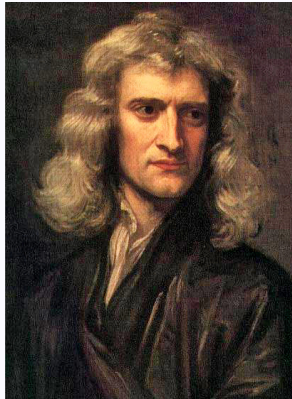# The classical n-body problem in the context of curved space

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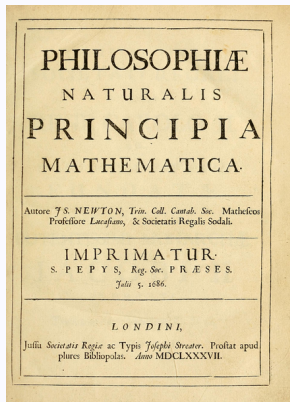CANADA**

**Karlstad, Sweden
10 november 2016**

Isaac Newton (1642-1726) at age 47
(oil painting by Godfrey Kneller)

1687: Newton publishes the first edition of

Philosophiae Naturalis Principia Mathematica

The equations of motion of the classical n-body problem in Euclidean space $\mathbb{R}^3$:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = \overline{1, n},$$

where $m_i, i = \overline{1, n}$, are the masses, $\mathbf{r}_i = (x_i, y_i, z_i), i = \overline{1, n}$, represent the position vectors, and

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|, i, j \in \{1, \ldots, n\}$$

denote the distances between the bodies m_i and m_j. We take the units such that the gravitational constant is 1.
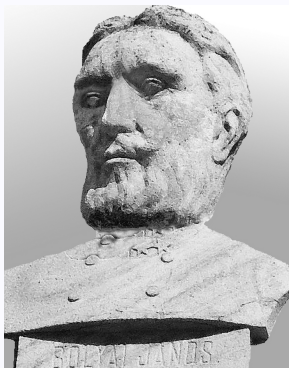
Integrals of motion

- energy (1): $\sum_{1 \le i < j \le n} \frac{m_i m_j}{r_{ij}} + \frac{1}{2} \sum_{i=1}^{n} m_i |\mathbf{r}_i|^2 = h$ (constant)

- centre of mass (3): $\sum_{i=1}^{n} m_i \mathbf{r}_i = \mathbf{0}$

- linear momentum (3): $\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \mathbf{0}$

- angular momentum (3): $\sum_{i=1}^{n} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{c}$ (constant)

1887: Heinrich Bruns shows that there are no other algebraic linearly independent integrals of motion
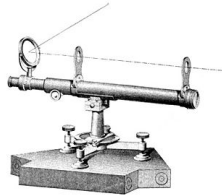
Carl Friedrich Gauss (1777-1855) and his heliotrope

János Bolyai (1802-1860) and Nikolai Lobachevsky (1792-1856)

Setting

The space in which the motion of the bodies takes place is:

$$\mathbb{M}_\kappa^3 = \{(w,x,y,z) | w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1}(z > 0 \text{ if } \kappa < 0)\},$$

where σ is the *signum function*

$$\sigma = \begin{cases} +1, & \text{for } \kappa > 0 \\ -1, & \text{for } \kappa < 0 \end{cases}$$

notice that

$$\mathbb{M}_1^3 = \mathbb{S}^3 \text{ and } \mathbb{M}_{-1}^3 = \mathbb{H}^3$$

Notations

Consider $m_1, \ldots, m_n > 0$ in $\mathbb{R}^4$ for $\kappa > 0$ and $\mathbb{M}^{3,1}$ (Minkowski space) for $\kappa < 0$, with positions given by

$$\mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = \overline{1, n}$$

$\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ is the configuration of the system

$\nabla_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma \partial_{z_i}), \ \ \nabla := (\nabla_{\mathbf{q}_1}, \ldots, \nabla_{\mathbf{q}_n})$ is the gradient

For $\mathbf{a} := (a_w, a_x, a_y, a_z), \mathbf{b} := (b_w, b_x, b_y, b_z),$

$$\mathbf{a} \cdot \mathbf{b} := (a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z)$$

is the inner product

Potential

For $\kappa \neq 0$, the force function is

$$U_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}}$$

$-U_\kappa$ is the potential (a homogeneous function of degree 0).

Euler's formula for homogeneous functions:

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = \overline{1, n}.$$

Equations of motion

Using variational methods (constrained Lagrangian dynamics), we obtain the equations of motion:

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q}_i = \kappa^{-1}, \ \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \ \kappa \neq 0, \ \ i = \overline{1, n}$$

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \cdot \mathbf{q}_j)[(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)\mathbf{q}_j - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}},$$

$$i = \overline{1, n}$$

Elimination of κ

Coordinate and time-rescaling transformations

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i, \ i = \overline{1,n} \ \text{ and } \ \tau = |\kappa|^{3/4}t$$

lead to the equations of motion

$$\mathbf{r}_i'' = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{r}_j - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j)\mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j)^2]^{3/2}} - \sigma(\mathbf{r}_i' \cdot \mathbf{r}_i')\mathbf{r}_i, \ \ i = \overline{1,n},$$

where

$$' = \frac{d}{d\tau}, \ \ \mathbf{r}_i \cdot \mathbf{r}_i = |\kappa|\mathbf{q}_i \cdot \mathbf{q}_i = |\kappa|\kappa^{-1} = \sigma$$

The positive case and the negative case

Equations of motion in $\mathbb{S}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Equations of motion in $\mathbb{H}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = -1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Hamiltonian form

$\mathbf{p} := (\mathbf{p}_1, \ldots, \mathbf{p}_n)$, $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$, $i = \overline{1,n}$, momenta

$T(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i)(\sigma \mathbf{q}_i \cdot \mathbf{q}_i)$, kinetic energy

$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q})$, Hamiltonian function

$$\begin{cases} \dot{\mathbf{q}}_i = \nabla_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\nabla_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - \sigma m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \cdot \mathbf{p}_i = 0, \quad i = \overline{1,n} \end{cases}$$

The wedge product

Consider the basis

$$\mathbf{e}_w = (1,0,0,0), \ \mathbf{e}_x = (0,1,0,0), \ \mathbf{e}_y = (0,0,1,0), \ \mathbf{e}_z = (0,0,0,1)$$

The wedge product of
$\mathbf{u} = (u_w, u_x, u_y, u_z), \mathbf{v} = (v_w, v_x, v_y, v_z) \in \mathbb{R}^4$ is defined as

$$\begin{aligned}
\mathbf{u} \wedge \mathbf{v} := & (u_w v_x - u_x v_w) e_w \wedge e_x + (u_w v_y - u_y v_w) e_w \wedge e_y + \\
& (u_w v_z - u_z v_w) e_w \wedge e_z + (u_x v_y - u_y v_x) e_x \wedge e_y + \\
& (u_x v_z - u_z v_x) e_x \wedge e_z + (u_y v_z - u_z v_y) e_y \wedge e_z,
\end{aligned}$$

where $\mathbf{e}_w \wedge \mathbf{e}_x, \mathbf{e}_w \wedge \mathbf{e}_y, \mathbf{e}_w \wedge \mathbf{e}_z, \mathbf{e}_x \wedge \mathbf{e}_y, \mathbf{e}_x \wedge \mathbf{e}_z, \mathbf{e}_y \wedge \mathbf{e}_z$ represent the bivectors that form a canonical basis of the exterior Grassmann algebra over $\mathbb{R}^4$

Integrals of the total angular momentum

$$\sum_{i=1}^{n} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c},$$

where $\mathbf{c} =$

$c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z,$

with the coefficients $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$

– on components, 6 integrals:

$$\sum_{i=1}^{n} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \quad \sum_{i=1}^{n} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy},$$

$$\sum_{i=1}^{n} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \quad \sum_{i=1}^{n} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy},$$

$$\sum_{i=1}^{n} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \quad \sum_{i=1}^{n} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}$$

Isometries in $\mathbb{S}^3$

In some suitable basis, rotations can be written as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \theta, \phi \in [0, 2\pi)$$

– simple rotations (elliptic): lead to new solutions
– double rotations (elliptic-elliptic): lead to new solutions

Isometries in $\mathbb{H}^3$

In some suitable basis, rotations can be written as

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{pmatrix}, \theta \in [0, 2\pi), \phi \in \mathbb{R},$$

– simple rotations (elliptic): lead to new solutions
– simple rotations (hyperbolic): lead to new solutions
– double rotations (elliptic-hyperbolic): lead to new solutions

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1-\xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1+\xi^2/2 \end{pmatrix}, \xi \in \mathbb{R}.$$

– simple rotations (parabolic): lead to no solutions

Relative equilibria (RE) in $\mathbb{S}^3$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = \overline{1, n},$$

[positive elliptic] :
$$\begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ (\text{constant}) \\ z_i(t) = z_i \ (\text{constant}), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ r_i^2 + y_i^2 + z_i^2 = 1, \ i = \overline{1, n}$

[positive elliptic$-$elliptic] :
$$\begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \rho_i \cos(\beta t + b_i) \\ z_i(t) = \rho_i \sin(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ y_i^2 + z_i^2 = \rho_i^2, \ r_i^2 + \rho_i^2 = 1, \ i = \overline{1, n}$

Relative equilibria (RE) in $\mathbb{H}^3$

$$\text{[negative elliptic]} : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \text{ (constant)} \\ z_i(t) = z_i \text{ (constant)}, \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2$, $r_i^2 + y_i^2 - z_i^2 = -1$, $i = \overline{1, n}$

$$\text{[negative hyperbolic]} : \begin{cases} w_i(t) = w_i \text{ (constant)} \\ x_i(t) = x_i \text{ (constant)} \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $y_i^2 - z_i^2 = -\eta_i^2$, $w_i^2 + x_i^2 - \eta_i^2 = -1$, $i = \overline{1, n}$

$$\text{[negative elliptic} - \text{hyperbolic]} : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2$, $y_i^2 - z_i^2 = -\eta_i^2$, so $r_i^2 - \eta_i^2 = -1$, $i = \overline{1, n}$

Fixed points (FP) in $\mathbb{S}^3$

– equilateral triangle on a great circle of a great sphere (equal masses, 3BP)
– any scalene acute triangle on a great circle of a great sphere (non-equal masses, 3BP)
– regular tetrahedron in a great sphere (equal masses, 4BP)
– two equilateral triangles, each on complementary great circles (equal masses, 6 BP):

$$
\begin{array}{llll}
w_1 = 1, & x_1 = 0, & y_1 = 0, & z_1 = 0, \\
w_2 = -1/2, & x_2 = \sqrt{3}/2, & y_2 = 0, & z_2 = 0, \\
w_3 = -1/2, & x_3 = -\sqrt{3}/2, & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -1/2, & z_5 = \sqrt{3}/2, \\
w_6 = 0, & x_6 = 0, & y_6 = -1/2, & z_6 = -\sqrt{3}/2, \\
\end{array}
$$

– two, not necessarily congruent, scalene acute triangles, each on one of two complementary great circles (non-equal masses, 6 BP)

Complementary circles in $\mathbb{S}^3$

Definition 1

Two great circles, C_1 and C_2, of two different great spheres of $\mathbb{S}^3$ are called *complementary* if there is a coordinate system $wxyz$ such that

$$C_1 = \mathbf{S}^1_{wx} = \{(0, 0, y, z) | y^2 + z^2 = 1\},$$
$$C_2 = \mathbf{S}^1_{yz} = \{(w, x, 0, 0) | w^2 + x^2 = 1\}.$$

Complementary circles form a Hopf link in a Hopf fibration,

$$h \colon \mathbb{S}^3 \to \mathbb{S}^2, \ h(w, x, y, z) = (w^2 + x^2 - y^2 - z^2, 2(wz + xy), 2(xz - wy)),$$

which takes circles of $\mathbb{S}^3$ to points of $\mathbb{S}^2$. Using the stereographic projection, it can be shown that the circles C_1 and C_2 are linked.

Distance between complementary circles

Since, in $\mathbb{S}^3$, the distance between two points, $\mathbf{a}$ and $\mathbf{b}$, is

$$d(\mathbf{a}, \mathbf{b}) = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

it follows that if $\mathbf{a} \in C_1$ and $\mathbf{b} \in C_2$, then

$$d(\mathbf{a}, \mathbf{b}) = \pi/2 = \text{constant}$$

Therefore if the body m_1 is on C_1 and the body m_2 is on C_2, the magnitude of the attraction between them is the same, no matter where each of them lies on the respective circle

Clifford tori in $\mathbb{S}^3$

A remarkable family of surfaces in $\mathbb{R}^4$ are the Clifford tori

$$\mathbf{T}^2_{r\rho} = \{(r\cos\theta, r\sin\theta, \rho\cos\phi, \rho\sin\phi) \mid r^2 + \rho^2 = 1, 0 \leq \theta, \phi < 2\pi\},$$

which lie in $\mathbb{S}^3$. Indeed, the Euclidean distance from the origin of the coordinate system to any point of a Clifford torus is
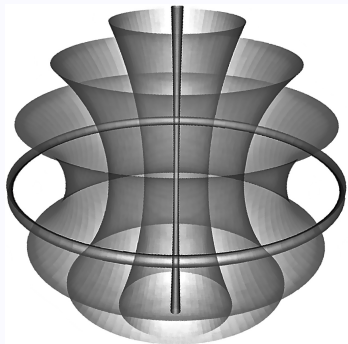
$$(r^2\cos^2\theta + r^2\sin^2\theta + \rho^2\cos^2\phi + \rho^2\sin^2\phi)^{1/2} = (r^2 + \rho^2)^{1/2} = 1$$

Unlike the standard torus, the Clifford torus is a flat surface, which divides $\mathbb{S}^3$ into two solid tori, for which it forms the boundary

Heegaard splitting of $\mathbb{S}^3$

The Clifford torus with $r = \rho = 1/\sqrt{2}$ provides the standard genus 1 splitting of $\mathbb{S}^3$, a case in which the two solid tori are congruent.

A 3D projection of a 4D foliation of $\mathbb{S}^3$ into Clifford tori

Theorem 2

Assume that, in the curved n-body problem in $\mathbb{S}^3$, $n \geq 2$, with bodies of masses $m_1, \ldots, m_n > 0$, positive elliptic and positive elliptic-elliptic relative equilibria exist. Then the corresponding solution $\mathbf{q}$ may have one of the following properties:

(i) it is a (simply rotating) positive elliptic RE, with the body of mass m_i moving on a (not necessarily geodesic) circle $\mathcal{C}_i, i = \overline{1, n}$, of a 2-sphere in $\mathbb{S}^3$; in the hyperplanes wxy and wxz, the circles $\mathcal{C}_i$ are parallel with the plane wx; another possibility is that some bodies rotate on a great circle of a great sphere, while the other bodies stay fixed on a complementary great circle of another great sphere.

(ii) it is a (doubly rotating) positive elliptic-elliptic RE, with some bodies rotating on a great circle of a great sphere and the other bodies rotating on a complementary great circle of another great sphere; another possibility is that each body m_i is moving on the Clifford torus $\mathbf{T}^2_{r_i \rho_i}, i = \overline{1, n}$.

Lagrangian RE as in (i)

$$w_1(t) = r\cos\omega t, \qquad\qquad x_1(t) = r\sin\omega t,$$
$$y_1(t) = y \text{ (constant)}, \qquad\qquad z_1(t) = z \text{ (constant)},$$
$$w_2(t) = r\cos(\omega t + 2\pi/3), \qquad x_2(t) = r\sin(\omega t + 2\pi/3),$$
$$y_2(t) = y \text{ (constant)}, \qquad\qquad z_2(t) = z \text{ (constant)},$$
$$w_3(t) = r\cos(\omega t + 4\pi/3), \qquad x_3(t) = r\sin(\omega t + 4\pi/3),$$
$$y_3(t) = y \text{ (constant)}, \qquad\qquad z_3(t) = z \text{ (constant)}.$$

Given $m := m_1 = m_2 = m_3 > 0$, $r \in (0,1)$, and y, z with $r^2 + y^2 + z^2 = 1$, we can always find two frequencies,

$$\alpha^+ = \frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}} \text{ and } \alpha^- = -\frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}};$$

$$c_{wx} = 3m\omega \neq 0 \text{ and } c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0.$$

Stability of Lagrangian RE in $\mathbb{S}^2$

Regina Martínez and Carles Simó: On $\mathbb{S}^2$, the Lagrangian RE with masses $m_1 = m_2 = m_3 = 1$ are linearly stable for $r \in (r_1, r_2) \cup (r_3, 1)$, where $r = \sqrt{1 - z^2}$,

$$r_1 = 0.55778526844099498188467226566148375,$$

$$r_2 = 0.68145469725865414807206661241888645,$$

$$r_3 = 0.92893280143637470996280353121615412,$$

truncated to 35 decimal digits.

Place the bodies $m_1 = m_2 = m_3 = m_4$ at the vertices of a regular tetrahedron. Then m_1 and m_2 move on the Clifford torus with $r = 0$ and $\rho = 1$, which is the only Clifford torus in the class of a given foliation of $\mathbb{S}^3$ that is also a great circle of $\mathbb{S}^3$. The bodies of mass m_3 and m_4 move on the Clifford torus with $r = \frac{\sqrt{6}}{3}$ and $\rho = \frac{\sqrt{3}}{3}$:

$$w_1 = 0, \ x_1 = 0, \ y_1 = \cos(\alpha t + \pi/2), \ z_1 = \sin(\alpha t + \pi/2),$$

$$w_2 = 0, \ x_2 = 0, \ y_2 = \cos(\alpha t + b_2), \ z_2 = \sin(\alpha t + b_2),$$

with $\sin b_2 = -\frac{1}{3}$ and $\cos b_2 = \frac{2\sqrt{2}}{3}$,

Example of RE as in (ii) on Clifford tori

$$w_3 = \frac{\sqrt{6}}{3}\cos(\alpha t + 3\pi/2), \ x_3 = \frac{\sqrt{6}}{3}\sin(\alpha t + 3\pi/2),$$

$$y_3 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_3), \ z_3 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_3),$$

with $\cos b_3 = -\frac{\sqrt{6}}{3}$ and $\sin b_3 = -\frac{\sqrt{3}}{3}$, and

$$w_4 = \frac{\sqrt{6}}{3}\cos(\alpha t + \pi/2), \ x_4 = \frac{\sqrt{6}}{3}\sin(\alpha t + \pi/2),$$

$$y_4 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_4), \ z_4 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_4),$$

with $\cos b_4 = -\frac{\sqrt{6}}{3}$ and $\sin b_4 = -\frac{\sqrt{3}}{3}$. notice that $b_3 = b_4$.

RE generated from FP configurations in $\mathbb{S}^3$

Theorem 3

Consider the bodies of masses $m_1, \ldots, m_n > 0, n \geq 2$, in $\mathbb{S}^3$. Then an RE generated from a fixed point configuration may have one of the following properties:

(i) it is a (simply rotating) positive elliptic RE for which all bodies rotate on the same great circle of a great sphere of $\mathbb{S}^3$;

(ii) it is a (simply rotating) positive elliptic RE for which some bodies rotate on a great circle of a great sphere, while the other bodies are fixed on a complementary great circle of a different great sphere;

(iii) it is a (doubly rotating) positive elliptic-elliptic RE for which some bodies rotate with frequency $\alpha \neq 0$ on a great circle of a great sphere, while the other bodies rotate with frequency $\beta \neq 0$ on a complementary great circle of a different sphere; the frequencies may be different in size, i.e. $|\alpha| \neq |\beta|$;

(iv) it is a (doubly rotating) positive elliptic-elliptic RE with frequencies $\alpha, \beta \neq 0$ equal in size, i.e. $|\alpha| = |\beta|$.

Example of RE as in (ii)

This is a solution of the 6-body problem with two equilateral triangles, one inscribed in a great circle of a great sphere and the other inscribed in a complementary great circle of another great sphere. The first triangle rotates uniformly, while the second triangle is fixed:

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i \in \{1, 2, 3, 4, 5, 6\},$$

$$w_1 = \cos \alpha t, \qquad x_1 = \sin \alpha t, \qquad y_1 = 0, \qquad z_1 = 0,$$
$$w_2 = \cos(\alpha t + a), \qquad x_2 = \sin(\alpha t + a), \qquad y_2 = 0, \qquad z_2 = 0,$$
$$w_3 = \cos(\alpha t + b), \qquad x_3 = \sin(\alpha t + b), \qquad y_3 = 0, \qquad z_3 = 0,$$
$$w_4 = 0, \qquad x_4 = 0, \qquad y_4 = 1, \qquad z_4 = 0,$$
$$w_5 = 0, \qquad x_5 = 0, \qquad y_5 = -\frac{1}{2}, \qquad z_5 = \frac{\sqrt{3}}{2},$$
$$w_6 = 0, \qquad x_6 = 0, \qquad y_6 = -\frac{1}{2}, \qquad z_6 = -\frac{\sqrt{3}}{2},$$

where $a = 2\pi/3$ and $b = 4\pi/3$.

Example of RE as in (iii)

In general, the orbit described below is quasiperiodic:

$$w_1 = \cos \alpha t, \qquad\qquad x_1 = \sin \alpha t,$$
$$y_1 = 0, \qquad\qquad z_1 = 0,$$
$$w_2 = \cos(\alpha t + 2\pi/3), \qquad\qquad x_2 = \sin(\alpha t + 2\pi/3),$$
$$y_2 = 0, \qquad\qquad z_2 = 0,$$
$$w_3 = \cos(\alpha t + 4\pi/3), \qquad\qquad x_3 = \sin(\alpha t + 4\pi/3),$$
$$y_3 = 0, \qquad\qquad z_3 = 0,$$
$$w_4 = 0, \qquad\qquad x_4 = 0,$$
$$y_4 = \cos \beta t, \qquad\qquad z_4 = \sin \beta t,$$
$$w_5 = 0, \qquad\qquad x_5 = 0,$$
$$y_5 = \cos(\beta t + 2\pi/3), \qquad\qquad z_5 = \sin(\beta t + 2\pi/3),$$
$$w_6 = 0, \qquad\qquad x_6 = 0,$$
$$y_6 = \cos(\beta t + 4\pi/3), \qquad\qquad z_6 = \sin(\beta t + 4\pi/3).$$

$$c_{wx} = 3m\alpha \neq 0, \quad c_{yz} = 3m\beta \neq 0, \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

Qualitative behaviour of RE in $\mathbb{H}^3$

Theorem 4

In the curved n-body problem in $\mathbb{H}^3$, $n \geq 2$, with bodies of masses $m_1, \ldots, m_n > 0$, every RE may have one of the following properties:
(i) it is a (simply rotating) negative elliptic RE, with the body of mass m_i moving on a circle $\mathcal{C}^i$, $i = \overline{1, n}$, of a hyperbolic 2-sphere in $\mathbb{H}^3$; in the hyperplanes wxy and wxz, the planes of the circles $\mathcal{C}^i$ are parallel with the plane wx;
(ii) it is a (simply rotating) negative hyperbolic relative equilibrium, with the body of mass m_i moving on some (not necessarily geodesic) hyperbola $\mathcal{H}_i$ of a hyperbolic 2-sphere in $\mathbb{H}^3$, $i = \overline{1, n}$; in the hyperplanes wyz and xyz, the planes of the hyperbolas $\mathcal{C}^i$ are parallel with the plane yz;
(iii) it is a (doubly rotating) negative elliptic-hyperbolic relative equilibrium, with the body of mass m_i moving on the hyperbolic cylinder

$$\mathbf{C}^2_{r_i \rho_i} = \{(r_i \cos \theta, r_i \sin \theta, \eta_i \sinh \iota, \eta_i \cosh \iota) \mid r_i^2 - \eta_i^2 = -1, \ \theta \in [0, 2\pi), \iota \in \mathbb{R}\},$$

$i = \overline{1, n}$.

Eulerian RE as in (ii)

The motion described below takes place on a hyperbolic 2-sphere, and is not periodic:

$$w_1 = 0, \quad x_1 = 0, \qquad\qquad y_1 = \sinh \beta t, \qquad z_1 = \cosh \beta t,$$
$$w_2 = 0, \quad x_2 = x \ (\text{constant}), \quad y_2 = \eta \sinh \beta t, \quad z_2 = \eta \cosh \beta t,$$
$$w_3 = 0, \quad x_3 = -x \ (\text{constant}), \quad y_3 = \eta \sinh \beta t, \quad z_3 = \eta \cosh \beta t,$$

Given $m := m_1 = m_2 = m_3 > 0, x > 0, \eta > 0$ with $x^2 - \eta^2 = -1$, there exist two non-zero frequencies,

$$\beta^+ = \frac{1}{2\eta} \sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}} \ \text{ and } \ \beta^- = -\frac{1}{2\eta} \sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}};$$

$$c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0, \ c_{yz} = m\beta(1 - 2\eta^2)$$

Example of RE as in (iii)

The motion described below takes place on a hyperbolic cylinder, and is not periodic:

$$w_1 = 0, \qquad x_1 = 0, \qquad y_1 = \sinh \beta t, \quad z_1 = \cosh \beta t,$$
$$w_2 = r \cos \alpha t, \quad x_2 = r \sin \alpha t, \quad y_2 = \eta \sinh \beta t, \quad z_2 = \eta \cosh \beta t,$$
$$w_3 = -r \cos \alpha t, \quad x_3 = -r \sin \alpha t, \quad y_3 = \eta \sinh \beta t, \quad z_3 = \eta \cosh \beta t.$$

$$c_{wx} = 2m\alpha r^2, c_{yz} = -1 - 2\beta\eta^2, c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

Extension of the equations to $\kappa = 0$

$$m_i \ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^{n} \frac{m_i m_j \left[\mathbf{q}_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) \mathbf{q}_i \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \;\; i = \overline{1, n},$$

where $m_1, m_2, \ldots, m_n > 0$ represent the masses, the vectors $\mathbf{r}_i$ are given by

$$\mathbf{q}_i = \mathbf{r}_i + (0, 0, 0, (\sigma \kappa)^{1/2}), \; \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \; i = \overline{1, n},$$

and

$$r_{ij} := \begin{cases} [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + (\omega_i - \omega_j)^2]^{1/2}, \kappa > 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}, \quad\quad\quad \kappa = 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (\omega_i - \omega_j)^2]^{1/2}, \kappa < 0. \end{cases}$$

The explicit equations

$$\begin{cases} \ddot{x}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j \left[x_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) x_i \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) x_i \\[4ex]

\ddot{y}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j \left[y_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) y_i \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) y_i \\[4ex]

\ddot{z}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j \left[z_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) z_i \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) z_i \\[4ex]

\ddot{\omega}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j \left[\omega_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) \omega_i + \frac{\sigma(\sigma\kappa)^{\frac{1}{2}} r_{ij}^2}{2} \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \left[\kappa\omega_i + \sigma(\sigma\kappa)^{\frac{1}{2}} \right], \end{cases}$$

$i = \overline{1, n}.$

Constraints

$$\kappa(x_i^2 + y_i^2 + z_i^2 + \sigma\omega_i^2) + 2(\sigma\kappa)^{1/2}\omega_i = 0,$$
$$\kappa(x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i + \sigma\omega_i\dot{\omega}_i) + (\sigma\kappa)^{1/2}\dot{\omega}_i = 0, \quad i = \overline{1,n}.$$

newtonian equations

For $\kappa = 0$ we recover the Newtonian equations:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = \overline{1, n},$$

with $\mathbf{r}_i = (x_i, y_i, z_i, 0), i = \overline{1, n}$

Bifurcation of the first integrals

– Integral of energy:
      for all $\kappa \in \mathbb{R}$: 1 integral (no bifurcation)

– Integrals of the centre of mass:
      $\kappa = 0$: 3 integrals
      $\kappa \neq 0$: 0 integrals

– Integrals of the linear momentum:
      $\kappa = 0$: 3 integrals
      $\kappa \neq 0$: 0 integrals

– Integrals of the total angular momentum:
      $\kappa = 0$: 3 integrals
      $\kappa \neq 0$: 6 integrals

Thank you very much!