PDE based image analysis

joint work with George Baravdish, Olof Svensson at LIU, Freddie Åström at Heidelberg

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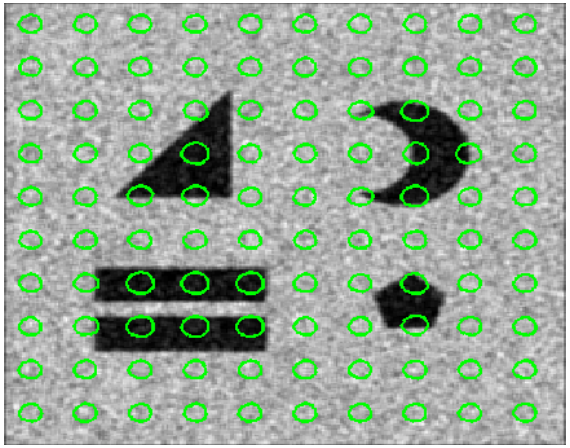
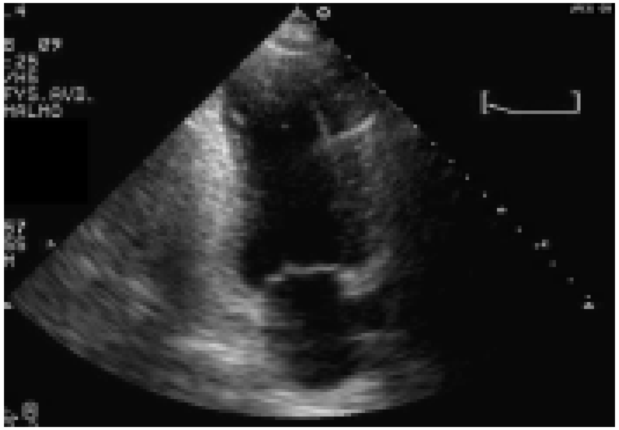
Karlstad, 2016 – 05 -18

Outline of talk

- Introduction of image analysis
- Filter and Linear PDE model
- Mean value averaging and Nonlinear PDE
- Variational approach and evolution equation
- Anisotropic evolution
- Viscosity solution
- Numerical algarithm
- Some results

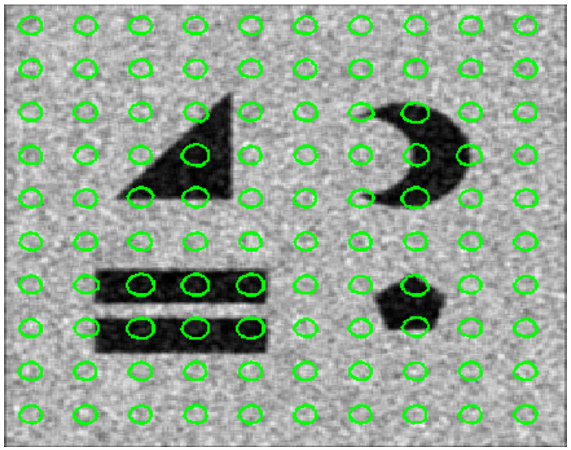
Example of image analysis





Find useful info from the image

Example of image analysis



K Fundana Thesis, 2010

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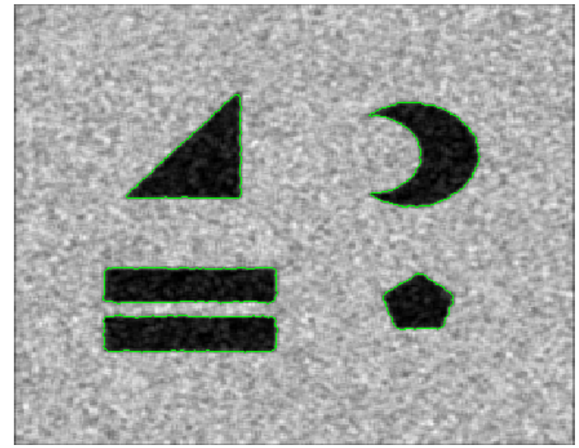
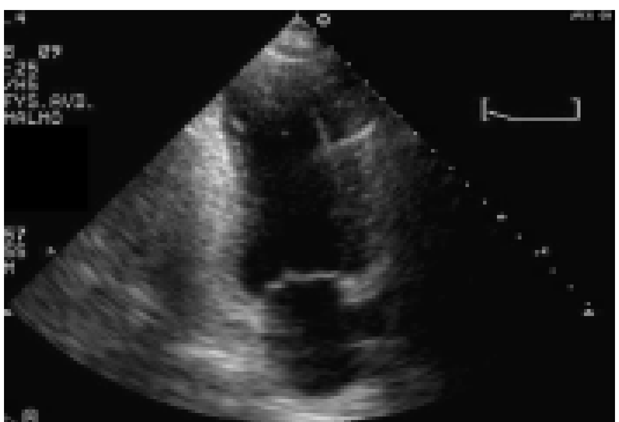
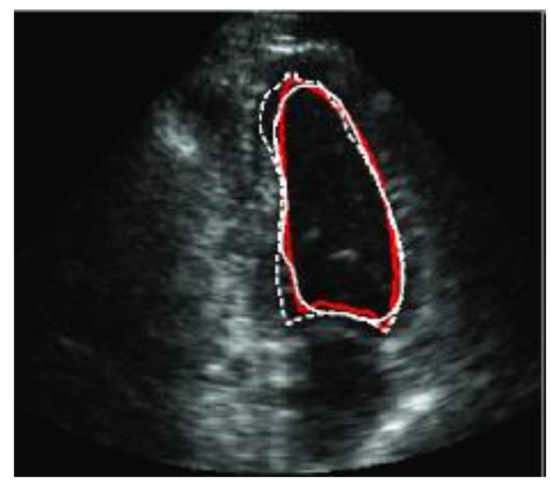
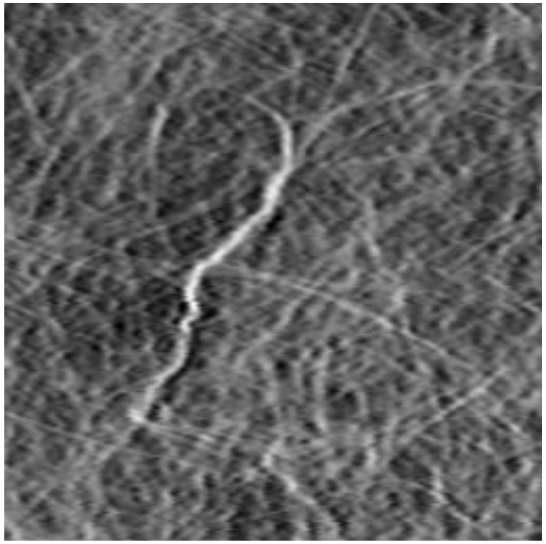




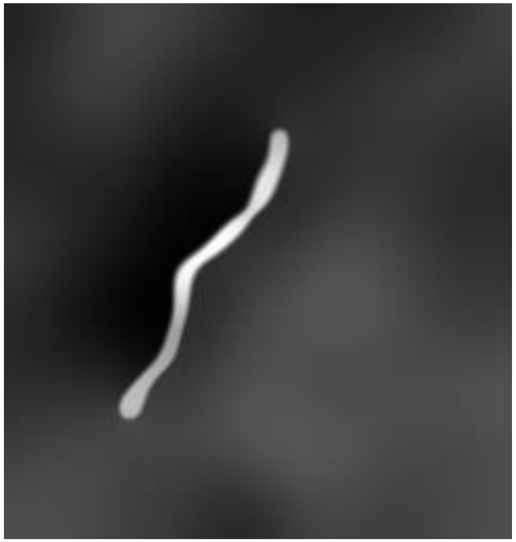


M Hansson Thesis, 2012

Some applications



Preprocessing of fabric image



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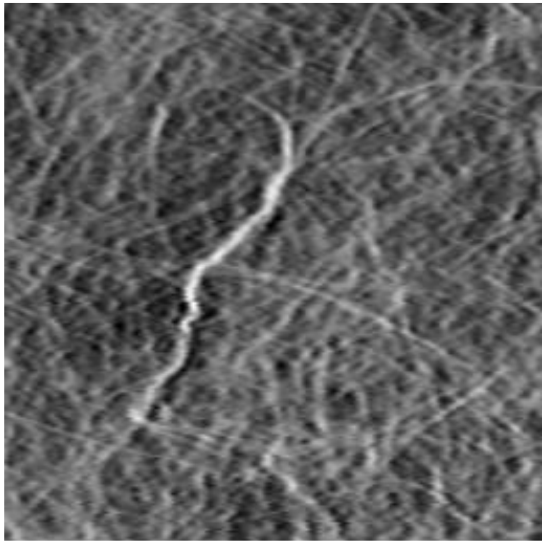

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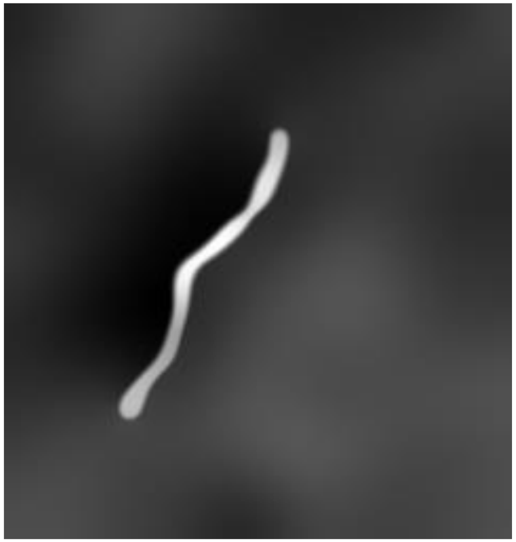

Defect detection in wood

Smoothing and Linear PDE

- Let $u(x)$ be the grey scale of image with noise
- Let $u_0(x)$ be the grey scale of original image
- $u(x) = u_0(x) + \sigma(x)$

$\boldsymbol{u}$

Smoothing and Linear PDE

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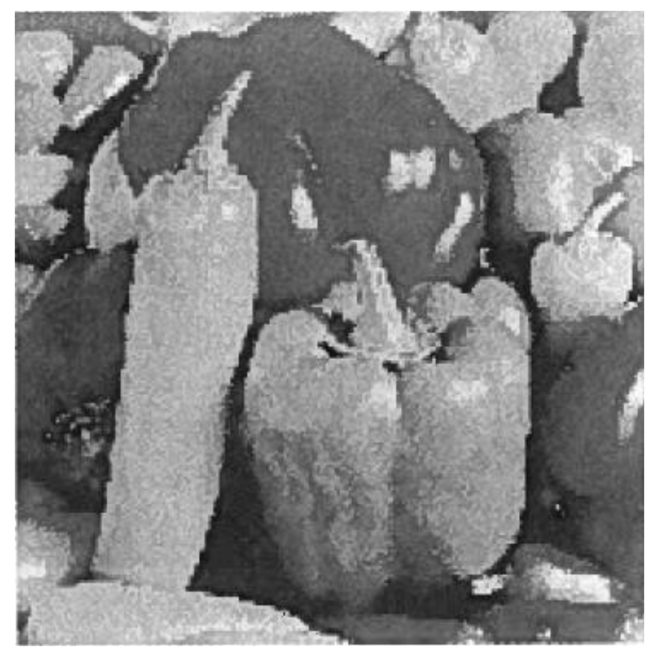
$\boldsymbol{u}$

$\xrightarrow{?}$

$\boldsymbol{u_0}$



Smoothing and Linear PDE

$$\tilde{u}(x) = \frac{1}{|B|}\int_{x+B} u(y)$$

B is a ball of radius *r* at origin

$$\tilde{u}(x,y) \approx u_0(x,y)$$

Smoothing and Linear PDE

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Smoothing and Linear PDE

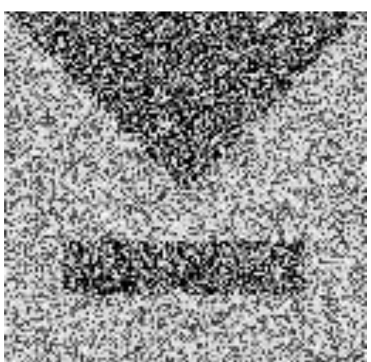
1. Idea of Witkin: Embed the original image u in a family of derived images obtained by convolving the original image u_0 with G_t a Gaussian kernal of variance t
2. $\mathrm{u}(x,\mathrm{t}) = \mathrm{c}(\mathrm{t}) \int \mathrm{u}_0(x-y) \exp\left(-\frac{y^2}{4\mathrm{t}^2}\right) \mathrm{d}y$
3. Larger value of t corresponds to image at coarser resolution

Smoothing and Linear PDE

- Idea of Koenderink: $u(x,t)$ satisfies the heat equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

- Linear PDE method is a low-pass filtering
- It can eliminate the noise
- But at weanwhile it also smooths the edges.



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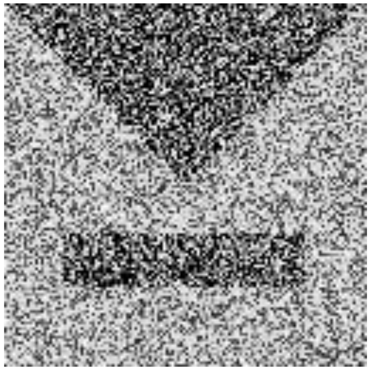
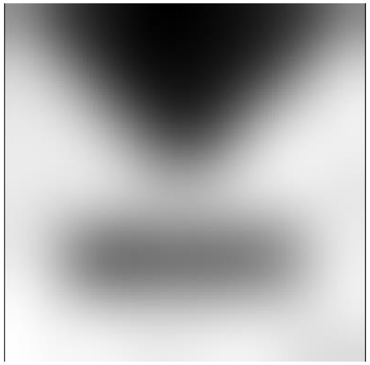
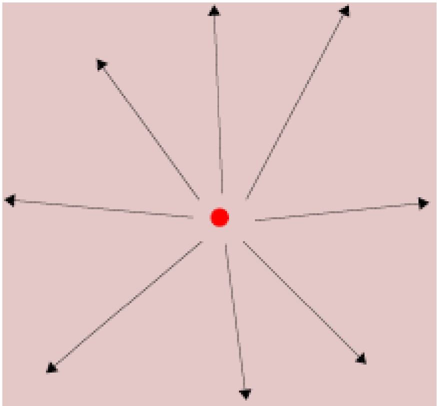




Smoothing and Linear PDE

Physical consideration

1. Causality: any feature at a coarse level of resolution is required to possess a ”cause” at a finer level of resolution
2. Homogeneity and Isotropy: The blurring is reqired to be space invariant
3. Fick's law : Flux $J(x) = D(x)\nabla u$, where D is called flux tensor
4. Invariance of Δ: uniformly diffusion in all directions



Smoothing and Linear PDE

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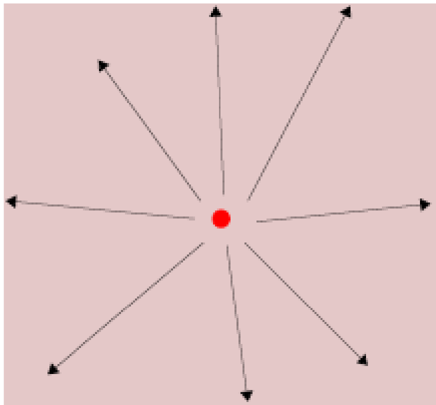
Isotropic diffusion

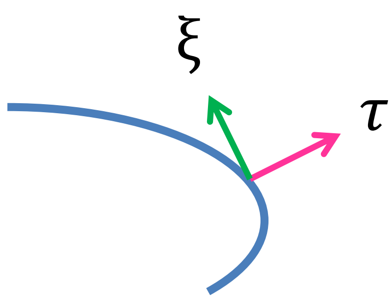


Smoothing and Linear PDE

Physical consideration

1. **Causality:** any feature at a coarse level of resolution is required to possess a "cause" at a finer level of resolution
2. **Homogeneity and Isotropy:** The blurring is reqired to be space invariant
3. **Fick's law** : Flux $J(x) = D(x)\nabla u$, where D is called flux tensor
4. **Invariance of Δ**: uniformly diffusion in all directions
5. **Gauge coordinates**: $\Delta u = u_{\xi\xi} + u_{\tau\tau}$

   Where $\xi \perp \nabla u, \quad \tau \parallel \nabla u$



Relation between smoothing and Linear PDE

$$M_r(u) = \frac{1}{|B|}\int_{x+B} u(y)$$

$$M_{n,r} = M_r \circ M_r \circ \cdots \circ M_r$$

If $\boldsymbol{nr = t}$ is fixed, then $u_n = M_{n,r}(u) \to v(t,x)$ as $\mathrm{n} \to \infty$, then $v(t,x)$ solves the heat equation

$$\begin{cases} v_t(t,x) = \Delta v(t,x) \\ v(0,x) = u(x) \end{cases}$$

Averaging and Nonlinear PDE

$$Tu(x) = \sup_{B} \inf_{y \in B} u(x+y)$$

B is a ball of radius r at origin

$$\mathrm{Tu}(x) \approx u_0(x)$$

Averaging and Nonlinear PDE

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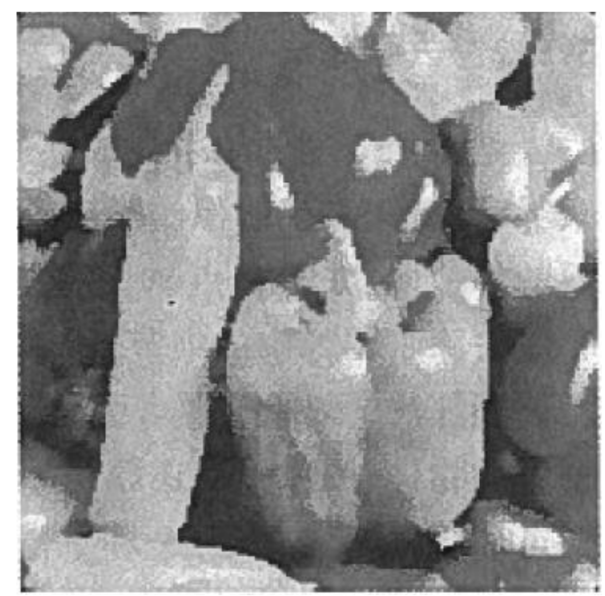


Averaging and Nonlinear PDE

Image with noise (left) and level set of image (right)

Averaging image (left) and Level set of treated image (right)

Averaging and Nonlinear PDE

$$T_r(u) = \sup_B \inf_{y \in B} u(x+y)\,, T_{n,r} = T_r \circ T_r \circ \cdots \circ T_r$$

If $\boldsymbol{nr = t}$ is fixed, then $u_n = T_{n,r}(u) \to v(x,t)$

and $v(x,t)$ solves the nonlinear equation

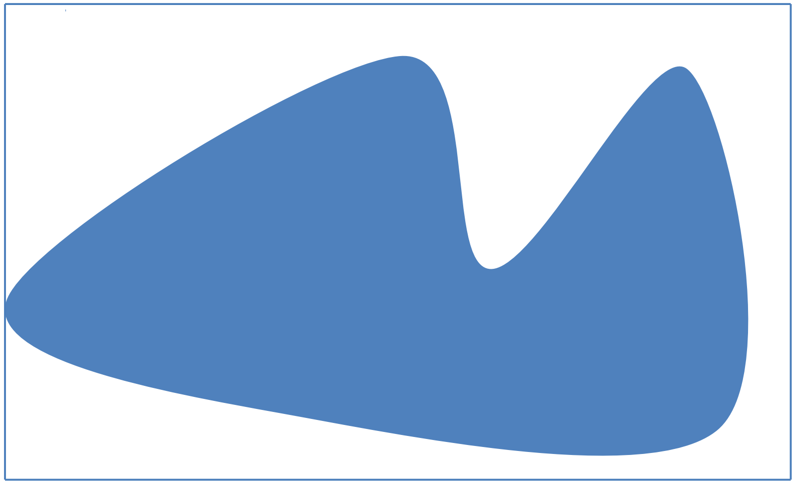
$$\begin{cases} v_t(x,t) = |\nabla v|(div\left(\frac{\nabla v}{|\nabla v|}\right))^{\frac{1}{3}} \\ v(x,0) = u(x) \end{cases}$$

Anisotropic Nonlinear model

- Fick's law: Flux $J(x) = D(x)\nabla u$,
- Perona-Malik model: $D(x) = g(|\nabla u|)$
- $u_t = div(g(|\nabla u|)\nabla u)$
- Idea of Perona-Malik: Encourage smoothing ***within*** the region in preference to smoothing ***across*** the bounday
- Could be done: Put

  D = 1 in interior

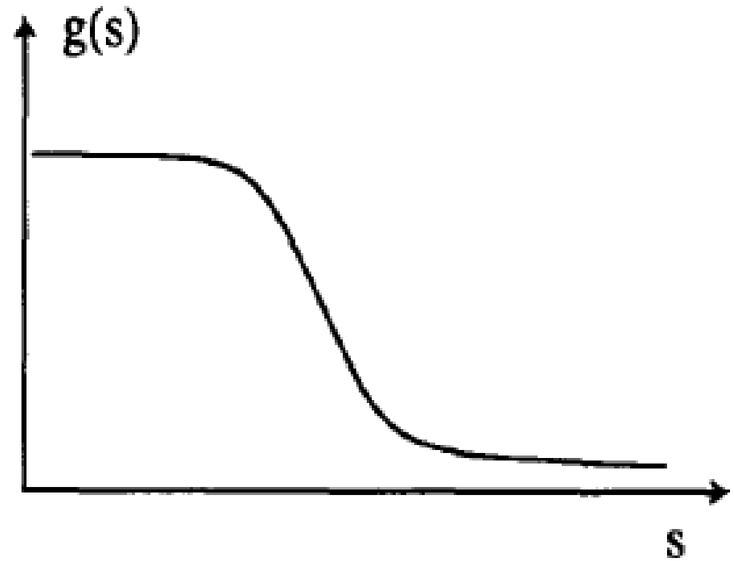
  D = 0 on boundary



Anisotropic Nonlinear model

- Fick's law: Flux $J(x) = D(x)\nabla u$,
- Perona-Malik model: $u_t = div(g(|\nabla u|)\nabla u)$

- Two typical functions of g
- $g(s) = \exp\left(-\frac{s^2}{K^2}\right)$
- $g(s) = \frac{1}{1+\frac{s^2}{K^2}}$



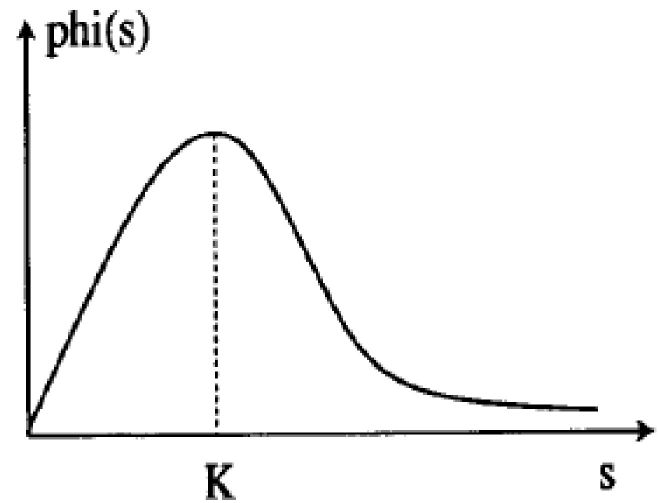
Anisotropic nonlinear model

- Perona – Malik model
- $u_t = div(g(|\nabla u|)\nabla u) = g\, u_{\xi\xi} + \varphi' u_{\tau\tau}$

  where $\varphi(s) = sg(s)$
- The first term is a diffusion along the edges
- The second term corresponds to local inversion of heat equation

  i.e. backward heat equation

  thus enhance edges



Perona - Malik model

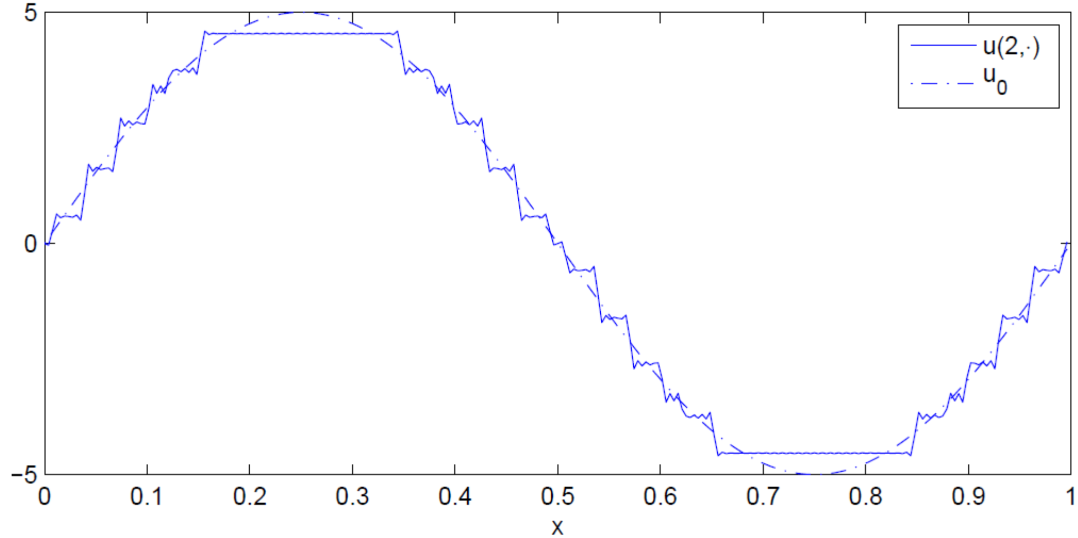
- Original image (left)
- Edges detected by Perona- Malik method (above)
- Edges detected by Gaussian smoothing (below)

P. Perona, J. Malik, IEEE Transaction PAMI, 12, 1990.

Perona-Malik model: Staircasing problem

- One problem in Perona-Malik model is the staircasing

P Guidotti, Advanced studies in pure math, 201x



- Another problem is the existence of multiple local maximum for the associated energy functional

$$E(u) = \int G(|\nabla u|) + \vartheta(u - u_0)^2$$

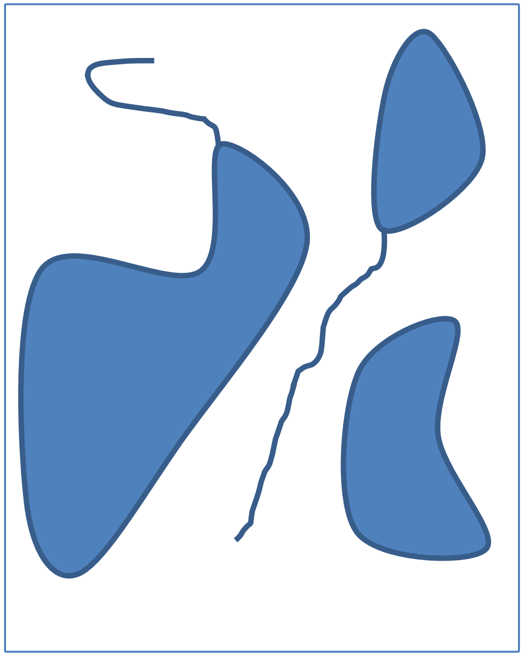
Variational formulation of segmentation

- Mumford -Shah model

$$E(u,K) = \beta \int_{\Omega} (u-u_0)^2 + \int_{\Omega \setminus K} |\nabla u|^{\mathbf{2}} + \alpha |\mathrm{K}|$$

K is the set of edges and u_0 is the given image

- The data term
- The regularity term of u
- The penalty term for the boundary



D. Mumford, J. Shah, CPAM , 1989

Variational formulation of segmentation

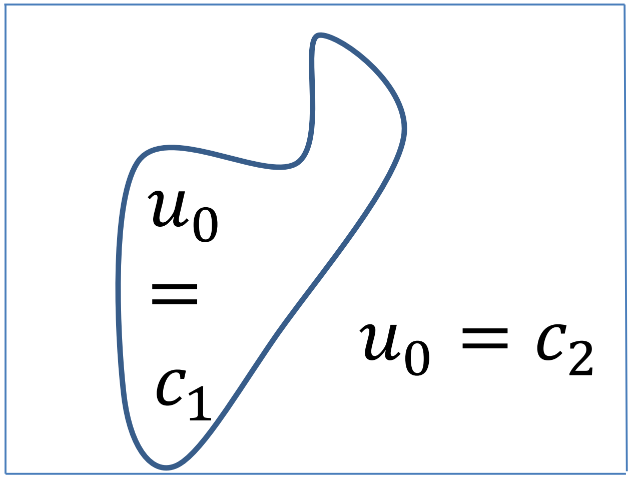
- Rudin-Osher-Fatemi total variation model

$$E(u) = \beta \int_{\Omega}(u - u_0)^2 + \int_{\Omega \setminus K} |\nabla u| + \alpha |K|$$

L. Rudin, S. Osher, E. Fatemi, Physica D , 1992

- Chan-Vese binary model

$$\mathrm{E}(u) = \int_{\Omega_1}(u - c_1)^2 + \int_{\Omega_2}(u - c_2)^2 + \beta \int_{\Omega} |\nabla u|$$



T. Chan, L. Vese,
IEEE, Transaction on
Image Processing , 2002

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K. Fundana, Thesis, 2010

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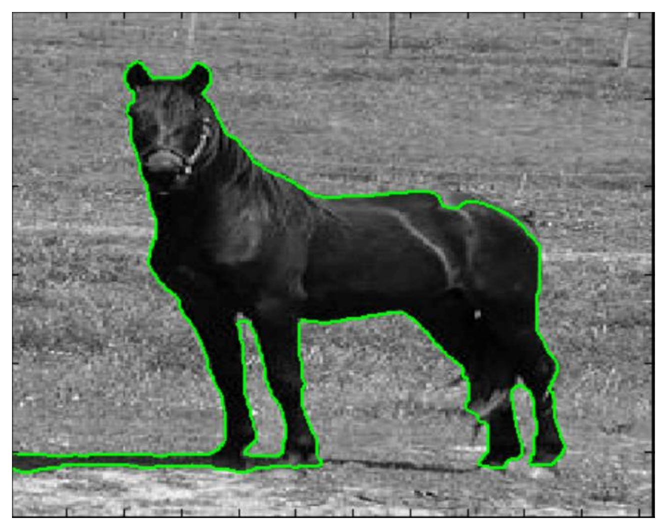


Segmentation with shape prior

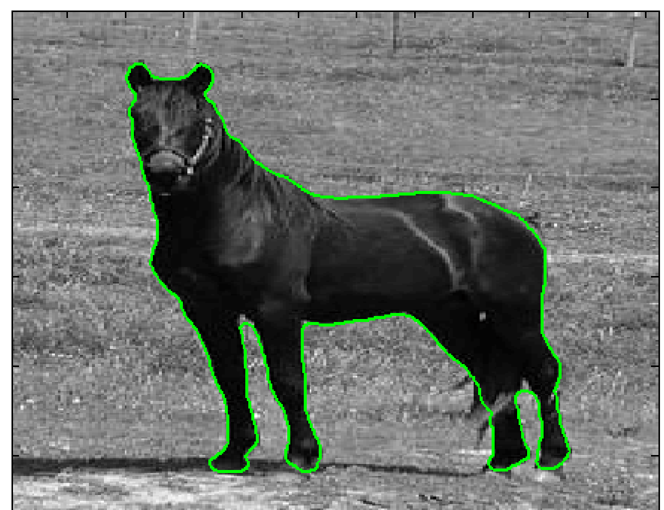
- $\varphi(x)$ is a reference shape prior

$$E(u) = \beta \int (u - u_0)^2 + \int |\nabla u| + \alpha \int (u - \varphi)^2$$

- The associated evolution equation

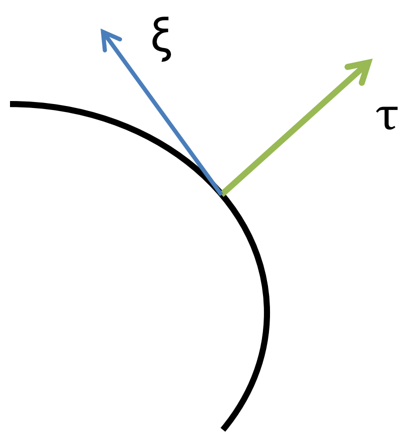
K. Fundana, Thesis, 2010

$$u_t = \Delta_1 u + \beta(u - u_0) + \alpha(u - \varphi)$$



Directional diffusion

- $\Delta_1(u) = div\left(\frac{\nabla u}{|\nabla u|}\right)$ describes the diffusion along the tangential direction, $(\nabla u)^{\perp}$, of level surface, i.e. along the edges
- $\Delta_\infty(u) = Hu\frac{\nabla u}{|\nabla u|}\cdot\frac{\nabla u}{|\nabla u|}$, H is the Hessian of u describes the diffusion along the normal direction, ∇u, of level surface.
- New idea: proper combination of Δ_1, Δ_∞.



Some relationships

Gauge coordinate (ξ , τ)

$$u_{\xi\xi} = |\nabla u| \Delta_1 u, \qquad u_{\tau\tau} = \Delta_\infty u$$

Laplacian Δ and infinity Laplacian Δ_∞

$$\Delta u = |\nabla u| \Delta_1 u + \Delta_\infty u$$

or

$$|\nabla u| \Delta_1 u = \Delta u - \Delta_\infty u$$

Anisotropic evolution

- Weighted directional evolution

$$\boxed{u_t = s\, u_{\xi\xi} + q\, u_{\tau\tau}}$$

where s, q has the property

$$0 < q \ll s, \;\; if \;\; |\nabla u| \gg 1$$

- Positivity of s, q ensures that the PDE is elliptic

- Condition $q \ll s$ provides us opportunity to have different evolution straitegies at different points based on geometric character of the level set

Anisotropic evolution

- PDE in Cartesian coordinates

$$\boxed{u_t = s|\nabla u|\Delta_1 u + q\Delta_\infty u}$$

- If $q = 0$, then the above equation is just *mean caurvature* equation
- If $q = (p-1)|\nabla u|^{p-2}, s = |\nabla u|^{p-2}$, then the above equation becomes *p-Laplace* equation. i.e.,

$$u_t = div\{|\nabla u|^{p-2}\nabla u\}$$

- For proper s, q, we have (p, q)-equation

$$u_t = div\{(\alpha|\nabla u|^{p-2}+\beta|\nabla u|^{q-2})\nabla u\}$$

Anisotropic evolution

Three potential candidates

- $L_1(u) = |\nabla u|^q \Delta_1(u) + (\mathrm{p} - 1)|\nabla u|^{p-2} \Delta_\infty(u)$

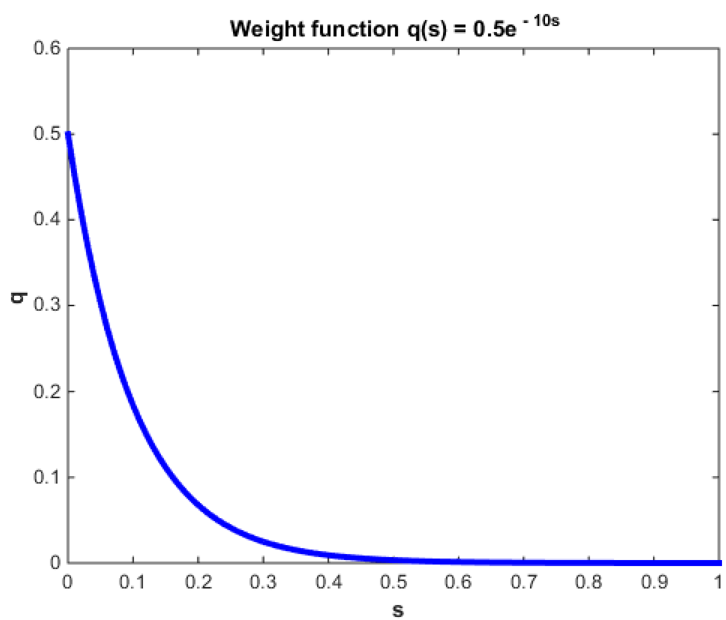
- $L_2(u) = |\nabla u|^{q(x)}(|\nabla u| \Delta_1(u) + q(x) \Delta_\infty(u))$

- $L_3(u) = |\nabla u|^{s(x)}(|\nabla u| \Delta_1(u) + q(x) \Delta_\infty(u))$

Viscosity solution

- A simple equation $u_t - F(\nabla u, D^2 u) = 0$

  $\mathrm{F}(p, X) = Tr(X) + (\mathrm{q}(|\mathrm{p}|) - 1)\bar{p} \cdot X\bar{p}, \quad \bar{p} = \frac{p}{|p|}$
- Function q: $[0, +\infty) \rightarrow (0,1)$ is continuous



- For any symmetric matrix X, $\lambda(X)$, $\Lambda(X)$ denote the smallest and largest eigenvalues of X.

Viscosity solution

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  $\mathrm{F}(p, X) = Tr(X) + (\mathrm{q}(|\mathrm{p}|) - 1)\bar{p} \cdot X\bar{p}, \quad \bar{p} = \frac{p}{|p|}$

- Subsolution u: for every point (t_0, x_0), all test function φ such that $u - \varphi$ has local maximum at (t_0, x_0), then

$$\begin{cases} \varphi_t - F(p, X) \le 0, & p \neq 0 \\ \varphi_t - Tr(X) - (q(0) - 1)\Lambda(X) \le 0, & p = 0 \end{cases}$$

- Supersolution u: for every point (t_0, x_0), all function φ s.t. u - φ has local minimum at (t_0, x_0), then

$$\begin{cases} \varphi_t - F(p, X) \ge 0, & p \neq 0 \\ \varphi_t - Tr(X) - (q(0) - 1)\lambda(X) \ge 0, & p = 0 \end{cases}$$

  where $p = \nabla\varphi(t_0, x_0)$, $X = D^2\varphi(t_0, x_0)$.

- Viscosity solution $\boldsymbol{u}$, if it's both sub- and supersolution

Numerical algorithms

- Rewritting

$$u_t = \sum a_{ij}(u)\partial_{ij}u, \qquad a_{ij} = \delta_{ij} + (q(|p|) - 1)\frac{p_i p_j}{|p|}$$

- Regularization

$$u_t = \sum a_{ij}^{\varepsilon}(u)\partial_{ij}u, \quad a_{ij}^{\varepsilon} = \delta_{ij} + (q(|p|) - 1)\frac{p_i p_j}{\sqrt{|p|^2 + \varepsilon^2}}$$

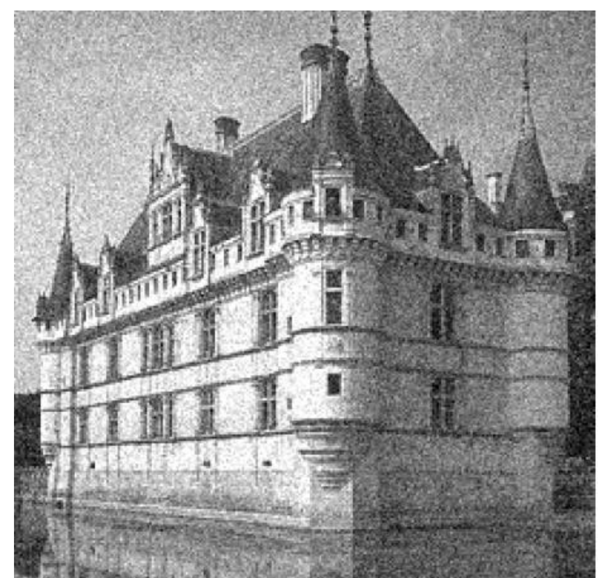
- Iteration scheme: Given u_k at step k, then update u_{k+1} at next step $k+1$ by solving the equation

$$u_t = \sum a_{ij}^{\varepsilon}(u_k)\partial_{ij}u, \quad a_{ij}^{\varepsilon} = \delta_{ij} + (q(|p_k|) - 1)\frac{p_{ki} p_{kj}}{\sqrt{|p_k|^2 + \varepsilon^2}}$$

i.e., freezing the coefficients

Numerical results

Numerical results



Quality measurements

- Structural similarity: SSIM

  I, K are original resp noisy images, μ_I, μ_K are the meanvalues σ_I, σ_K are the variances, σ_{IK} is the covariance

$$\boldsymbol{SSIM = \frac{(2\mu_I\mu_K + c_1)(2\sigma_{IK} + c_2)}{(\mu_I^2 + \mu_K^2 + c_1)(\sigma_I^2 + \sigma_K^2 + c_2)}}$$

  where c_1, c_2 are two stabilization constants

- Peak signal-to-noise ratio: PSNR

  I, K are original resp noise images of dimension $m \times n$, the mean squared errors $MSE = \frac{1}{mn}\Sigma(I_{ij} - K_{ij})^2$, MAX_I is the maximum value of image I,

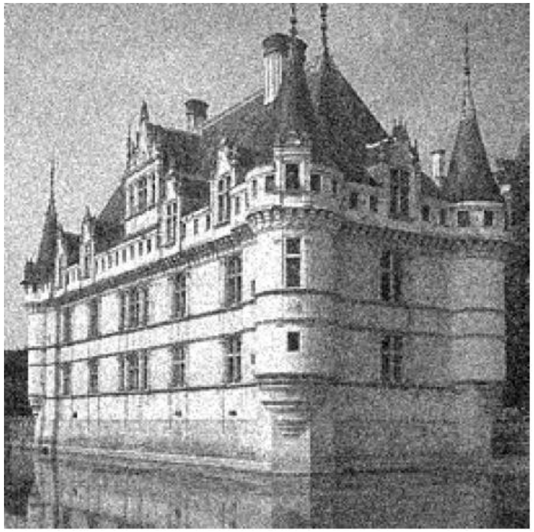
$$PSNR = 20\log(MAX_I) - 10\log(MSE).$$

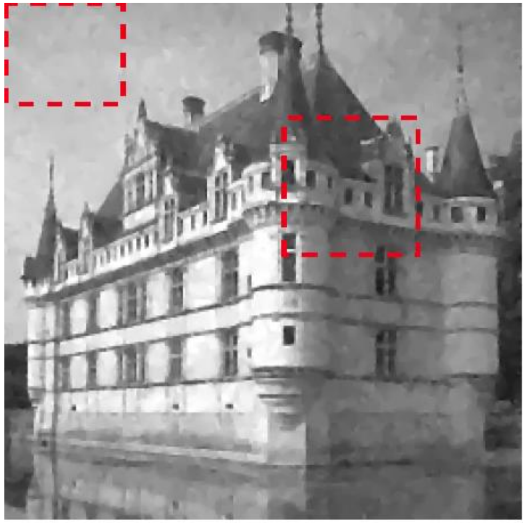
Numerical results

TV PSNR=27.2, SSIM=0.83

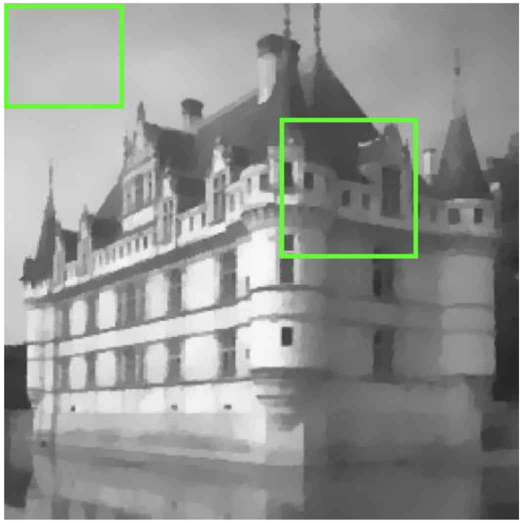
$\Delta_{(p(x),q(x))}$ PSNR=29.2, SSIM=0.87

Numerical results





$\Delta_{(p(x),q(x))}$ PSNR=25.9, SSIM = 0.81



TV PSNR=24.0, SSIM = 0.74