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Half-Space Problems for a Linearized Discrete Quantum Kinetic Equation

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Abstract We study typical half-space problems of rarefied gas dynamics, including the problems of Milne and Kramer, for a general discrete model of a quantum kinetic equation for excitations in a Bose gas. In the discrete case the plane stationary quantum kinetic equation reduces to a system of ordinary differential equations. These systems are studied close to equilibrium and are proved to have the same structure as corresponding systems for the discrete Boltzmann equation. Then a classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete kinetic equation can be made. The number of additional conditions that need to be imposed for well-posedness is given by some characteristic numbers. These characteristic numbers are calculated for discrete models axially symmetric with respect to the x-axis. When the characteristic numbers change is found in the discrete as well as the continuous case. As an illustration explicit solutions are found for a small-sized model.

Keywords Bose-Einstein condensate · Low temperature kinetics · Discrete kinetic equation · Milne problem · Kramer problem

1 Introduction

Half-space problems have an important role in the study of the asymptotic behavior of the solutions of boundary value problems of kinetic equations for small Knudsen numbers [4, 15, 16, 26, 27]. In this paper we study half-space problems related to a quantum kinetic equation [23, 33], for the distribution function of excited atoms interacting with a Bose-Einstein condensate. Motivated by the work

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of Arkeryd and Nouri [2] we are interested in the equation

$$\begin{cases} p^1 \dfrac{dF}{dx} = C_{12}\left(F\right) + \Gamma C_{22}\left(F\right), \\ F\left(0, \mathbf{p}\right) = F_0\left(\mathbf{p}\right) \text{ for } p^1 > 0, \end{cases} \tag{1}$$

where $F = F\left(x, \mathbf{p}\right)$ denotes the distribution function of the excitations, $\Gamma \in \mathbb{R}_+ = \left\{x \in \mathbb{R} \mid x \geq 0\right\}$ is constant, $x \in \mathbb{R}_+$, $\mathbf{p} = \left(p^1, p^2, p^3\right) \in \mathbb{R}^3$, and $F_0 = F_0\left(\mathbf{p}\right)$ is given, with the collision integrals

$$C_{12}\left(F\right) = n \int \delta_0 \delta_3 \left[\left(1+F_*\right) F' F'_* - F_* \left(1+F'\right)\left(1+F'_*\right)\right] d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}'_*,$$

with

$$\delta_0 = \delta\left(\mathbf{p}_* - \mathbf{p}' - \mathbf{p}'_*\right) \delta\left(\mathbf{p}_*^2 + n - \left(\mathbf{p}'\right)^2 - \left(\mathbf{p}'_*\right)^2\right) \text{ and}$$
$$\delta_3 = \delta\left(\mathbf{p}_* - \mathbf{p}\right) - \delta\left(\mathbf{p}' - \mathbf{p}\right) - \delta\left(\mathbf{p}'_* - \mathbf{p}\right),$$

and

$$C_{22}\left(F\right) = \int \delta_1 \left[\left(1+F\right)\left(1+F_*\right) F' F'_* - F F_* \left(1+F'\right)\left(1+F'_*\right)\right] d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}'_*$$

with

$$\delta_1 = \delta\left(\mathbf{p} + \mathbf{p}_* - \mathbf{p}' - \mathbf{p}'_*\right) \delta\left(\mathbf{p}^2 + \mathbf{p}_*^2 - \left(\mathbf{p}'\right)^2 - \left(\mathbf{p}'_*\right)^2\right).$$

Here and below we use the notation $F'_* = F\left(x, \mathbf{p}'_*\right)$ etc.. The density of the condensate, n_c, is assumed to be constant, $n_c = n$ (cf. [2]). In the Nordheim-Boltzmann [24] (or the Uehling-Uhlenbeck [28]) collision integral $C_{22}\left(F\right)$ binary collisions between excited atoms are considered, while in the collision integral $C_{12}\left(F\right)$ binary collisions involving one condensate atom are considered [33].

If the distribution function F is close to an equilibrium distribution, i.e. a Planckian

$$P = \frac{1}{e^{\alpha\left(|\mathbf{p}|^2 + n\right) + \beta \cdot \mathbf{p}} - 1} = \frac{1}{e^{\alpha\left(|\mathbf{p} - \mathbf{p}_0|^2 + n_0\right)} - 1},$$

with $\alpha > 0$, $\beta \in \mathbb{R}^3$, $\mathbf{p}_0 = -\dfrac{\beta}{2}$, and $n_0 = n - |\mathbf{p}_0|^2$, cf. [2], then the non-linear equation (1) can be approximated by the linearized equation

$$\begin{cases} p^1 \dfrac{df}{dx} + Lf = 0, \; f = f\left(x, \mathbf{p}\right) \\ f\left(0, \mathbf{p}\right) = f_0\left(\mathbf{p}\right) \text{ for } p^1 > 0, \end{cases} \tag{2}$$

where

$$F = P + (P(1+P))^{1/2} f, \; F_0 = P + (P(1+P))^{1/2} f_0, \text{ and } L = L_{12} + \Gamma L_{22},$$

with

$$L_{12} f = \int \delta_0 \delta_3 \Big[\left(P_* - P'\right)\left(P'_*(1+P'_*)\right)^{1/2} f'_* + \left(P_* - P'_*\right)\left(P'(1+P')\right)^{1/2} f' +$$
$$\left(1 + P' + P'_*\right)\left(P_*(1+P_*)\right)^{1/2} f_* \Big] d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}'_*$$

and

$$L_{22}f = \int \delta_1 \Big[\left(PP_* - P'(1+P+P_*)\right) \left(P'_*(1+P'_*)\right)^{1/2} f'_* + \\ \left(PP_* - P'_*(1+P+P_*)\right) \left(P'(1+P')\right)^{1/2} f' + \\ \left(P(1+P'+P'_*) - P'P'_*\right) \left(P_*(1+P_*)\right)^{1/2} f_* + \\ \left(P_*(1+P'+P'_*) - P'P'_*\right) \left(P(1+P)\right)^{1/2} f \Big] \, d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}'_*.$$

It can be shown (cf. [2] for L_{12} and, for example, [16] for the linearized Boltzmann operator) that the linearized operators L_{12} and L_{22}, and so also L, (all acting in the velocity space) are symmetric and positive semi-definite operators on L^2.

In the paper [2], Arkeryd and Nouri studied the Milne problem for the linearized equation (2), with $\Gamma = 0$, $F = P(1+f)$, and a cut-off at $\lambda > 0$ in the integrand of L, such that $|\mathbf{p}|, |\mathbf{p}_*|, |\mathbf{p}'|, |\mathbf{p}'_*| \geq \lambda$. The corresponding linearized half-space problems for the Boltzmann equation is well-studied [3,17,20], see also [4] and references therein.

In this paper we discretize the variable $\mathbf{p}$ and obtain a general discrete model for Eq.(1), which is similar to the discrete Boltzmann equation (a general discrete velocity model, DVM, for the Boltzmann equation) [14]. It is a well-known fact that the Boltzmann equation can be approximated up to any order of DVMs [11, 25, 18], which motivated us to introduce discrete models also for this equation. By the discretization, Eq.(1) reduces to a system of ordinary differential equations. We find that the discrete linearized quantum kinetic equation (the discrete version of Eq.(2)) has the same structure as the linearized discrete Boltzmann equation. This means that the linearized operator is symmetric and positive semi-definite, and that the null-space is non-trivial. One difference is that the mass flow is not constant (with respect to the variable x) as for the discrete Boltzmann equation. However, this cause us no difficulties, in difference to in the continuous case in [2], since the structure will still be the same. A classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete Boltzmann equation has been made in [5] (which is a continuation of the paper [9]), based on the dimensions of the stable, unstable and center manifolds of the singular points (Maxwellians for DVMs). We establish similar results in our case. This means, that we, in addition to adding the Nordheim-Boltzmann (or Uehling-Uhlenbeck) collision integral $C_{22}(F)$, also can introduce an inhomogeneous term and more general boundary conditions. Similar results can also be established for the discrete Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation and the discrete anyon Boltzmann equation (see Remark 6 and 7 in Section 4).

Furthermore, we have, for axially symmetric discrete models with respect to the x-axis, made a table of some characteristic numbers, from which we, by Theorem 1, can obtain the dimensions of the stable, unstable and center manifolds of the singular points (Planckians in our case). This includes determining when the characteristic numbers change, not only in the discrete, but also in the continuous case (cf. [7,5] for DVMs and [17] for the Boltzmann equation).

Nonlinear half-space problems for the Boltzmann equation have also been studied for small perturbations of the singular points (Maxwellians for the Boltzmann equation), see for example [6,21,22,29] for the discrete Boltzmann equation and $[30,19,32]$ for the continuous Boltzmann equation. In the discrete case similar results to the ones in [6] can be obtained for the quantum kinetic equation (1).

We want to make clear that the aim of the paper is not the study of the general half-space problem we obtain, since it is already well studied for the discrete Boltzmann equation [9,5]. The novelty of the paper is instead the introduction of discrete models for the equation for the distribution function of excited atoms interacting with a Bose-Einstein condensate and the studies of those models. These studies includes that we by the right linearization end up with a system having similar properties as the one obtained for the discrete Boltzmann equation. It makes it, as mentioned above, possible to extend the results in [2] obtained for the continuous equation. The same is true also for the discrete Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation and the discrete anyon Boltzmann equation (see Remark 6 and 7 in Section 4). However, in concrete situations, it will look different depending on which equation we study. One difference is the characteristic numbers, studied for axially symmetric models in Section 5. Our experience from the Boltzmann equation, make us believe that these numbers (also calculated in the continuous case) are as important in the continuous case as in the studied discrete case. Another difference between our equation and the Boltzmann equation, is that in our case we will have a non-constant mass-flow. To illustrate this we created a model that we solved explicitly and was able to give an explicit expression for the non-constant mass flow for (see Section 6).

The remaining part of this paper is organized as follows. In Section 2 we introduce a general discrete model for Eq.(1) and derive some of its properties. By a transformation around a Planckian, we obtain a linearized operator and a nonlinear part presented in Section 3. It is shown that the system has the same structure (the linearized operator and the non-linear part have similar properties) as the corresponding system for DVMs of the Boltzmann equation. Then some results for the linearized discrete Boltzmann equation can be applied for the problem of our study. These results are presented in Section 4. In Section 5 some characteristic numbers, from which we, by Theorem 1, can obtain the dimensions of the stable, unstable and center manifolds of the singular points (Planckians), are obtained for axially symmetric discrete models with respect to the x-axis. When the characteristic numbers change, are determined both in the discrete as well as the continuous case. A linearized half-space problem (with $\Gamma = 0$) is explicitly solved for a small-sized discrete model in Section 6.

2 Discrete model

We introduce a general discrete model for Eq.(1)

$$p_i^1 \frac{dF_i}{dx} = C_{12i}(F) + \Gamma C_{22i}(F), x \in \mathbb{R}_+, \ i = 1, ..., N, \tag{3}$$

where $\mathscr{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set, $F_i = F_i(x) = F(x, \mathbf{p}_i)$, where $F = F(x, \mathbf{p})$ is the distribution function of the excitations, and $\Gamma \in \mathbb{R}_+$ is constant. For generality, we allow $\mathbf{p}$ to be of dimension d, rather than of dimension 3. We assume that

$$p_i^1 \neq 0, \text{ for } i = 1, ..., N.$$

The collision operators $C_{12i}(F)$ are given by

$$C_{12i}(F) = \sum_{j,k,l=1}^{N} (\delta_{il} - \delta_{ij} - \delta_{ik}) \Gamma_{jk}^{l} ((1+F_l) F_j F_k - F_l (1+F_j)(1+F_k)),$$

where

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases},$$

with $\Gamma_{jk}^{i} = 1$ if

$$\mathbf{p}_i = \mathbf{p}_j + \mathbf{p}_k \text{ and } |\mathbf{p}_i|^2 = |\mathbf{p}_j|^2 + |\mathbf{p}_k|^2 + n, \tag{4}$$

and $\Gamma_{jk}^{i} = 0$ otherwise. Furthermore, the collision operators $C_{22i}(F)$ are given by

$$C_{22i}(F) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} ((1+F_i)(1+F_j) F_k F_l - F_i F_j (1+F_k)(1+F_l)),$$

with $\Gamma_{ij}^{kl} = 1$ if

$$\mathbf{p}_i + \mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l \text{ and } |\mathbf{p}_i|^2 + |\mathbf{p}_j|^2 = |\mathbf{p}_k|^2 + |\mathbf{p}_l|^2, \tag{5}$$

and $\Gamma_{ij}^{kl} = 0$, otherwise.

Remark 1 For a function $g = g(\mathbf{p})$ (possibly depending on more variables than $\mathbf{p}$), we will, as we consider the discrete case, identify g with its restrictions to the points $\mathbf{p} \in \mathscr{P}$, i.e.

$$g = (g_1, ..., g_N), \text{ with } g_i = g(\mathbf{p}_i).$$

Then Eq.(3) can be rewritten as

$$B\frac{dF}{dx} = C_{12}(F) + \Gamma C_{22}(F), \text{ with } x \in \mathbb{R} \text{ and } B = \text{diag}(p_1^1, ..., p_N^1). \tag{6}$$

The collision operator $C_{12}(F)$ in (6) is also given by the expression

$$C_{12}(F) = n\widetilde{L}F + n\widetilde{Q}(F,F), \tag{7}$$

where

$$\left(\widetilde{L}F\right)_i = \sum_{j,k=1}^{N} 2\Gamma_{ij}^{k} F_k - \Gamma_{jk}^{i} F_i \text{ and}$$

$$\widetilde{Q}_i(F,G) = \sum_{j,k=1}^{N} \Gamma_{jk}^{i} Q_{jk}^{i}(F,G) - 2\Gamma_{ij}^{k} Q_{ij}^{k}(F,G),$$

with

$$Q_{jk}^{i}(F,G) = \frac{1}{2}(F_j G_k + G_j F_k - F_i (G_j + G_k) - G_i (F_j + F_k)),$$

and the collision operator $C_{22}(F)$ in (6) is given by the expression

$$C_{22}(F) = Q(F,F) + \widehat{Q}(F,F,F), \tag{8}$$

where

$$Q_i(F,G) = \frac{1}{2}\sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl}\left((G_k H_l + H_k G_l) - (G_i H_j + H_j G_i)\right)$$

and

$$\widehat{Q}_i(F,G,H) =$$
$$\frac{1}{2}\sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl}\left((F_i + F_j)(G_k H_l + H_k G_l) - (F_k + F_l)(G_i H_j + H_j G_i)\right).$$

A function $\phi = \phi(\mathbf{p})$ is a collision invariant, if and only if,

$$\phi_i = \phi_j + \phi_k, \tag{9}$$

for all indices such that $\Gamma_{jk}^{i} \neq 0$, if $\Gamma = 0$, with the additional condition

$$\phi_i + \phi_j = \phi_k + \phi_l, \tag{10}$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$, if $\Gamma \neq 0$. We have the trivial collision invariants ("the physical collision invariants")

$$\phi^1 = p^1, ..., \phi^d = p^d, \phi^{d+1} = |\mathbf{p}|^2 + n \tag{11}$$

including all linear combinations of these. We want to stress that by Remark 1 and in correspondence with Eqs.(9), (10) the collision invariants $\phi^i = \phi^i(\mathbf{p})$ in Eq.(11) are vectors.

In the discrete case, in difference to the continuous case, there can be spurious (or non-physical) collision invariants. We consider below (even if this restriction is not necessary in our general context) only normal discrete models. That is, discrete models without spurious collision invariants, i.e. any collision invariant is of the form

$$\phi = \phi(\mathbf{p}) = -\alpha\left(|\mathbf{p}|^2 + n\right) - \beta \cdot \mathbf{p} \tag{12}$$

for some constant $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$. Construction of normal discrete kinetic models and especially DVMs have been extensively studied, see for example [10, 12, 13] and references therein. A Maxwellian distribution or just Maxwellian is on the form

$$M = e^{\phi} = e^{-\alpha\left(|\mathbf{p}|^2+n\right)-\beta\cdot\mathbf{p}}$$

or

$$M_i = e^{\phi_i} = e^{-\alpha\left(|\mathbf{p}_i|^2+n\right)-\beta\cdot\mathbf{p}_i},\ i = 1, ..., N,$$

where $\phi = (\phi_1, ..., \phi_N)$ is a collision invariant, and a Planckian distribution or just Planckian is given by

$$P = \frac{M}{1-M} = \frac{1}{M^{-1}-1} = \frac{1}{e^{\alpha\left(|\mathbf{p}|^2+n\right)+\beta\cdot\mathbf{p}} - 1}$$

or

$$P_i = \frac{M_i}{1-M_i} = \frac{1}{e^{\alpha\left(|\mathbf{p}_i|^2+n\right)+\beta\cdot\mathbf{p}_i}-1} \text{ for } i=1,...,N.$$

One can easily see that

$$\langle H, C_{12}(F)\rangle = n\sum_{i,j,k=1}^{N} \Gamma_{jk}^{i}\left(\left(1+F_i\right)F_jF_k - F_i\left(1+F_j\right)\left(1+F_k\right)\right)\left(H_i-H_j-H_k\right), \quad (13)$$

and so

$$\left\langle \log\frac{F}{1+F}, C_{12}(F)\right\rangle = n\sum_{i,j,k=1}^{N} \Gamma_{jk}^{i}\left(1+F_i\right)\left(1+F_j\right)\left(1+F_k\right) \left(\frac{F_j}{1+F_j}\frac{F_k}{1+F_k} - \frac{F_i}{1+F_i}\right)\left(\log\frac{F_i}{1+F_i} - \log\left(\frac{F_j}{1+F_j}\frac{F_k}{1+F_k}\right)\right) \leq 0,$$

with equality if and only if

$$\frac{F_i}{1+F_i} = \frac{F_j}{1+F_j}\frac{F_k}{1+F_k}, \quad (14)$$

for all indices such that $\Gamma_{jk}^{i} \neq 0$. Here and below, we denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^n$. Hence, there is equality in Eq.(14), if and only if, $\dfrac{F}{1+F}$ is a Maxwellian, or equivalently, if and only if, F is a Planckian.

Moreover, one can easily obtain that

$$\langle H, C_{22}(F)\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{N} \Gamma_{ij}^{kl}\left(H_i+H_j-H_k-H_l\right) \left(\left(1+F_i\right)\left(1+F_j\right)F_kF_l - F_iF_j\left(1+F_k\right)\left(1+F_l\right)\right), \quad (15)$$

and so

$$\left\langle \log\frac{F}{1+F}, C_{22}(F)\right\rangle = \frac{1}{4}\sum_{i,j,k=1}^{N} \Gamma_{ij}^{kl}\left(1+F_i\right)\left(1+F_j\right)\left(1+F_k\right)\left(1+F_l\right) \left(\log\left(\frac{F_i}{1+F_i}\frac{F_j}{1+F_j}\right) - \log\left(\frac{F_k}{1+F_k}\frac{F_l}{1+F_l}\right)\right) \left(\frac{F_k}{1+F_k}\frac{F_l}{1+F_l} - \frac{F_i}{1+F_i}\frac{F_j}{1+F_j}\right) \leq 0, \quad (16)$$

with equality if and only if

$$\frac{F_i}{1+F_i}\frac{F_j}{1+F_j} = \frac{F_k}{1+F_k}\frac{F_l}{1+F_l}, \quad (17)$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$. There is equality in Eq.(16), if and only if, $\dfrac{F}{1+F}$ is a Maxwellian, or equivalently, if and only if, F is a Planckian.

By the relations (13) and (15)

$$\langle \phi, C_{12}(F) + \Gamma C_{22}(F) \rangle, \tag{18}$$

is zero, independently of our choice of non-negative function F, if and only if, ϕ is a collision invariant, and so (for normal models) the equation

$$\langle \phi, C_{12}(F) + \Gamma C_{22}(F) \rangle = 0,$$

has the general solution (12).

3 Linearized operator

Given a Planckian

$$P = \frac{1}{e^{\alpha\left(|\mathbf{p}|^2+n\right)+\beta\cdot\mathbf{p}} - 1} = \frac{1}{e^{\alpha\left(|\mathbf{p}-\mathbf{p}_0|^2+n_0\right)} - 1}, \tag{19}$$

with $\alpha > 0$, $\beta \in \mathbb{R}^d$, $\mathbf{p}_0 = \dfrac{\beta}{2}$ and $n_0 = n - |\mathbf{p}_0|^2$, we denote

$$F = P + R^{1/2} f, \text{ with } R = P(1+P), \tag{20}$$

in Eq.(6), and obtain

$$B\frac{df}{dx} + Lf = S(f),$$

where $L = L_{12} + \Gamma L_{22}$ is the linearized collision operator ($N \times N$ matrix) given by

$$L_{12} f = -2nR^{-1/2}\widetilde{Q}(P, R^{1/2} f) - n\widetilde{L}R^{1/2} f \tag{21}$$

and

$$L_{22} f = -R^{-1/2}\left(2Q(P, R^{1/2} f) + \widehat{Q}(R^{1/2} f, P, P) + 2\widehat{Q}(P, R^{1/2} f, P)\right). \tag{22}$$

The nonlinear part $S(f) = S_{12}(f,f) + \Gamma S_{22}(f,f,f)$ is given by

$$S_{12}(f,g) = nR^{-1/2}\widetilde{Q}(R^{1/2} f, R^{1/2} g) \tag{23}$$

and

$$S_{22}(f,g,h) = R^{-1/2}\Big(Q(R^{1/2} f, R^{1/2} g) + \widehat{Q}(P + R^{1/2} f, R^{1/2} g, R^{1/2} h) + \widehat{Q}(R^{1/2} f, P, R^{1/2} h) + \widehat{Q}(R^{1/2} f, R^{1/2} g, P)\Big). \tag{24}$$

In more explicit forms, the operators (21) and (23) read

$$(L_{12} f)_i = n \sum_{j,k=1}^{N} \frac{\Gamma_{jk}^{i} L_{jk}^{i} f - 2\Gamma_{ij}^{k} L_{ij}^{k} f}{R_i^{1/2}}, \; i = 1, \ldots, N, \tag{25}$$

where

$$L^i_{jk}f = (1+P_j+P_k)R_i^{1/2}f_i - (P_k-P_i)R_j^{1/2}f_j - (P_j-P_i)R_k^{1/2}f_k,$$

and

$$S_{12i}(f,g) = n\sum_{j,k=1}^{N}\frac{\Gamma^i_{jk}S^i_{jk}(f,g) - 2\Gamma^k_{ij}S^k_{ij}(f,g)}{R_i^{1/2}},\ i=1,...,N,$$

with

$$S^i_{jk}(f,g) = \frac{1}{2}\Big(R_j^{1/2}R_k^{1/2}(f_jg_k+g_jf_k) - R_i^{1/2}R_j^{1/2}(f_ig_j+g_if_j) - \\ R_i^{1/2}R_k^{1/2}(f_ig_k+g_if_k)\Big).$$

Moreover, the operators (22) and (24) read, in more explicit forms,

$$(L_{22}f)_i = \sum_{j,k,l=1}^{N}\frac{\Gamma^{kl}_{ij}}{R_i^{1/2}}(P^{kl}_{ij}f_i + P^{kl}_{ji}f_j - P^{ij}_{kl}f_k - P^{ij}_{lk}f_l),\ i=1,...,N \qquad (26)$$

where

$$P^{kl}_{ij} = (P_j(1+P_k+P_l) - P_kP_l)R_i^{1/2},$$

and

$$S_{22i}(f,f,f) = \sum_{j,k,l=1}^{N}\frac{\Gamma^{kl}_{ij}}{R_i^{1/2}}\Big(S^{kl}_{ij}(f,f,f) - S^{ij}_{kl}(f,f,f)\Big),\ i=1,...,N,$$

with

$$S^{kl}_{ij}(f,f,f) = (1+P_i+P_j)R_k^{1/2}R_l^{1/2}f_kf_l + \Big(R_i^{1/2}f_i + R_j^{1/2}f_j\Big) \\ \Big(P_kR_l^{1/2}f_l + P_lR_k^{1/2}f_k + R_k^{1/2}R_l^{1/2}f_kf_l\Big).$$

By Eqs.(4),(25), and the relations

$$P_j(1+P_j)(P_k-P_i) = P_k(1+P_k)(P_j-P_i) = P_i(1+P_j)(1+P_k), \\ P_i(1+P_j+P_k) = P_2P_3 = P_i(1+P_j)(1+P_k)$$

for $\Gamma^i_{jk} \neq 0$, we obtain the equality

$$\langle g, L_{12}f\rangle = n\sum_{i,j,k=1}^{N}\Gamma^i_{jk}P_i(1+P_j)(1+P_k)\left(\frac{f_i}{R_i^{1/2}} - \frac{f_j}{R_j^{1/2}} - \frac{f_k}{R_k^{1/2}}\right) \\ \left(\frac{g_i}{R_i^{1/2}} - \frac{g_j}{R_j^{1/2}} - \frac{g_k}{R_k^{1/2}}\right).$$

Similarly, by Eqs.(5),(26), and the relations

$$P_iP_j(1+P_k)(1+P_l) = P_kP_l(1+P_i)(1+P_j),$$
$$P_{ij}^{kl} = P_kP_l(1+P_j)\frac{\sqrt{1+P_i}}{\sqrt{P_i}}$$

for $\Gamma_{ij}^{kl} \neq 0$, we obtain the equality

$$\langle g, L_{22}f\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{N} \Gamma_{ij}^{kl} P_iP_j(1+P_k)(1+P_l) \left(\frac{f_i}{R_i^{1/2}}+\frac{f_j}{R_j^{1/2}}-\frac{f_k}{R_k^{1/2}}-\frac{f_l}{R_l^{1/2}}\right)\left(\frac{g_i}{R_i^{1/2}}+\frac{g_j}{R_j^{1/2}}-\frac{g_k}{R_k^{1/2}}-\frac{g_l}{R_l^{1/2}}\right).$$

It is easy to see that the matrix L is symmetric and positive semi-definite, i.e.

$$\langle g, Lf\rangle = \langle Lg, f\rangle \text{ and } \langle f, Lf\rangle \geq 0,$$

for all functions $g = g(\xi)$ and $f = f(\xi)$.

Furthermore, $\langle f, Lf\rangle = 0$ if and only if

$$\frac{f_i}{R_i^{1/2}} = \frac{f_j}{R_j^{1/2}} + \frac{f_k}{R_k^{1/2}} \tag{27}$$

for all indices satisfying $\Gamma_{jk}^{i} \neq 0$ if $\Gamma = 0$.

If $\Gamma \neq 0$, $\langle f, Lf\rangle = 0$ if and only if also, additionally to Eq.(27),

$$\frac{f_i}{R_i^{1/2}} + \frac{f_j}{R_j^{1/2}} = \frac{f_k}{R_k^{1/2}} + \frac{f_l}{R_l^{1/2}} \tag{28}$$

for all indices satisfying $\Gamma_{ij}^{kl} \neq 0$. We denote $f = R^{1/2}\phi$ in Eq.(27) and Eq.(28) and obtain Eq.(9) and Eq.(10) respectively. Hence, since L is semi-positive,

$$Lf = 0 \text{ if and only if } f = R^{1/2}\phi,$$

where ϕ is a collision invariant (12).

Then also

$$\left\langle S(f), R^{1/2}\phi \right\rangle = \langle C_{12}(F) + \Gamma C_{22}(F), \phi\rangle + \left\langle F, LR^{1/2}\phi \right\rangle = 0$$

for all collision invariants ϕ.

The system (6) transforms in

$$B\frac{df}{dx} + Lf = S(f). \tag{29}$$

The diagonal matrix B (6) (under our assumptions) has no zero diagonal elements and is non-singular. We denote $f(0) = f_0$. Then the formal solution of Eq.(29) reads

$$f(x) = e^{-xB^{-1}L} f_0 + \int_0^x e^{(\sigma - x)B^{-1}L} B^{-1} \left[S(f) \right](\sigma)\, d\sigma.$$

As in the case of DVMs for the Boltzmann equation, the linearized operator L is symmetric, positive semi-definite, and have a non-trivial null-space, and by assumption, the matrix B is non-singular. Therefore, we can apply a result obtained by Bobylev and Bernhoff in [9] (see also [5]), that we will present below.

We denote by $n^{\pm}$, where $n^+ + n^- = N$, and $m^{\pm}$, with $m^+ + m^- = q$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices B and $B^{-1}L$ respectively, and by m^0 the number of zero eigenvalues of $B^{-1}L$. Moreover, we denote by k^+, k^-, and l the numbers of positive, negative, and zero eigenvalues of the $\rho \times \rho$ matrix K ($\rho = d + 1$ for normal discrete models), with entries

$$k_{ij} = \left\langle y_i, y_j \right\rangle_B = \left\langle y_i, B y_j \right\rangle,$$

such that $\left\{ y_1, \ldots, y_\rho \right\}$ is a basis of the null-space of L, i.e. in our case

$$\text{span}\left(y_1, \ldots, y_\rho \right) = N(L) = \text{ span}\left(R^{1/2} p^1, \ldots, R^{1/2} p^d, R^{1/2} (|\mathbf{p}|^2 + n) \right).$$

Here and below, we denote $\langle \cdot, \cdot \rangle_B = \langle \cdot, B \cdot \rangle$ and by $N(L)$ the null-space of L.

In applications, the number ρ of collision invariants is usually relatively small compared to N (note that formally $N = \infty$ for the continuous equation whereas $\rho \leq 4$). Also, the matrix B is diagonal and therefore all its eigenvalues are known. This explains the importance of the following result [9]. The theorem is valid for any real symmetric matrices L and B, such that L is semi-positive, B is invertible, and $\dim(N(L)) = \rho \geq 1$.

Theorem 1 *The numbers of positive, negative and zero eigenvalues of $B^{-1}L$ are given by*

$$\begin{cases} m^+ = n^+ - k^+ - l \\ m^- = n^- - k^- - l \\ m^0 = \rho + l. \end{cases}$$

In the proof of Theorem 1 a basis

$$\left\{ u_1, \ldots, u_q, y_1, \ldots, y_k, z_1, \ldots, z_l, w_1, \ldots, w_l \right\} \tag{30}$$

of $\mathbb{R}^N$, such that

$$y_i, z_r \in N(L),\ B^{-1} L w_r = z_r \text{ and } B^{-1} L u_\alpha = \lambda_\alpha u_\alpha, \tag{31}$$

and

$$\begin{gathered} \left\langle u_\alpha, u_\beta \right\rangle_B = \lambda_\alpha \delta_{\alpha\beta}, \text{ with } \lambda_1, \ldots, \lambda_{m^+} > 0 \text{ and } \lambda_{m^+ + 1}, \ldots, \lambda_q < 0, \\ \left\langle y_i, y_j \right\rangle_B = \gamma_i \delta_{ij}, \text{ with } \gamma_1, \ldots, \gamma_{k^+} > 0 \text{ and } \gamma_{k^+ + 1}, \ldots, \gamma_k < 0, \\ \langle u_\alpha, z_r \rangle_B = \langle u_\alpha, w_r \rangle_B = \langle u_\alpha, y_i \rangle_B = \langle w_r, y_i \rangle_B = \langle z_r, y_i \rangle_B = 0, \\ \langle w_r, w_s \rangle_B = \langle z_r, z_s \rangle_B = 0 \text{ and } \langle w_r, z_s \rangle_B = \delta_{rs}, \end{gathered} \tag{32}$$

is constructed.

4 Half-space problems

We consider the inhomogeneous (or homogeneous if $g = 0$) linearized problem

$$B\frac{df}{dx} + Lf = g, \tag{33}$$

where $g = g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$, with one of the boundary conditions

(O) the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty;$$

(P) the solution is bounded, i.e.

$$|f(x)| < \infty \text{ for all } x \in \mathbb{R}_+;$$

(Q) the solution can be slowly increasing at infinity, i.e.

$$|f(x)|\, e^{-\varepsilon x} \to 0 \text{ as } x \to \infty, \text{ for all } \varepsilon > 0.$$

In case of boundary condition (O) we additionally assume that

$$g(x) \in N(L)^{\perp} \text{ for all } x \in \mathbb{R}_+. \tag{34}$$

Remark 2 The boundary condition (O) corresponds to the case when we have made the expansion (20) around a Planckian P, such that $F \to P$ as $x \to \infty$. The boundary conditions (P) and (Q) are the boundary conditions in the Milne and Kramers problem respectively.

We can (without loss of generality) assume that

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \tag{35}$$

where

$$B_+ = \operatorname{diag}\left(p_1^1, ..., p_{n^+}^1\right) \text{ and } B_- = -\operatorname{diag}\left(p_{n^++1}^1, ..., p_N^1\right), \text{ with}$$
$$p_1^1, ..., p_{n^+}^1 > 0 \text{ and } p_{n^++1}^1, ..., p_N^1 < 0.$$

We also define the projections $R_+ : \mathbb{R}^N \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^N \to \mathbb{R}^{n^-}$, by

$$R_+ s = s^+ = (s_1, ..., s_{n^+}) \text{ and } R_- s = s^- = (s_{n^++1}, ..., s_N)$$

for $s = (s_1, ..., s_N)$.

At $x = 0$ we assume the boundary condition

$$f^+(0) = h_0, \tag{36}$$

where $h_0 \in \mathbb{R}^{n^+}$.

The solutions of the system (33) with one of the boundary conditions (O) (together with condition (34)), (P), and (Q) reads

$$f(x) = \boldsymbol{\Psi}^+(x) + \boldsymbol{\Psi}^-(x) + \boldsymbol{\Phi}^+(x) + \boldsymbol{\Phi}^-(x),$$

where (in the notations of (30)-(32))

$$\Psi^{+}(x)=\sum_{r=1}^{m^{+}}u_r\left(\beta_r(0)e^{-\lambda_r x}+\int_0^x e^{(\sigma-x)\lambda_r}\frac{\langle g(\sigma),u_r\rangle}{\lambda_r}\,d\sigma\right),$$

$$\Psi^{-}(x)=-\sum_{r=m^{+}+1}^{q}u_r\int_x^{\infty}e^{(\sigma-x)\lambda_r}\frac{\langle g(\sigma),u_r\rangle}{\lambda_r}\,d\sigma,$$

$$\Phi^{+}(x)=\sum_{i=1}^{k^{+}}y_i\left(\mu_i(0)+\int_0^x\frac{\langle g(\sigma),y_i\rangle}{\langle y_i,y_i\rangle_B}\,d\sigma\right)$$

$$+\sum_{j=1}^{l}z_j\left(\eta_j(0)+\int_0^x\left\langle g(\sigma),w_j\right\rangle d\sigma-x\left(\alpha_j(0)+\int_0^x\left\langle g(\sigma),z_j\right\rangle d\sigma\right)\right),$$

$$\Phi^{-}(x)=\sum_{i=k^{+}+1}^{k}y_i\left(\mu_i(0)+\int_0^x\frac{\langle g(\sigma),y_i\rangle}{\langle y_i,y_i\rangle_B}\,d\sigma\right)$$

$$+\sum_{j=1}^{l}w_j\left(\alpha_j(0)+\int_0^x\left\langle g(\sigma),z_j\right\rangle d\sigma\right),$$

with for the case with boundary condition (O) (note that $\langle g(x),y_i\rangle=\left\langle g(\sigma),z_j\right\rangle=0$ for $i=1,...,k$ and $j=1,...,l$, by condition (34))

$$\mu_i(0)=\alpha_j(0)=0 \text{ for } i=1,...,k \text{ and } j=1,...,l, \text{ and}$$

$$\eta_j(0)=-\int_0^{\infty}\left\langle g(\sigma),w_j\right\rangle d\sigma \text{ for } j=1,...,l,$$

and for the case with boundary condition (P)

$$\alpha_j(0)=-\int_0^{\infty}\left\langle g(\sigma),z_j\right\rangle d\sigma \text{ for } j=1,...,l.$$

By the boundary condition (36), or equivalently

$$R_+\Psi^{+}(0)+R_+\Phi^{+}(0)=h_0-R_+\Psi^{-}(0)-R_+\Phi^{-}(0),$$

and that

$$\{R_+u_1,...,R_+u_{m^+},R_+y_1,...,R_+y_{k^+},R_+z_1,...,R_+z_l\}$$

is a basis of $\mathbb{R}^{n^+}$ (see [9,5,6]), we obtain the following theorem (cf. [5]), where we denote

$$\begin{cases} \mu_i^\infty = \mu_i(0) + \int\limits_0^\infty \frac{\langle g(\sigma), y_i\rangle}{\langle y_i, y_i\rangle_B} \, d\sigma \text{ for } i = 1, ..., k \\ \alpha_j^\infty = \alpha_j(0) + \int\limits_0^\infty \left\langle g(\sigma), z_j \right\rangle d\sigma \text{ for } j = 1, ..., l \\ \eta_j^\infty = \eta_j(0) + \int\limits_0^\infty \left\langle g(\sigma), w_j \right\rangle d\sigma \text{ for } j = 1, ..., l \end{cases} .$$

Theorem 2 *(i) Let*

$$U_+ = span(u_1, ..., u_{m^+}) = span\left\{ u \,|\, B^{-1}Lu = \lambda u \, for\ some\ \lambda > 0 \right\}.$$

Assume that the condition (34) *is fulfilled and that*

$$h_0, R_+ e^{xB^{-1}L} B^{-1} g(x) \in R_+ U_+ \ for\ all\ x \in \mathbb{R}_+. \tag{37}$$

Then the system (33) *with the boundary conditions (O) and* (36) *has a unique solution.*

(ii) Assume that

$$\lim_{x\to\infty} x \int\limits_x^\infty \left\langle g(\sigma), z_j \right\rangle d\sigma = 0 \, for\ j = 1, ..., l. \tag{38}$$

Then the system (33) *with the boundary conditions (P) and* (36) *has a unique solution with the asymptotic flow*

$$f_A = \sum_{i=1}^{k} \mu_i^\infty y_i + \sum_{j=1}^{l} \eta_j^\infty z_j,$$

if the k^- parameters $\mu_{k^+ +1}^\infty, ..., \mu_k^\infty$ are prescribed.

(iii) The system (33) *with the boundary conditions (Q) and* (36) *has a unique solution with the asymptotic flow*

$$f_A(x) = \sum_{i=1}^{k} \mu_i^\infty y_i + \sum_{j=1}^{l} \left(\left(\eta_j^\infty - x\alpha_j^\infty \right) z_j + \alpha_j^\infty w_j \right),$$

if the $k^- + l$ parameters $\mu_{k^+ +1}^\infty, ..., \mu_k^\infty$ and $\alpha_1^\infty, ..., \alpha_l^\infty$ are prescribed.

Especially, for the homogeneous system

$$B\frac{df}{dx} + Lf = 0,$$

condition (38) is fulfiled and condition(37) is reduced to

$$h_0 \in U_+.$$

Remark 3 We can also, before prescribing the set of velocities, make the change of variables $\mathbf{p} \to \mathbf{p} + \mathbf{p}_0$ (cf. Eq.(19)). We then, instead of relations (4), obtain the relations

$$\mathbf{p}_i = \mathbf{p}_j + \mathbf{p}_k + \mathbf{p}_0 \text{ and } |\mathbf{p}_i|^2 = |\mathbf{p}_j|^2 + |\mathbf{p}_k|^2 + n_0,$$

and the collision invariants

$$\phi = \mathbf{a}\cdot(\mathbf{p} + \mathbf{p}_0) + b\left(|\mathbf{p}|^2 + n_0\right).$$

Moreover

$$N(L) = \text{ span}\left(R^{1/2}\left(p^1 + p_0^1\right), ..., R^{1/2}\left(p^d + p_0^d\right), R^{1/2}\left(|\mathbf{p}|^2 + n_0\right)\right),$$

and if $p_0^1 \neq 0$, then the matrix B have to be replaced with $B + p_0^1 I$.

Remark 4 Our results can be generalized to more general boundary conditions

$$f^+(0) = Cf^-(0) + h_0,$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$ (cf. [5,6]).

In order to be able to obtain existence and uniqueness of solutions of the linearized half-space problems we will then need to assume that the matrix C fulfills the condition

$$\dim\left(R_+ - CR_-\right)U_+ = m^+,$$

with $U_+ = $span$(u_1, ..., u_{m^+})$, as we consider boundary condition (O), the condition

$$\dim\left(R_+ - CR_-\right)X_+ = n^+, \tag{39}$$

with $X_+ = $span$(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1, ..., z_l)$, as we consider boundary condition (P), and the condition (39) or the condition

$$\dim\left(R_+ - CR_-\right)\widetilde{X}_+ = n^+,$$

with $\widetilde{X}_+ = $span$(u_1, ..., u_{m^+}, y_1, ..., y_{k^+}, z_1 + w_1, ..., z_l + w_l)$, as we consider boundary condition (Q). Furthermore, R_+ need to be replaced by $R_+ - CR_-$ in assumption (37).

Remark 5 All our results can be extended in a natural way (cf. [5,6]), to yield for singular matrices B, if

$$N(L) \cap N(B) = \{0\}.$$

Remark 6 Similar results can be established for the discrete Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation

$$p_i^1 \frac{dF_i}{dx} = Q_i^\varepsilon(F), x \in \mathbb{R}_+, i = 1, ..., N,$$

where

$$Q_i^\varepsilon(F) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl}(F_k F_l (1 + \varepsilon F_i)(1 + \varepsilon F_j) - F_i F_j (1 + \varepsilon F_k)(1 + \varepsilon F_l)),$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0,$$

with equality unless the conservation laws (5) are satisfied.

Here $\varepsilon = 0$ corresponds to the discrete Boltzmann equation, and we have $\varepsilon = 1$ for bosons and $\varepsilon = -1$ for fermions.

The singular points are

$$P = \frac{M}{1 - \varepsilon M},$$

where $M = e^{a + \mathbf{b} \cdot \mathbf{p} + c|\mathbf{p}|^2}$, with $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$, is a Maxwellian (note that for $\varepsilon = 0$, $P = M$).

By denoting (cf. Eq.(20))

$$f = P + \sqrt{R} F, \text{ with } R = P(1 + \varepsilon P),$$

we obtain a system, where the linearized operator L is symmetric and positive semi-definite, with a non-trivial null-space $N(L)$, which for normal models will be

$$N(L) = \text{ span} \left(R^{1/2}, R^{1/2} p^1, ..., R^{1/2} p^d, R^{1/2} |\mathbf{p}|^2 \right).$$

For $\varepsilon = 0$ this is a well-known fact, see for example [5], and for $\varepsilon = 1$, $L = L_{22}$, where L_{22} is given in Eq.(22), in the particular case $\Gamma_{ij}^{kl} = 1$ for all non-zero Γ_{ij}^{kl}.

Remark 7 We introduce a discrete version of the anyon Boltzmann equation [8, 1]:

$$p_i^1 \frac{dg_i}{dx} = Q_i^{\alpha}(g), x \in \mathbb{R}_+, i = 1, ..., N,$$

where $0 \leq \alpha \leq 1$ ($\alpha = 0$ for bosons and $\alpha = 1$ for fermions) and

$$Q_i^{\alpha}(g) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} \left(g_k g_l F(g_i) F(g_j) - g_i g_j F(g_k) F(g_l) \right),$$

with

$$F(h) = (1 - \alpha h)^{\alpha} (1 + (1 - \alpha) h)^{1-\alpha},$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0,$$

with equality unless the conservation laws (5) are satisfied.

The singular points g_0 are given by

$$\frac{g_0}{F(g_0)} = M$$

where $M = e^{a + \mathbf{b} \cdot \mathbf{p} + c|\mathbf{p}|^2}$, with $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$, is a Maxwellian, or (cf. [31])

$$g_0 = \frac{1}{\omega(\mathbf{p}) + \alpha},$$

where

$$\omega(\mathbf{p})^{\alpha}(1+\omega(\mathbf{p}))^{1-\alpha} = M^{-1}.$$

By denoting (cf. Eq.(20))

$$f = g_0 + \sqrt{R}g, \text{ with } R = \frac{g_0 F(g_0)}{F(g_0) + (2\alpha - 1) g_0},$$

we can, with a similar approach as in Section 3, obtain a system, where the linearized operator L again is symmetric and positive semi-definite, with a non-trivial null-space $N(L)$, which for normal models will be

$$N(L) = \text{ span}\left(R^{1/2}, R^{1/2}p^1, ..., R^{1/2}p^d, R^{1/2}|\mathbf{p}|^2\right),$$

and then Theorem 2 can be applied also in this case.

To our knowledge there are no similar results in the literature before, neither in the discrete or the continuous case, as the ones indicated by Remarks 6, 7, and Theorem 2.

5 Axially symmetric discrete models

In this section we consider only such symmetric sets $\mathscr{P}$, such that

$$\text{if } \mathbf{p}_i = (p_i^1, p_i^2, ..., p_i^d) \in \mathscr{P}, \text{ then also } (-p_i^1, p_i^2, ..., p_i^d) \in \mathscr{P}. \quad (40)$$

Then the equations (3) admit a class of solutions satisfying

$$F_i = F_{i'} \text{ if } p_i^1 = p_{i'}^1 \text{ and } |\mathbf{p}_i|^2 = |\mathbf{p}_{i'}|^2. \quad (41)$$

This reduces the number N of equations (3) to the number $2\widetilde{N} \leq N$ of different combinations $(p_i^1, |\mathbf{p}_i|^2)$. The structure of the collision terms (7) and (8) (in slightly different notations) remains unchanged. However, to be able to keep the structure, we might need to add equal equations (instead of just taking them away). Hence, the elements in the diagonal matrix (35) might change, but will still be multiples (with positive multipliers) of the previous ones. The multipliers will also be the same independently of the sign. Below we will omit the tildes, and just write N instead of $\widetilde{N}$.

We can, without loss of generality, assume that

$$(p_{i+N}^1, |\mathbf{p}_{i+N}|^2) = (-p_i^1, |\mathbf{p}_i|^2) \text{ and } p_i^1 > 0$$

for $i = 1, ..., N$.

We now assume that (i) we have a symmetric set $\mathscr{P}$ (40); (ii) the reduction (41) is done with the multipliers $m_i, i = 1, ..., N$; (iii) our discrete model is normal; and (iv)

$$B = \text{diag}(m_1 p_1^1, ..., m_N p_N^1, -m_1 p_1^1, ..., -m_N p_N^1), \text{ with } p_1^1, ..., p_N^1 > 0.$$

Here we study instead of Eq.(33), cf. Remark 3, the equation

$$(B+p_0^1 I)\frac{df}{dx}+Lf=g.$$

The linearized collision operator L has the null-space

$$N(L)=\operatorname{span}(\phi_1,\phi_2),$$

where

$$\begin{cases}\phi_1=R^{1/2}\left(p^1+p_0^1\right)=R^{1/2}\cdot(p_1^1+p_0^1,\dots,p_N^1+p_0^1,-p_1^1+p_0^1,\dots,-p_N^1+p_0^1)\\ \phi_2=R^{1/2}\left|\mathbf{p}\right|^2=R^{1/2}\cdot(\left|\mathbf{p}_1\right|^2,\dots,\left|\mathbf{p}_N\right|^2,\left|\mathbf{p}_1\right|^2,\dots,\left|\mathbf{p}_N\right|^2),\ R=P(1+P)\end{cases}.$$

Then

$$K=(\langle\phi_i,\phi_j\rangle_{B+p_0^1 I})=\begin{pmatrix}\left(p_0^1\right)^3\chi_0+3p_0^1\chi_1 & \chi_2+\left(p_0^1\right)^2\chi_4\\ \chi_2+\left(p_0^1\right)^2\chi_4 & p_0^1\chi_3\end{pmatrix},$$

where

$$\chi_0=\left\langle R^{1/2},R^{1/2}\right\rangle,\chi_1=\left\langle R^{1/2}p^1,R^{1/2}p^1\right\rangle=\left\langle R^{1/2},R^{1/2}p^1\right\rangle_B,$$
$$\chi_2=\left\langle R^{1/2}p^1,R^{1/2}\left|\mathbf{p}\right|^2\right\rangle_B,\chi_3=\left\langle R^{1/2}\left|\mathbf{p}\right|^2,R^{1/2}\left|\mathbf{p}\right|^2\right\rangle,\chi_4=\left\langle R^{1/2},R^{1/2}\left|\mathbf{p}\right|^2\right\rangle.$$

Hence,

$$\det(K)=\left(p_0^1\right)^4\chi_0\chi_3+3\left(p_0^1\right)^2\chi_1\chi_3-\left(\chi_2+\left(p_0^1\right)^2\chi_4\right)^2,$$

and the degenerate values of p_0^1, i.e. the values of p_0^1 for which $l\geq 1$, are

$$p_{0\pm}^1=\pm\sqrt{\frac{3\chi_1\chi_3-2\chi_2\chi_4+\sqrt{\left(3\chi_1\chi_3-2\chi_2\chi_4\right)^2+4\left(\chi_0\chi_3-\chi_4^2\right)\chi_2^2}}{2\left(\chi_0\chi_3-\chi_4^2\right)}}.$$

We obtain the following table for the values of k^+, k^- and l

	$p_0^1<p_{0-}^1$	$p_0^1=p_{0-}^1$	$p_{0-}^1<p_0^1<p_{0+}^1$	$p_0^1=p_{0+}^1$	$p_{0+}^1<p_0^1$
k^+	0	0	1	1	2
k^-	2	1	1	0	0
l	0	1	0	1	0

.

If we consider symmetric sets $\mathscr{P}$, such that

$$\text{if } \mathbf{p}_i=(p_i^1,p_i^2,\dots,p_i^d)\in\mathscr{P},\text{ then also }(\pm p_i^1,\pm p_i^2,\dots,\pm p_i^d)\in\mathscr{P},$$

then 0 is added to the degenerate values and the values of k^+, k^- and l are

		$p_0^1=-p_{0+}^1$		$p_0^1=0$		$p_0^1=p_{0+}^1$	
k^+	0	0	1	1	d	d	$d+1$
k^-	$d+1$	d	d	1	1	0	0
l	0	1	0	$d-1$	0	1	0

.

Similar numbers have been calculated for axially symmetric DVMs around the x-axis in [7,5].

For the continuous Boltzmann equation the degenerate values are 0 and $\pm\sqrt{\frac{5T}{3}}$, where T is the temperature of the Maxwellian M, that the linearization is made around (cf. Eq.(20)) [17]. The values of k^+, k^- and l for the Boltzmann equation are given by the table (cf. [17])

		$u=-\sqrt{\frac{5T}{3}}$		$u=0$		$u=\sqrt{\frac{5T}{3}}$	
k^+	0	0	1	1	4	4	5
k^-	5	4	4	1	1	0	0
l	0	1	0	3	0	1	0

.

These values are of as great importance in the continuous case as in the discrete case [5,6,4,17,30,32]. Therefore it is natural to believe that these numbers should have the same importance also for the continuous version of the equation of our studies (cf. Theorem 2).

In the continuous case $\langle f,g\rangle = \int fg\,d\mathbf{p}$ and $\langle f,g\rangle_{p^1} = \int p^1 fg\,d\mathbf{p}$ corresponds to $\langle f,g\rangle_B$. Assuming that

$$P = \frac{1}{e^{\frac{|\mathbf{p}|^2}{T}}-1},$$

we have

$$R = P(1+P) = \frac{e^{\frac{|\mathbf{p}|^2}{T}}}{\left(e^{\frac{|\mathbf{p}|^2}{T}}-1\right)^2},$$

and hence, by a change to spherical coordinates, we obtain

$$\chi_0 = \int\limits_{|\mathbf{p}|\geq\lambda\sqrt{T}} R\,d\mathbf{p} = 4\pi T\,I_2, \chi_1 = \int\limits_{|\mathbf{p}|\geq\lambda\sqrt{T}} R\left(p^1\right)^2 d\mathbf{p} = \frac{4\pi}{3}T^2 I_4,$$

$$\chi_2 = \int\limits_{|\mathbf{p}|\geq\lambda\sqrt{T}} R\left(p^1\right)^2 |\mathbf{p}|^2\,d\mathbf{p} = \frac{4\pi}{3}T^3 I_6, \chi_3 = \int\limits_{|\mathbf{p}|\geq\lambda\sqrt{T}} R\,|\mathbf{p}|^4\,d\mathbf{p} = 4\pi T^3 I_6, \text{ and}$$

$$\chi_4 = \int\limits_{|\mathbf{p}|\geq\lambda\sqrt{T}} R\,|\mathbf{p}|^2\,d\mathbf{p} = 4\pi T^2 I_4, \text{ with } I_n = \int_\lambda^\infty r^n \frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\,dr.$$

Here we have considered the restriction $|\mathbf{p}| \geq \lambda\sqrt{T}$, for some $\lambda > 0$, as in [2]. Hence, in the continuous case the degenerate values are

$$p^1_{0\pm} = \pm\sqrt{\frac{T}{6}}\sqrt{\frac{I_4 I_6 + I_6\sqrt{4I_2I_6-3I_4^2}}{I_2I_6-I_4^2}}, \text{ with } I_n = \int_\lambda^\infty r^n \frac{e^{r^2}}{\left(e^{r^2}-1\right)^2}\,dr.$$

We want to stress that $I_2 I_6 \geq I_4^2$ by the Cauchy-Schwarz inequality. Furthermore, the values of k^+, k^- and l are given by the table

		$p_0^1 = -p_{0+}^1$		$p_0^1 = 0$		$p_0^1 = p_{0+}^1$	
k^+	0	0	1	1	3	3	4
k^-	4	3	3	1	1	0	0
l	0	1	0	2	0	1	0

In [2] the case $p_0^1 = 0$ is considered for the continuous equation in a symmetric setting (corresponding to the symmetric reduction above), such that $l = 0$, but still $k^+ = k^- = 1$

6 Explicit solutions for a small-sized model

We consider the case $\Gamma = 0$ in Eq.(6), assuming that the change of variables in Remark 3 is made for $\mathbf{p}_0 = (0, \sqrt{\frac{n}{2}}, \sqrt{\frac{n}{2}})$ (note that $p_0^1 = 0$ and $n_0 = n - |\mathbf{p}_0|^2 = 0$, cf. [2]), and introduce the model with

$$\begin{aligned} &p_1 = (2p,0,0), p_2 = (p,q^+,q^-), p_3 = (p,q^-,q^+), \\ &p_4 = (-2p,0,0), p_5 = (-p,q^+,q^-), p_6 = (-p,q^-,q^+), \end{aligned}$$

where

$$q^{\pm} = \frac{-\sqrt{2n} \pm \sqrt{8p^2 - 2n}}{4},$$

in space. Then the nonlinear system (33) reads

$$\begin{cases} 2p\dfrac{d\widetilde{F}_1}{dx} = 2n\left(\left(1+\widetilde{F}_1\right)\widetilde{F}_2\widetilde{F}_3 - \widetilde{F}_1\left(1+\widetilde{F}_2\right)\left(1+\widetilde{F}_3\right)\right) \\ p\dfrac{d\widetilde{F}_2}{dx} = -2n\left(\left(1+\widetilde{F}_1\right)\widetilde{F}_2\widetilde{F}_3 - \widetilde{F}_1\left(1+\widetilde{F}_2\right)\left(1+\widetilde{F}_3\right)\right) \\ p\dfrac{d\widetilde{F}_3}{dx} = -2n\left(\left(1+\widetilde{F}_1\right)\widetilde{F}_2\widetilde{F}_3 - \widetilde{F}_1\left(1+\widetilde{F}_2\right)\left(1+\widetilde{F}_3\right)\right) \\ -2p\dfrac{d\widetilde{F}_4}{dx} = 2n\left(\left(1+\widetilde{F}_4\right)\widetilde{F}_5\widetilde{F}_6 - \widetilde{F}_4\left(1+\widetilde{F}_5\right)\left(1+\widetilde{F}_6\right)\right) \\ -p\dfrac{d\widetilde{F}_5}{dx} = -2n\left(\left(1+\widetilde{F}_4\right)\widetilde{F}_5\widetilde{F}_6 - \widetilde{F}_4\left(1+\widetilde{F}_5\right)\left(1+\widetilde{F}_6\right)\right) \\ -p\dfrac{d\widetilde{F}_6}{dx} = -2n\left(\left(1+\widetilde{F}_4\right)\widetilde{F}_5\widetilde{F}_6 - \widetilde{F}_4\left(1+\widetilde{F}_5\right)\left(1+\widetilde{F}_6\right)\right) \end{cases}, \tag{42}$$

For a flow axially symmetric around the x-axis we can reduce the system (42) to

$$\begin{cases} 2p\dfrac{dF_1}{dx} = 2n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right) \\ 2p\dfrac{dF_2}{dx} = -4n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right) \\ -2p\dfrac{dF_3}{dx} = 2n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right) \\ -2p\dfrac{dF_4}{dx} = -4n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right) \end{cases}. \tag{43}$$

with $F_1 = \widetilde{F}_1$, $F_2 = \widetilde{F}_2 = \widetilde{F}_3$, $F_3 = \widetilde{F}_4$, and $F_4 = \widetilde{F}_5 = \widetilde{F}_6$. Note that the collision invariants are p^1 and $|\mathbf{p}|^2$.

We define the projections $R_+ : \mathbb{R}^4 \to \mathbb{R}^2$ and $R_- : \mathbb{R}^4 \to \mathbb{R}^2$, by

$$R_+h = h^+ = (h_1, h_2) \text{ and } R_-h = h^- = (h_3, h_4),$$

where $h = (h_1, h_2, h_3, h_4)$.

We consider the problem

$$\begin{cases} B\dfrac{dF}{dx} = C_{12}(F), \\ F^+(0) = \widetilde{h}_0 \end{cases},$$

where

$$B = (2p, 2p, -2p, -2p)$$

and

$$C_{12}(F) = 2n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right)(1,-2,0,0) + \\ 2n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right)(0,0,1,-2).$$

If we denote

$$F = P + R^{1/2}f,$$

with

$$R = P(1+P) \text{ and } P = \frac{1}{e^{2p^2}-1}\left(\frac{1}{e^{2p^2}+1}, 1, \frac{1}{e^{2p^2}+1}, 1\right),$$

in Eq.(43) we obtain

$$\begin{cases} B\dfrac{df}{dx} + Lf = S(f) \\ f^+(0) = h_0 \end{cases},$$

where L is the linearized collision operator, $S(f)$ the nonlinear part, and $h_0 \in \mathbb{R}^2$. The linearized problem reads

$$\begin{cases} B\dfrac{df}{dx} + Lf = 0 \\ f^+(0) = h_0 \end{cases}, \tag{44}$$

where the linearized collision operator

$$L = \frac{2n}{\sinh(p^2)} \begin{pmatrix} \cosh(p^2) & -1 & 0 & 0 \\ -1 & \frac{1}{\cosh(p^2)} & 0 & 0 \\ 0 & 0 & \cosh(p^2) & -1 \\ 0 & 0 & -1 & \frac{1}{\cosh(p^2)} \end{pmatrix}$$

is symmetric and positive semi-definite, and have the null-space

$$N(L) = \mathrm{span}(R^{1/2}p^1, R^{1/2}|\mathbf{p}|^2) = \mathrm{span}(\left(1,\cosh(p^2),0,0\right), \left(0,0,1,\cosh(p^2)\right)).$$

Since

$$K = \begin{pmatrix} \langle y_1,y_1\rangle_B & \langle y_1,y_2\rangle_B \\ \langle y_2,y_1\rangle_B & \langle y_2,y_2\rangle_B \end{pmatrix} = \left(1+\cosh^2(p^2)\right) \begin{pmatrix} 2p & 0 \\ 0 & -2p \end{pmatrix},$$

where

$$y_1 = \left(1,\cosh(p^2),0,0\right) \text{ and } y_2 = \left(0,0,1,\cosh(p^2)\right),$$

$$k^+ = k^- = 1 \text{ and } l = 0,$$

and hence the matrix $B^{-1}L$ has one positive and one negative eigenvalue.

Explicitly, the non-zero eigenvalues of the matrix $B^{-1}L$ are

$$\lambda_\pm = \pm\frac{n}{p\sinh(p^2)}\left(\frac{1}{\cosh(p^2)} + \cosh(p^2)\right),$$

with the corresponding eigenvectors

$$v_+ = (\cosh(p^2),-1,0,0) \text{ and } v_- = (0,0,\cosh(p^2),-1).$$

The bounded solution of problem (44) is

$$f = \alpha e^{-\lambda_+ x}v_+ + \beta y_1 + \gamma y_2.$$

By the boundary condition

$$f_0^+ = (h_{01},h_{02}),$$

we obtain the system

$$\begin{cases} \alpha\cosh(p^2) + \beta = h_{01} \\ -\alpha + \beta\cosh(p^2) = h_{02} \end{cases},$$

and hence

$$\begin{cases} \alpha = \frac{h_{01}\cosh(p^2)-h_{02}}{1+\cosh^2(p^2)} \\ \beta = \frac{h_{01}+h_{02}\cosh(p^2)}{1+\cosh^2(p^2)} \end{cases}.$$

The parameter γ can be fixed if we assume the additional condition

$$\left\langle y, R^{1/2}f\right\rangle_B = \beta\langle y,y_1\rangle_B + \gamma\langle y,y_2\rangle_B = \mathscr{E},$$

where

$$y = R^{1/2}|\mathbf{p}|^2 = \frac{p^2}{\sinh(p^2)}\left(\frac{1}{\cosh(p^2)},1,\frac{1}{\cosh(p^2)},1\right).$$

Then

$$\gamma = \frac{\mathscr{E} - \beta \langle y, y_1 \rangle_B}{\langle y, y_2 \rangle_B} = \beta - \frac{\mathscr{E} \sinh(p^2)\cosh(p^2)}{2p^3\left(1+\cosh^2(p^2)\right)} = \frac{2p^3\left(h_{01}+h_{02}\cosh(p^2)\right) - \mathscr{E}\sinh(p^2)\cosh(p^2)}{2p^3\left(1+\cosh^2(p^2)\right)}.$$

The non-constant mass flow is given by

$$\left\langle R^{1/2}, f \right\rangle_B = \frac{p}{2\sinh(p^2)}\left((\beta-\gamma)\left(\frac{1}{\cosh(p^2)}+2\cosh(p^2)\right) - \alpha e^{-\lambda_+ x}\right) = \frac{\mathscr{E}\sinh(p^2)\left(1+2\cosh^2(p^2)\right)+2p^3\left(h_{02}-h_{01}\cosh(p^2)\right)e^{-\lambda_+ x}}{4p^2\sinh(p^2)\left(1+\cosh^2(p^2)\right)}.$$

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