper is organized as follows. First we state under which assumptions our results are obtained and present the main results. Then we give a very brief presentation of the proof. Though we don't present all details and parts of the proof here, all results are mathematically rigorous.

ASSUMPTIONS AND MAIN RESULTS

We make the following assumptions on our DVMs.

1. There is a number c_0, with the following properties:
   [i] rank$(K)=p-1$, where K is the $p\times p$ matrix with the elements

   $$k_{ij}=\left\langle M_+\phi_i,\phi_j\right\rangle_{B-c_0I}.$$

   The rank of K is independent of the choice of the basis $\left\{\phi_1,...,\phi_p\right\}$. In other words, there is a unique (up to its sign) vector $\phi_\perp$ in span$(\phi_1,...,\phi_p)$, such that $\langle M_+\phi_\perp,\phi_\perp\rangle=1$ and

   $$\langle M_+\phi_\perp,\phi\rangle_{B-c_0I}=0 \text{ for all } \phi\in\mathrm{span}(\phi_1,...,\phi_p). \tag{4}$$

   [ii] $c_0\neq\xi_i^1$ for $i=1,...,n$, or, equivalently, $\det(B-c_0I)\neq 0$.

2. The vector(s) $\phi_\perp$ fulfilling Eqs.(4), also satisfy $\left\langle M_+\phi_\perp, \phi_\perp^2 \right\rangle_{B-c_0 I} \neq 0$. We choose the sign of the vector $\phi_\perp$, such that $\left\langle M_+\phi_\perp, \phi_\perp^2 \right\rangle_{B-c_0 I} > 0$.
3. For each c (at least in a neighborhood of c_0), the number of Maxwellians M, such that the relations

$$\langle M, \phi_i \rangle_{B-cI} = \langle M_+, \phi_i \rangle_{B-cI}, \; i = 1, ..., p, \tag{5}$$

are fulfilled, is finite.

Remark 1 *Let M_+ be a Maxwellian with zero bulk velocity ($\mathbf{u} = \mathbf{0}$). Then, for the "continuous" Boltzmann equation, $M_+ = \dfrac{\rho}{(2\pi T)^{d/2}} e^{-|\xi|^2/(2T)}$. In this case (with $d = 3$) $c_0 = \pm\sqrt{\dfrac{5T}{3}}$ (note that the assumption 1 [ii] never is fulfilled in the continuous case), $\phi_\perp = \dfrac{1}{\sqrt{2\rho T}}(\xi^1 \pm \dfrac{|\xi|^2}{\sqrt{15T}})$ and $\left\langle M_+\phi_\perp, \phi_\perp^2 \right\rangle_{B-c_0 I} = \dfrac{2}{3}\sqrt{\dfrac{2T}{\rho}} > 0$. Assumption 3 is also fulfilled in the continuous case. There is at most one more Maxwellian M, besides M_+, which fulfills Eqs.*(5).

Remark 2 *Assume that we have an axially symmetric normal model (if $\left(\xi^1, ..., \xi^d\right) \in \mathrm{V}$, then $\left(\pm\xi^1, ..., \pm\xi^d\right) \in \mathrm{V}$) with $n = 2N$. Let $M = Ke^{c|\xi|^2}$ and denote*

$$\begin{cases} \phi_1 = (1, ..., 1) \\ \phi_2 = (\xi_1^1, ..., \xi_N^1, -\xi_1^1, ..., -\xi_N^1) \\ \phi_{i+1} = (\xi_1^i, ..., \xi_{2N}^i), \; i = 2, ..., d, \\ \phi_{d+2} = (|\xi_1|^2, ..., |\xi_N|^2, |\xi_1|^2, ..., |\xi_N|^2) \end{cases}.$$

Then

$$c_0 = c_\pm = \pm\sqrt{\frac{\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5 - \chi_3^2)}}, \text{ where}$$

$$\chi_1 = \langle \phi_1, M\phi_1 \rangle, \chi_2 = \langle \phi_2, M\phi_2 \rangle, \chi_3 = \langle \phi_1, M\phi_3 \rangle, \chi_4 = \langle \phi_2, M\phi_3 \rangle_B \text{ and } \chi_5 = \langle \phi_3, M\phi_3 \rangle.$$

In the remaining part of this chapter we will assume that assumptions 1-3 are fulfilled. We denote

$$\|h\| = \|h(y)\| = \sup_{y\in\mathbb{R}} |h(y)|$$

for any bounded (vector or scalar) function $h(y): \mathbb{R} \to \mathbb{R}^k$, where k is a positive integer.

A proof for existence of weak shock wave solutions for DVMs was already presented in 1998 by Bose, Illner and Ukai [3]. In their technical proof Bose et al. are following the lines of the pioneering work for the continuous Boltzmann equation by Caflisch and Nicolaenko [4] (for more resent research in the continuous case see [5]).

In this work, we follow a more straightforward way, suiting the discrete case. We use results by the authors [6] on the main characteristics (dimensions of corresponding stable, unstable and center manifolds) for singular points to general dynamical systems of the same type as in the shock wave problem for DVMs. We want to stress that our proof is constructive, and that it can also (at least implicitly) be shown how close to the typical speed c_0, the shock speed must be for our results to be valid.

Theorem 1 *For any given positive Maxwellian M_+, there exists a family of Maxwellians $M_- = M_-(\varepsilon)$ and shock speeds $c = c(\varepsilon) = c_0 + \varepsilon$, such that the shock wave problem* (3) *has a non-negative locally unique (with respect to the norm $\|\cdot\|$ and up to a shift in the independent variable) non-trivial bounded solution for each sufficiently small $\varepsilon > 0$. Furthermore, M_- is determined by M_+ and c.*

Remark 3 *The arguments in this paper can be changed, so that we can interchange M_- and M_+ in Theorem* 1 *(under slightly changed assumptions and with $\varepsilon < 0$).*

Remark 4 *The approach of this paper can also be applied for the DBE for mixtures.*

BRIEF PRESENTATION OF THE PROOF

We consider
$$(B-cI)\frac{dF}{dy}=Q(F,F)\text{, where }F\to M_{+}\text{ as }y\to\infty,$$
and denote
$$F=M+M^{1/2}h\text{, with }M=M_{+}.$$
We obtain
$$(B-cI)\frac{dh}{dy}+Lh=S(h,h)\text{, where }h\to 0\text{ as }y\to\infty, \tag{6}$$
with
$$Lh=-2M^{-1/2}Q(M,M^{1/2}h)\text{ and }S(g,h)=M^{-1/2}Q(M^{1/2}g,M^{1/2}h).$$
The linear operator ($n\times n$ matrix) L is symmetric and semi-positive (i.e. $\langle h,h\rangle_{L}\geq 0$ for all $h\in\mathbb{R}^{n}$) and have the null-space
$$N(L)=\operatorname{span}(M^{1/2}\phi_{1},...,M^{1/2}\phi_{p})=\operatorname{span}(e_{1},...,e_{p}),$$
where $\{e_{1},...,e_{p}\}$ can be chosen such that
$$\langle e_{i},e_{j}\rangle=\delta_{ij}\text{ and }\langle e_{i},e_{j}\rangle_{B-cI}=(\gamma_{i}-c)\delta_{ij}\text{, with }\gamma_{i}=\langle e_{i},e_{i}\rangle_{B}. \tag{7}$$
The quadratic part $S(h,h)$ is orthogonal to $N(L)$ (i.e. $\langle\phi,S(h,h)\rangle=0$ if $\phi\in N(L)$).

By assumption 1 [i], there is a number $c=c_{0}$, such that (after possible renumbering)
$$\gamma_{p}=c_{0}\text{ and }\gamma_{i}\neq c_{0}\text{ for }i=1,...,p-1. \tag{8}$$
We study Eqs.(6) for
$$c=c_{0}+\varepsilon,\ 0<\varepsilon\leq s,$$
where s is chosen such that
$$\det(B-cI)\neq 0\text{ and }\gamma_{i}\neq c,\ i=1,...,p,\text{ if }0<\varepsilon\leq s. \tag{9}$$
Clearly, (for a finite number n) such a number s exists by assumption 1.

Then Eqs.(6) are equivalent with the system
$$\frac{dh}{dy}+(B-cI)^{-1}Lh=(B-cI)^{-1}S(h,h).$$

Let $n^{\pm}$, with $n^{+}+n^{-}=n$, and $m^{\pm}$, denote the numbers of the positive and negative eigenvalues of the matrices $B-cI$ and $(B-cI)^{-1}L$ respectively. Moreover, let k^{+}, k^{-}, and l be the numbers of positive, negative and zero eigenvalues of the $p\times p$ matrix K, with entries $k_{ij}=\langle y_{i},y_{j}\rangle_{B-cI}$, such that $N(L)=\operatorname{span}(y_{1},...,y_{p})$ (the numbers k^{+}, k^{-}, and l are independent of the choice of the basis $\{y_{1},...,y_{p}\}$ of $N(L)$ [6]). Then $m^{\pm}=n^{\pm}-k^{\pm}-l$, and the matrix $(B-cI)^{-1}L$ is diagonalizable if and only if $l=0$ (see [6] for details).

Remark 5 *Eqs.(7)-(9), imply that $l=1$ if $\varepsilon=0$, and $l=0$ if $0<\varepsilon\leq s$, while n^{+} and k^{+} do not change for $0\leq\varepsilon\leq s$. Therefore, $(B-cI)^{-1}L$ has exactly one more positive eigenvalue, for $0<\varepsilon\leq s$, than for $\varepsilon=0$.*

The matrix $(B-cI)^{-1}L$ has (for $0<\varepsilon\leq s$) exactly $n-p$ non-zero (real) eigenvalues. Moreover, (see Ref.[6] for details) there is a basis $\{u_{0},...,u_{m}\}$, with $m=n-p-1$, of $\operatorname{Im}((B-cI)^{-1}L)$, such that
$$(B-cI)^{-1}Lu_{i}=\lambda_{i}u_{i},\ \lambda_{i}\neq 0,\ \langle u_{i},u_{j}\rangle_{B-cI}=\lambda_{i}\delta_{ij},\ u_{i}=(B-cI)^{-1}L^{1/2}w_{i},\ \langle w_{i},w_{j}\rangle=\delta_{ij},\ i,j=0,...,m.$$

We choose u_{0}, such that λ_{0} is the minimal positive eigenvalue and $\langle u_{0},M^{1/2}\phi_{\perp}\rangle\geq 0$.

Remark 6 *The real eigenvalues of $(B-cI)^{-1}L$ are continuous in ε. Moreover,*
$$\lambda_{0}=O(\varepsilon)\text{ and }\lambda_{i}=O(1),\ i=1,...,m,\text{ as }\varepsilon\to 0^{+}. \tag{10}$$
The smallness of λ_{0}, compared to the other eigenvalues is essential in the proof.

We denote

$$h=\sum_{i=0}^{m} x_i u_i, \text{ where } x_i = x_i(y) = \langle h, u_i \rangle_{B-cI}.$$

Then,

$$\begin{aligned}\frac{dx_i}{dy}+\lambda_i x_i &= g_i(X,X), \text{ where } X=(x_0,...,x_m),\ g_i=g_i(X,X)=\sum_{j,k=0}^{m} x_j x_k g^i_{jk},\ i=0,...,m, \text{ with}\\ g^i_{jk} &= \frac{1}{\lambda_i}\left\langle u_i, S(u_j,u_k)\right\rangle = \left\langle L^{-1/2}w_i, S((B-cI)^{-1}L^{1/2}w_j,(B-cI)^{-1}L^{1/2}w_k)\right\rangle.\end{aligned}$$

We denote by $\widehat{g}_i$ the symmetric $(m+1)\times(m+1)$ matrix with entries

$$(\widehat{g}_i)_{jk} = g^i_{jk},\ 0\le j,k\le m,$$

and by $\mathscr{G}_i>0$ the maximum of the absolute values of the eigenvalues of the matrix $\widehat{g}_i$, or, equivalently, $\mathscr{G}_i=\sup\limits_{|X|=1}|\widehat{g}_i X|$. Then

$$g_i(X,X)=\langle X,\widehat{g}_i X\rangle \text{ and } |g_i(X,X)|\le \mathscr{G}_i|X|^2, \text{ for } i=0,...,m.$$

It is clear that $x_0=x_0(y)$ plays a special role for small values of the minimal positive eigenvalue λ_0 (and therefore also for small ε). We assume that $x_0\neq 0$ and substitute

$$\begin{cases} x_0(y)=\lambda_0 x(t)\\ x_i(y)=\lambda_0 x(t) z_i(t), \end{cases}, \text{ with } t=\lambda_0 y, \text{ for } i=1,...,m.$$

Denoting

$$Z=(1,z_1,...,z_m),\ z=(z_1,...,z_m),\ \theta=\theta(z)=g_0(Z,Z) \text{ and } \mu_i=\frac{\lambda_0}{\lambda_i-\lambda_0},\ i=1,...,m,$$

we obtain

$$\begin{cases} \dfrac{dx}{dt}+x=x^2\theta(z)\\ \dfrac{dz_i}{dt}+\dfrac{1}{\mu_i}z_i=x(g_i(Z,Z)-z_i\theta(z)) \end{cases},\quad i=1,...,m.$$

Solving the first equation we obtain, noting that $x(t)=O(e^{-t})$ as $t\to\infty$ and therefore $a=\lim\limits_{t\to\infty}\dfrac{1}{x(t)}e^{-t}\in\mathbb{R}$,

$$x=\frac{1}{ae^t+T(-1)\theta(z)}, \text{ where } T(b)f(t)=\int_0^\infty e^{-u}f(t-bu)\,du.$$

The parameter a reflects the invariance of our equation under shifts in the invariant variable t. The sign of a is, however, defined uniquely. It must be the same as the sign of

$$\theta_0=\lim_{t\to\infty}\theta(z)=g_0(\omega,\omega)=\frac{1}{\lambda_0}\langle u_0,S(u_0,u_0)\rangle, \text{ where } \omega=(1,0,...,0)\in\mathbb{R}^m.$$

Lemma 1 *If $\left\langle M_+\phi_\perp,\phi_\perp^2\right\rangle_{B-c_0I}>0$, where the vector $\phi_\perp$ is fulfilling Eqs.(4), then $\theta_0(0)=\lim\limits_{\varepsilon\to 0}\theta_0(\varepsilon)>0$.*

Remark 7 *By assumption 2 and Lemma 1, $\theta_0(0)=\lim\limits_{\varepsilon\to 0}\theta_0(\varepsilon)$ is positive. Hence, by continuity of θ_0 in ε, we can allow s (possibly by choosing it smaller than above) to be such that $\theta_0=\theta_0(\varepsilon)$ is positive for $0\le\varepsilon\le s$.*

We study only the case $0<\varepsilon\le s$ below, and therefore we choose $a=1$. Then $x(t)$ must satisfy

$$x(t)=\frac{1}{e^t+T(-1)\theta(z)}.$$

Furthermore, if the functions $z_i = z_i(t)$, $i = 1, ..., m$, are bounded, then they satisfy the integral equations

$$z_i = \mu_i T(\mu_i)[x(g_i(Z,Z) - z_i\theta(z))],\ i = 1, ..., m, \text{ where } T(b)f(t) = \int_0^\infty e^{-u} f(t - bu)\, du.$$

We denote

$$\|S\| = \sqrt{\sum_{i=1}^m \mathscr{G}_i^2} \text{ and } \|\theta\| = \mathscr{G}_0, \text{ with } \mathscr{G}_i = \sup_{|X|=1} |\widehat{g}_i X|\,,\ i = 0, ..., m,$$

and introduce the Banach space

$$\mathscr{X} = \{z = z(t) \in \mathscr{C}(\mathbb{R}, \mathbb{R}^m) \mid \|z\| < \infty\},$$

where $\mathscr{C}(\mathbb{R}, \mathbb{R}^m)$ denote the space of all continuous bounded functions on $\mathbb{R}$ into $\mathbb{R}^m$, and its closed convex subset

$$\mathscr{B}_R = \{z \in \mathscr{X} \mid \|z\| \le R\}, \text{ with } R < R_* = \sqrt{1 + \frac{\theta_0}{\|\theta\|}} - 1 \le \sqrt{2} - 1.$$

Furthermore, we introduce the mapping $\mathscr{Z} = \mathscr{Z}(z)$ of $\mathscr{B}_R$ into $\mathscr{X}$, defined by

$$\mathscr{Z}(z) = (Z_1(z), ..., Z_m(z)),\ Z_i(z) = \mu_i T(\frac{1}{\mu_i}) \frac{g_i(Z,Z) - z_i\theta(z)}{e^t + T(-1)\theta(z)},\ i = 1, ..., m.$$

We obtain the following lemma.

Lemma 2 *If $z, z' \in \mathscr{B}_R$, then*

$$\|\mathscr{Z}(z)\| \le \Phi(R) \text{ and } \left\|\mathscr{Z}(z) - \mathscr{Z}(z')\right\| \le \Phi'(R)\left\|z - z'\right\|,$$

where

$$\Phi(R) = \frac{\delta}{\Delta(R)}(\frac{\|S\|}{\|\theta\|} + R)(1+R)^2,$$

with

$$\delta = \max(|\mu_1|, ..., |\mu_m|) \text{ and } \Delta(R) = (1 + R_*)^2 - (1+R)^2,$$

and $\Phi'(R) = \dfrac{d\Phi(R)}{dR}$ *is the Fréchet derivative of* $\Phi(R)$*, i.e.*

$$\Phi'(R) = \frac{1}{\Delta(R)}[2\Phi(R)(1+R) + 2\delta(\frac{\|S\|}{\|\theta\|} + R)(1+R) + \delta(1+R)^2].$$

Then, (very briefly) the existence of a non-negative locally unique (up to a shift in the independent variable) bounded solution is proved by using contraction mapping arguments [4] (the mapping $\mathscr{Z}(z)$ has a fixed point if $\delta > 0$ is small enough and hence, if $\varepsilon > 0$ is small enough, by Eqs.(4)). Finally, this solution is shown to tend to a Maxwellian at minus infinity using arguments used in [5].

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REFERENCES

1. R. Gatignol, *Théorie Cinétique des Gaz à Répartition Discrète de Vitesses*, Springer-Verlag, 1975.
2. H. Cabannes, The discrete Boltzmann equation (1980 (2003)), lecture notes given at the University of California at Berkeley, 1980, revised with R. Gatignol and L-S. Luo, 2003.
3. C. Bose, R. Illner, and S. Ukai, *Transp. Th. Stat. Phys.* **27**, 35–66 (1998).
4. R. E. Caflisch, and B. Nicolaenko, *Comm. Math. Phys.* **86**, 161–194 (1982).
5. T.-P. Liu, and S.-H. Yu, *Comm. Math. Phys.* **246**, 133–179 (2004).
6. A. V. Bobylev, and N. Bernhoff, "Discrete velocity models and dynamical systems," in *Lecture Notes on the Discretization of the Boltzmann Equation*, edited by N. Bellomo, and R. Gatignol, World Scientific, 2003, pp. 203–222.