Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Error estimates for elliptic and parabolic equations with oscillating coefficients

Sina Reichelt

Karlstads Universitet

April 13, 2016

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 418 · sina.reichelt@wias-berlin.de

1 Introduction: periodic homogenization

2 Two-scale convergence

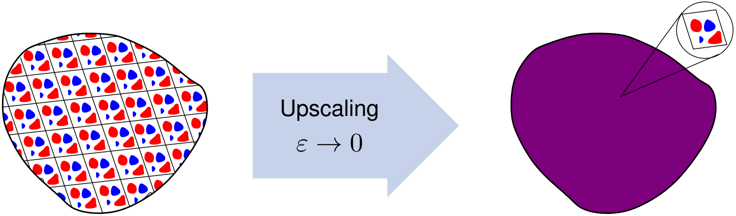
3 Two-scale homogenization of reaction-diffusion systems

4 Error estimates for elliptic equations

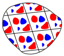
Set-up: Given is a problem (P_ε) involving effects on microscopic scales, where $\varepsilon > 0$ denotes the quotient $\varepsilon = \frac{\text{micro-length}}{\text{macro-length}}$.

Difficulties: Numerical and analytical treatments are very challenging. What happens if ε tends to zero?

Task: Find an effective model problem (P_0), which represents the properties of the original microscopic system well.

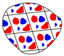


Let us consider the linear elliptic equation



$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P_ε}$$

Let us consider the linear elliptic equation

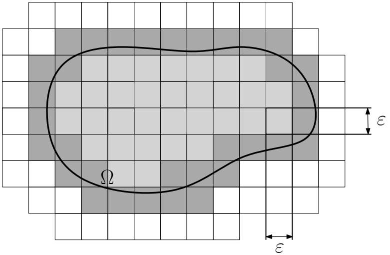


$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f && \text{in } \Omega, \\ u_\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

The coefficients a_ε are of the form

$$a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$$

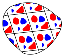
for a given matrix $a(y) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ which is Y-periodic and positive definite.



Notation:

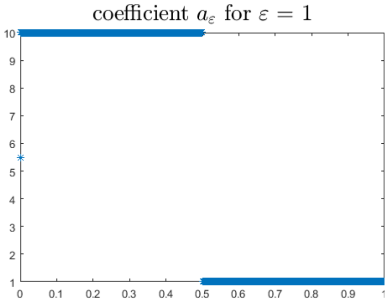
$\Omega \subset \mathbb{R}^d$	macroscopic domain
$Y = (0,1)^d$	unit-cell
$Y_{\mathrm{per}} = \mathbb{R}^d/\mathbb{Z}^d$	torus

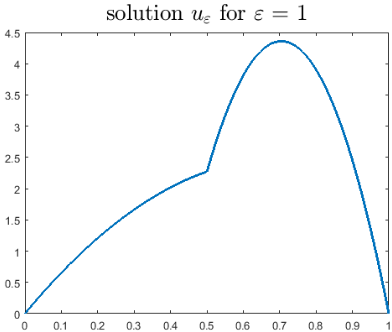
Let us consider the linear elliptic equation



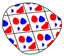
$$
\begin{aligned}
-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\
u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\qquad (\mathrm{P}_\varepsilon)
$$

Example: $\Omega = (0,1)$ one dimensional



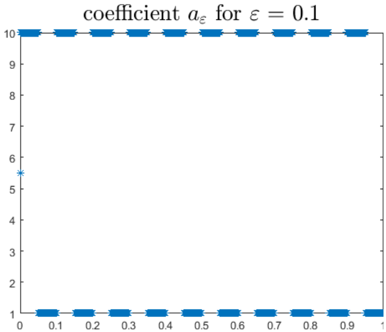


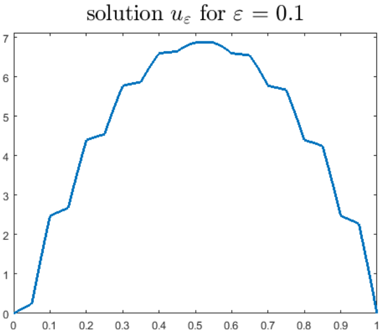
Let us consider the linear elliptic equation



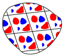
$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

Example: $\Omega = (0,1)$ one dimensional



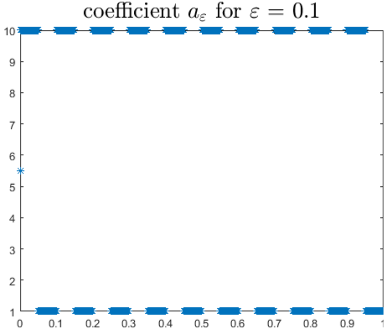


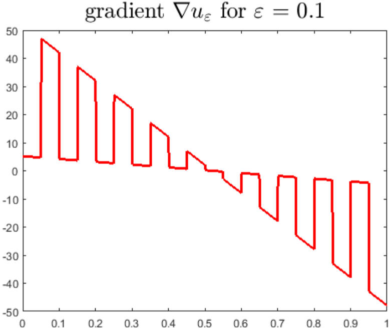
Let us consider the linear elliptic equation



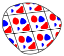
$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

Example: $\Omega = (0,1)$ one dimensional



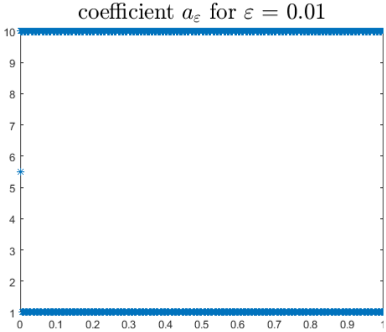


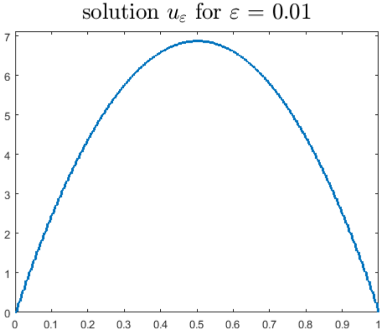
Let us consider the linear elliptic equation



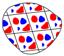
$$
\begin{aligned}
-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\
u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{P_ε}
$$

Example: $\Omega = (0,1)$ one dimensional



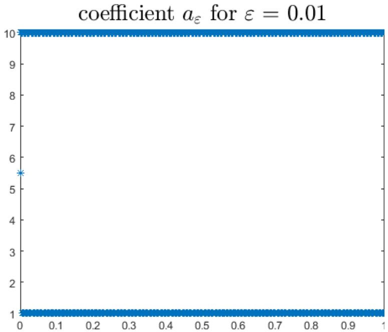


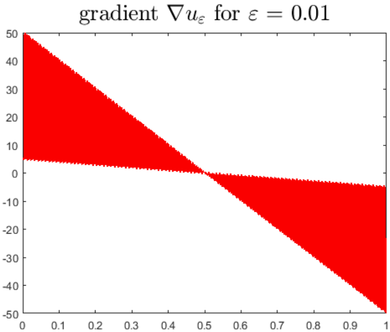
Let us consider the linear elliptic equation



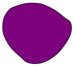
$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in} \quad \Omega, \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

Example: $\Omega = (0,1)$ one dimensional





The homogenized equation reads



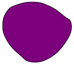
$$\begin{aligned} -\operatorname{div}(a_{\text{eff}} \nabla u) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P$_0$}$$

Example: $\Omega = (0,1)$ one dimensional

The effective coefficient a_{eff} is the harmonic mean, i.e.

$$a_{\text{eff}} = \left(\fint_Y \frac{1}{a(y)} \, \mathrm{d}y \right)^{-1}.$$

The homogenized equation reads



$$
\begin{aligned}
-\operatorname{div}(a_{\text{eff}} \nabla u) &= f && \text{in} \quad \Omega, \\
u &= 0 && \text{on} \quad \partial\Omega.
\end{aligned}
\tag{P_0}
$$

The effective matrix $a_{\text{eff}} \in \mathbb{R}^{d\times d}_{\text{sym}}$ is given by solving the unit-cell problem

$$
\xi \cdot a_{\text{eff}}\, \xi = \min_{\varphi \in \mathrm{H}^1_{\text{per}}(Y)} \fint_Y (\nabla_y \varphi + \xi) \cdot a(y) (\nabla_y \varphi + \xi)\, \mathrm{d}y \qquad \forall\, \xi \in \mathbb{R}^d.
$$

Ansatz of power series in ε for u_ε

$$u_\varepsilon(x) \approx u_0(x, \tfrac{x}{\varepsilon}) + \varepsilon u_1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \tfrac{x}{\varepsilon}) + \dots .$$

Inserting u_ε into (P_ε) and sorting for powers of ε gives

$$u_\varepsilon(x) = u(x) + \varepsilon U(x, \tfrac{x}{\varepsilon}) + O(\varepsilon^2),$$

where u solves the homogenized equation (P_0) and the corrector U is given via

$$U(x,y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x) \cdot z_i(y).$$

Here $z_i \in \mathrm{H}^1_{\mathrm{per}}(Y)$ with $\int_Y z_i \,\mathrm{d}y = 0$ solves the local problem

$$\mathrm{div}_y\left(a(y)(\nabla_y z_i(y) + \mathrm{e}_i)\right) = 0 \qquad \text{in } Y_{\mathrm{per}}.$$

(z_i = minimizers of unit-cell probem for $\xi = \mathrm{e}_i$.)

Ansatz of power series in ε for u_ε

$$u_\varepsilon(x) \approx u_0(x, \tfrac{x}{\varepsilon}) + \varepsilon u_1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \tfrac{x}{\varepsilon}) + \ldots .$$

Inserting u_ε into (P_ε) and sorting for powers of ε gives

$$u_\varepsilon(x) = u(x) + \varepsilon U(x, \tfrac{x}{\varepsilon}) + O(\varepsilon^2),$$

where u solves the homogenized equation (P_0) and the corrector U is given via

$$U(x,y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x) \cdot z_i(y).$$

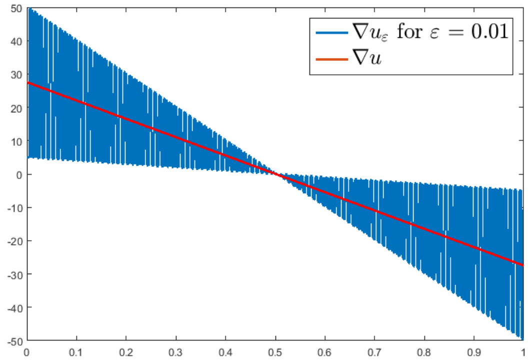
One can prove

$$u_\varepsilon \to u \quad \text{and} \quad \nabla u_\varepsilon \to \nabla u + \nabla_y U(x,y) \quad \text{in the two-scale sense.}$$

Ansatz of p

Inserting u

where u sc ... en via



One can prove

$$u_\varepsilon \to u \quad \text{and} \quad \nabla u_\varepsilon \to \nabla u + \nabla_y U(x,y) \quad \text{in the two-scale sense.}$$

Definition [Nguetseng 1989, Allaire 1992]

We say $u_\varepsilon \rightharpoonup U$ in the two-scale sense, if

$$\int_\Omega u_\varepsilon(x)\Phi(x, \tfrac{x}{\varepsilon})\,\mathrm{d}x \to \int_{\Omega\times Y} U(x,y)\Phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

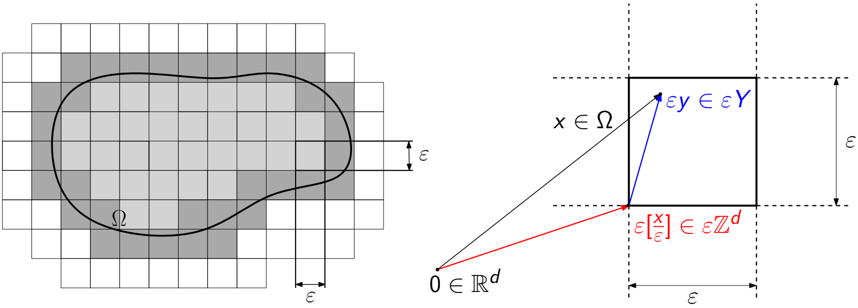
for all $\Phi \in \mathrm{C}_c^\infty(\Omega \times Y_{\text{per}})$.

- This is a weak notion of convergence.
- Strong two-scale convergence formulation via periodic unfolding operator $\mathcal{T}_\varepsilon$.
- The periodic unfolding method enables error estimates.

Definition [Cioranescu et al. 2002]

The **periodic unfolding operator** $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega \times Y_{\text{per}})$ is defined via

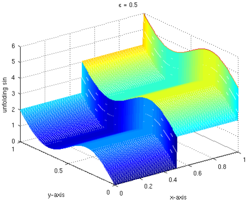
$$(\mathcal{T}_\varepsilon u)(x,y) = u\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right).$$



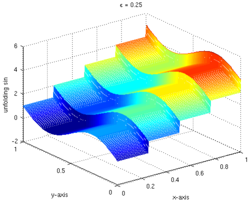
$$(\mathcal{T}_\varepsilon u)(x,y) = u\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)$$

Example: strong two-scale convergence of oscillating functions

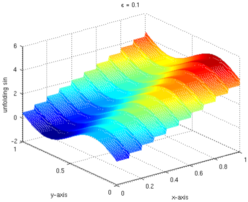
- $\Omega = Y = (0,1)$
- $u_\varepsilon(x) = \sin(2\pi\frac{x}{\varepsilon}) + 4x$
- $(\mathcal{T}_\varepsilon u_\varepsilon)(x,y) = \sin(2\pi y) + 4(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y)$ is not Y-periodic



$\varepsilon = 0.5$



$\varepsilon = 0.25$



$\varepsilon = 0.1$

- Limit $\lim_{\varepsilon\to 0}(\mathcal{T}_\varepsilon u_\varepsilon)(x,y) = \sin(2\pi y) + 4x$ is Y-periodic

$$(\mathcal{T}_\varepsilon u)(x,y) = u\left(\varepsilon\left[\tfrac{x}{\varepsilon}\right] + \varepsilon y\right)$$

Definition (weak and strong two-scale convergence) [Mielke/Timofte 2007]

Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{L}^2(\Omega)$.

$$u_\varepsilon \xrightarrow{2\mathrm{w}} U \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \quad :\overset{\mathrm{Def}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}})$$
$$u_\varepsilon \xrightarrow{2\mathrm{s}} U \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \quad :\overset{\mathrm{Def}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u_\varepsilon \to U \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}})$$

Consider for $\varepsilon > 0$ the following **system of coupled reaction-diffusion equations** with slow diffusion and no-flux boundary conditions in $[0,T] \times \Omega$,

$$\begin{aligned} \dot{u}_\varepsilon &= \mathrm{div}(D_1(\tfrac{x}{\varepsilon})\nabla u_\varepsilon) + F_1(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon), \\ \dot{v}_\varepsilon &= \mathrm{div}(\varepsilon^2 D_2(\tfrac{x}{\varepsilon})\nabla v_\varepsilon) + F_2(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon). \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

Consider for $\varepsilon > 0$ the following **system of coupled reaction-diffusion equations** with slow diffusion and no-flux boundary conditions in $[0, T] \times \Omega$,

$$\begin{aligned} \dot{u}_\varepsilon &= \operatorname{div}(D_1(\tfrac{x}{\varepsilon})\nabla u_\varepsilon) + F_1(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon), \\ \dot{v}_\varepsilon &= \operatorname{div}(\varepsilon^2 D_2(\tfrac{x}{\varepsilon})\nabla v_\varepsilon) + F_2(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon). \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

- Many applied math publications on systems with slow diffusion, i.e. [Hornung/Jäger/Mikelić 1994, Peter/Böhm 2008, Hanke 2011, Mielke/Rohan 2013, Graf/Peter 2014, Jäger/Neuss-Radu 2007, Ptashnyk/Roose 2010, Eck 2004, Fatima et al. 2011, Muntean/van Noorden 2013].

Consider for $\varepsilon > 0$ the following **system of coupled reaction-diffusion equations** with slow diffusion and no-flux boundary conditions in $[0,T] \times \Omega$,

$$\begin{aligned} \dot u_\varepsilon &= \mathrm{div}(D_1(\tfrac{x}{\varepsilon})\nabla u_\varepsilon) + F_1(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon), \\ \dot v_\varepsilon &= \mathrm{div}(\varepsilon^2 D_2(\tfrac{x}{\varepsilon})\nabla v_\varepsilon) + F_2(\tfrac{x}{\varepsilon}, u_\varepsilon, v_\varepsilon). \end{aligned} \qquad (\mathrm{P}_\varepsilon)$$

- Many applied math publications on systems with slow diffusion, i.e. [Hornung/Jäger/Mikelić 1994, Peter/Böhm 2008, Hanke 2011, Mielke/Rohan 2013, Graf/Peter 2014, Jäger/Neuss-Radu 2007, Ptashnyk/Roose 2010, Eck 2004, Fatima et al. 2011, Muntean/van Noorden 2013].
- Proof of $u_\varepsilon \to u$ and $v_\varepsilon \xrightarrow{2s} V$ in [Mielke/R./Thomas, NHM, 2014] for well-prepared initial values (⇒ improved time-regularity for solutions), i.e.

$$\sup_{\varepsilon>0} \| \mathrm{div}(\varepsilon^2 D_2(\tfrac{x}{\varepsilon})\nabla v_\varepsilon(0))\|_{\mathrm{L}^2(\Omega)} < \infty.$$

Theorem ([R., PhD thesis, 2015])

We assume $u_\varepsilon(0) \to u(0)$ *in* $\mathrm{L}^2(\Omega)$ *and* $v_\varepsilon(0) \xrightarrow{2s} V(0)$ *in* $\mathrm{L}^2(\Omega \times Y_{\mathrm{per}})$.
Then the solution $(u_\varepsilon, v_\varepsilon)$ *of* (P_ε) *converges in the two-scale sense to* (u, V) *which solves the* ***effective two-scale model*** (P_0).

Macroscopic equation in $[0,T] \times \Omega$:

$$\dot{u}(t,x) = \underbrace{\mathrm{div}(D_{\mathrm{eff}} \nabla u(t,x))}_{\textit{macro-diffusion}} + \underbrace{\int_Y F_1(y, u(t,x), V(t,x,y))\,\mathrm{d}y}_{\textit{macro-reaction}},$$

Two-scale equation in $[0,T] \times Y_{\mathrm{per}}$, $\quad \Omega \sim$ *"set of parameter":* $\qquad (\mathrm{P}_0)$

$$\dot{V}(t,x,y) = \underbrace{\mathrm{div}_y(D_2(y)\nabla_y V(t,x,y))}_{\textit{micro-diffusion}} + \underbrace{F_2(y, u(t,x), V(t,x,y))}_{\textit{two-scale reaction}}.$$

(1) Unfolding of the original problem (P_ε) to the two-scale space $\Omega \times Y_{\mathrm{per}}$ ...
The folding operator $\mathcal{F}_\varepsilon : \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \to \mathrm{L}^2(\Omega)$ is defined via

$$(\mathcal{F}_\varepsilon V)(x) := \fint_{\varepsilon\left([\frac{x}{\varepsilon}]+Y\right)} V\left(z, \{\tfrac{x}{\varepsilon}\}\right) \,\mathrm{d}z.$$

$$\Longrightarrow \qquad \mathcal{F}_\varepsilon V \xrightarrow{2\mathrm{s}} V \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \quad \text{and} \quad \mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^* .$$

(1) Unfolding of the original problem (P_ε) to the two-scale space $\Omega \times Y_{\mathrm{per}}$ ...
The folding operator $\mathcal{F}_\varepsilon : \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \to \mathrm{L}^2(\Omega)$ is defined via

$$(\mathcal{F}_\varepsilon V)(x) := \fint_{\varepsilon\left([\frac{x}{\varepsilon}]+Y\right)} V\left(z, \{\tfrac{x}{\varepsilon}\}\right) \,\mathrm{d}z.$$

$$\Longrightarrow \qquad \mathcal{F}_\varepsilon V \xrightarrow{2\mathrm{s}} V \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \quad \text{and} \quad \mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^* \,.$$

(2) Finding a test function for (P_ε) ...

$$V\left(x, \tfrac{x}{\varepsilon}\right) \quad \text{and} \quad (\mathcal{F}_\varepsilon V)(x) \quad \text{are not admissible.}$$

The gradient folding operator $\mathcal{G}_\varepsilon : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y)) \to \mathrm{H}^1(\Omega)$ is defined as the solution $\mathcal{G}_\varepsilon V := \widehat{v}_\varepsilon$ of the elliptic problem [Mielke/Timofte 2007, Hanke 2011]

$$\int_\Omega (\widehat{v}_\varepsilon - \mathcal{F}_\varepsilon V) \cdot \varphi + \left(\varepsilon \nabla \widehat{v}_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y V)\right) : \varepsilon \nabla \varphi \,\mathrm{d}x = 0 \quad \forall\, \varphi \in \mathrm{H}^1(\Omega).$$

$$\Longrightarrow \qquad \mathcal{G}_\varepsilon V \xrightarrow{2\mathrm{s}} V \quad \text{and} \quad \varepsilon \nabla(\mathcal{G}_\varepsilon V) \xrightarrow{2\mathrm{s}} \nabla_y V \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}).$$ [Hanke 2011]

(1) Unfolding of the original problem (P_ε) to the two-scale space $\Omega \times Y_{\mathrm{per}}$ ...
The folding operator $\mathcal{F}_\varepsilon : \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \to \mathrm{L}^2(\Omega)$ is defined via

$$(\mathcal{F}_\varepsilon V)(x) := \fint_{\varepsilon\left([\frac{x}{\varepsilon}]+Y\right)} V\left(z, \{\tfrac{x}{\varepsilon}\}\right)\,\mathrm{d}z.$$

$$\Longrightarrow \qquad \mathcal{F}_\varepsilon V \xrightarrow{2\mathrm{s}} V \quad \text{in } \mathrm{L}^2(\Omega \times Y_{\mathrm{per}}) \quad \text{and} \quad \mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^*\,.$$

(2) Finding a test function for (P_ε) ...

$$V\left(x, \tfrac{x}{\varepsilon}\right) \quad \text{and} \quad (\mathcal{F}_\varepsilon V)(x) \quad \text{are not admissible.}$$

The gradient folding operator $\mathcal{G}_\varepsilon : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y)) \to \mathrm{H}^1(\Omega)$ is defined as the solution $\mathcal{G}_\varepsilon V := \widehat{v}_\varepsilon$ of the elliptic problem [Mielke/Timofte 2007, Hanke 2011]

$$\int_\Omega (\widehat{v}_\varepsilon - \mathcal{F}_\varepsilon V)\cdot\varphi + \left(\varepsilon\nabla\widehat{v}_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y V)\right) : \varepsilon\nabla\varphi\,\mathrm{d}x = 0 \quad \forall\,\varphi \in \mathrm{H}^1(\Omega).$$

$$\Longrightarrow \qquad \mathcal{G}_\varepsilon V \xrightarrow{2\mathrm{s}} V \quad \text{and} \quad \varepsilon\nabla(\mathcal{G}_\varepsilon V) \xrightarrow{2\mathrm{s}} \nabla_y V \quad \text{in } \mathrm{L}^2(\Omega\times Y_{\mathrm{per}}).$$ [Hanke 2011]

(3) Finding a test function for (P_0) ... $\mathcal{T}_\varepsilon v_\varepsilon$ is not Y-periodic.

Theorem (R., JPCS, 2016)

Let $(u_\varepsilon, v_\varepsilon)$ *and* (u, V) *be the solution of* (P_ε) *and* (P_0), *respectively . If the initial values are well-prepared and satisfy*

$$\exists\, c \geq 0 : \quad \|u_\varepsilon(0) - u(0)\|_{\mathrm{L}^2(\Omega)} + \|\mathcal{T}_\varepsilon v_\varepsilon(0) - V(0)\|_{\mathrm{L}^2(\Omega\times Y)} \leq \varepsilon^{1/4} c,$$

then there exists a constant $C \geq 0$ *such that*

$$\|u_\varepsilon - u\|_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon) - [\nabla u + \nabla_y U]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times Y))} +$$
$$\|\mathcal{T}_\varepsilon v_\varepsilon - V\|_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega\times Y))} + \|\mathcal{T}_\varepsilon(\varepsilon\nabla v_\varepsilon) - \nabla_y V\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times Y))} \leq \varepsilon^{1/4} C.$$

Theorem (R., JPCS, 2016)

Let $(u_\varepsilon, v_\varepsilon)$ *and* (u, V) *be the solution of* (P_ε) *and* (P_0)*, respectively . If the initial values are well-prepared and satisfy*

$$\exists\, c \geq 0 : \quad \|u_\varepsilon(0) - u(0)\|_{\mathrm{L}^2(\Omega)} + \|\mathcal{T}_\varepsilon v_\varepsilon(0) - V(0)\|_{\mathrm{L}^2(\Omega\times Y)} \leq \varepsilon^{1/4} c,$$

then there exists a constant $C \geq 0$ *such that*

$$\|u_\varepsilon - u\|_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon) - [\nabla u + \nabla_y U]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times Y))} +$$
$$\|\mathcal{T}_\varepsilon v_\varepsilon - V\|_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega\times Y))} + \|\mathcal{T}_\varepsilon(\varepsilon\nabla v_\varepsilon) - \nabla_y V\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times Y))} \leq \varepsilon^{1/4} C.$$

- Work in progress (next section): estimates of order $O(\varepsilon^{1/2})$.
- Estimates for non well-prepared initial values of order $O(\varepsilon^{1/6})$ in [R., PhD thesis, 2015].

Given: elliptic equation with not exatly ε-periodic coefficients

$$v_\varepsilon - \operatorname{div}\left(\varepsilon^2 \mathbb{A}\left(x, \tfrac{x}{\varepsilon}\right) \nabla v_\varepsilon\right) = F_\varepsilon(x) \quad \text{in } \Omega \tag{P_ε}$$

supplemented with no-flux boundary conditions on $\partial\Omega$ and $F_\varepsilon(x) := (\mathcal{F}_\varepsilon \mathbb{F})(x)$.

Aim: prove error estimates under the minimal regularity assumptions

$$\mathbb{A} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{L}^\infty(Y_{\mathrm{per}})) \quad \text{and} \quad \mathbb{F} \in \mathrm{H}^1(\Omega; \mathrm{L}^2(Y_{\mathrm{per}})).$$

Theorem (unpublished)

Let $v_\varepsilon \in \mathrm{H}^1(\Omega)$ solve (P_ε). Then there exists $V \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ solving

$$V - \operatorname{div}_y(\mathbb{A}\nabla_y V) = \mathbb{F} \quad \textit{in } \Omega \times Y_{\mathrm{per}} \tag{P_0}$$

and a constant $C > 0$ such that

$$\| \mathcal{T}_\varepsilon v_\varepsilon - V \|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))} \leq \varepsilon^{1/2} C.$$

Step 1: Testing (P_0) with $\mathcal{T}_\varepsilon\varphi \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y)) \supsetneq \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y))$ is not allowed; thus we obtain the periodicity defect error $\Delta^\varepsilon_{\mathrm{per}}$

$$\left|\int_{\Omega\times Y} \mathbb{A}\nabla_y V\cdot\nabla_y(\mathcal{T}_\varepsilon\varphi) + V\,\mathcal{T}_\varepsilon\varphi - \mathbb{F}\,\mathcal{T}_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}y\right| \leq \Delta^\varepsilon_{\mathrm{per}}\|\varphi\|_\varepsilon,$$

where $\|\varphi\|_\varepsilon := \|\varphi\|_{\mathrm{L}^2(\Omega)} + \varepsilon\|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}$.

Step 1: Testing (P_0) with $\mathcal{T}_\varepsilon\varphi \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y)) \supsetneq \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y))$ is not allowed; thus we obtain the periodicity defect error$\Delta^\varepsilon_{\mathrm{per}}$

$$\left|\int_{\Omega\times Y} \mathbb{A}\nabla_y V\cdot\nabla_y(\mathcal{T}_\varepsilon\varphi) + V\,\mathcal{T}_\varepsilon\varphi - \mathbb{F}\,\mathcal{T}_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}y\right| \leq \Delta^\varepsilon_{\mathrm{per}}\|\varphi\|_\varepsilon,$$

where $\|\varphi\|_\varepsilon := \|\varphi\|_{\mathrm{L}^2(\Omega)} + \varepsilon\|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}$.

Step 2: Using $\mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^*$ and $\nabla_y(\mathcal{T}_\varepsilon\varphi) = \mathcal{T}_\varepsilon(\varepsilon\nabla\varphi)$ as well as estimating the approximation error $\Delta^\varepsilon_{\mathrm{app}}$ for $A_\varepsilon - \mathcal{F}_\varepsilon\mathbb{A}$ gives

$$\left|\int_\Omega A_\varepsilon\,\mathcal{F}_\varepsilon(\nabla_y V)\cdot\varepsilon\nabla\varphi + \mathcal{F}_\varepsilon V\varphi - F_\varepsilon\varphi\,\mathrm{d}x\right| \leq \Delta^\varepsilon_{\mathrm{per,app}}\|\varphi\|_\varepsilon.$$

Step 1: Testing (P_0) with $\mathcal{T}_\varepsilon\varphi \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y)) \supsetneqq \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y))$ is not allowed; thus we obtain the periodicity defect error$\Delta^\varepsilon_{\mathrm{per}}$

$$\left|\int_{\Omega\times Y} \mathbb{A}\nabla_y V\cdot\nabla_y(\mathcal{T}_\varepsilon\varphi) + V\,\mathcal{T}_\varepsilon\varphi - \mathbb{F}\,\mathcal{T}_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}y\right| \leq \Delta^\varepsilon_{\mathrm{per}}\|\varphi\|_\varepsilon,$$

where $\|\varphi\|_\varepsilon := \|\varphi\|_{\mathrm{L}^2(\Omega)} + \varepsilon\|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}$.

Step 2: Using $\mathcal{F}_\varepsilon = \mathcal{T}^*_\varepsilon$ and $\nabla_y(\mathcal{T}_\varepsilon\varphi) = \mathcal{T}_\varepsilon(\varepsilon\nabla\varphi)$ as well as estimating the approximation error $\Delta^\varepsilon_{\mathrm{app}}$ for $A_\varepsilon - \mathcal{F}_\varepsilon\mathbb{A}$ gives

$$\left|\int_\Omega A_\varepsilon\,\mathcal{F}_\varepsilon(\nabla_y V)\cdot\varepsilon\nabla\varphi + \mathcal{F}_\varepsilon V\varphi - F_\varepsilon\varphi\,\mathrm{d}x\right| \leq \Delta^\varepsilon_{\mathrm{per,app}}\|\varphi\|_\varepsilon.$$

Step 3: Testing (P_ε) with $\varphi_\varepsilon = v_\varepsilon - \mathcal{G}_\varepsilon V \in \mathrm{H}^1(\Omega)$ is admissible. By controling the folding mismatch $\Delta^\varepsilon_{\mathrm{fold}}$ of $\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon V$ and $\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon\nabla(\mathcal{G}_\varepsilon V)$, we obatin

$$\alpha\|v_\varepsilon - \mathcal{G}_\varepsilon V\|^2_\varepsilon \leq \Delta^\varepsilon_{\mathrm{per,app,fold}}\|v_\varepsilon - \mathcal{G}_\varepsilon V\|_\varepsilon.$$

It remains: quantitative estimates for the error terms

- approximation error: $\Delta^{\varepsilon}_{\mathrm{app}} \sim O(\varepsilon^{1/2}) \Rightarrow$ easy
- periodicity defect of $\mathcal{T}_\varepsilon \varphi$ for arbitrary $\varphi \in \mathrm{H}^1(\Omega)$: $\Delta^{\varepsilon}_{\mathrm{per}} \sim O(\varepsilon^{1/2})$
  $\Rightarrow$ estimates exist [Griso 2004/05]
- folding mismatch between $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$: $\Delta^{\varepsilon}_{\mathrm{fold}} \sim O(\varepsilon^{1/2}) \Rightarrow$ new estimate

Overall, we arrive at

$$\alpha \|v_\varepsilon - \mathcal{G}_\varepsilon V\|^2_\varepsilon \le \varepsilon^{1/2} C \|v_\varepsilon - \mathcal{G}_\varepsilon V\|_\varepsilon .$$

$\square$

It remains: quantitative estimates for the error terms

- approximation error: $\Delta^{\varepsilon}_{\mathrm{app}} \sim O(\varepsilon^{1/2}) \Rightarrow$ easy
- periodicity defect of $\mathcal{T}_\varepsilon \varphi$ for arbitrary $\varphi \in \mathrm{H}^1(\Omega)$: $\Delta^{\varepsilon}_{\mathrm{per}} \sim O(\varepsilon^{1/2})$ $\Rightarrow$ estimates exist [Griso 2004/05]
- folding mismatch between $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$: $\Delta^{\varepsilon}_{\mathrm{fold}} \sim O(\varepsilon^{1/2}) \Rightarrow$ new estimate

Overall, we arrive at

$$\alpha \|v_\varepsilon - \mathcal{G}_\varepsilon V\|^2_\varepsilon \leq \varepsilon^{1/2} C \|v_\varepsilon - \mathcal{G}_\varepsilon V\|_\varepsilon .$$

□

Related results in the literature:

- Error estimates for slow & classical diffusion based on asymptotic expansion [Eck 2004, Muntean/van Noorden 2013]
- Unfolding based error estimates for classical diffusion [Griso 2004/05, Fatima/Muntean/Ptashnyk 2012]

Proposition [R., PhD thesis, 2015]

For every $V \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ it holds

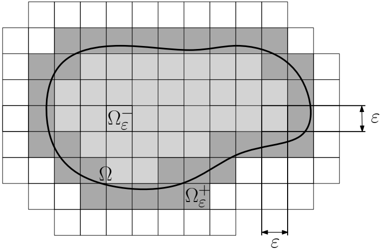
$$\| \mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon \nabla(\mathcal{G}_\varepsilon V) - \mathcal{F}_\varepsilon(\nabla_y V)\|_{\mathrm{L}^2(\Omega)} \leq (\varepsilon + \varepsilon^{1/2}) C \|V\|.$$

Proposition [R., PhD thesis, 2015]

For every $V \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ it holds

$$\|\mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla(\mathcal{G}_\varepsilon V) - \mathcal{F}_\varepsilon(\nabla_y V)\|_{\mathrm{L}^2(\Omega)} \leq (\varepsilon + \varepsilon^{1/2})C\|V\|.$$

Outline of proof: Step 1. Derive boundary estimates of order $O(\varepsilon^{1/2})$.



Assume $\Omega = \Omega_\varepsilon^+$ and redefine $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$ on Ω_ε^+.

Proposition [R., PhD thesis, 2015]

For every $V \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ it holds

$$\| \mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V \|_{\mathrm{L}^2(\Omega)} + \| \varepsilon \nabla (\mathcal{G}_\varepsilon V) - \mathcal{F}_\varepsilon (\nabla_y V) \|_{\mathrm{L}^2(\Omega)} \leq (\varepsilon + \varepsilon^{1/2}) C \|V\|.$$

Outline of proof: Step 1. Derive boundary estimates of order $O(\varepsilon^{1/2})$.

Step 2. Assume $V(x,y) = w(x) \cdot z(y)$ is a product.

- Introduce the scale-splitting operator $\mathcal{Q}_\varepsilon$. [Cioranescu/Damlamian/Griso 2008]
- Use that $x \mapsto (\mathcal{Q}_\varepsilon w)(x) \cdot z(\frac{x}{\varepsilon})$ is an admissible test function in $\mathrm{H}^1(\Omega)$.

Proposition [R., PhD thesis, 2015]

For every $V \in \mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y))$ it holds

$$\|\mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla(\mathcal{G}_\varepsilon V) - \mathcal{F}_\varepsilon(\nabla_y V)\|_{\mathrm{L}^2(\Omega)} \leq (\varepsilon + \varepsilon^{1/2})C\|V\|.$$

Outline of proof: Step 1. Derive boundary estimates of order $O(\varepsilon^{1/2})$.

Step 2. Assume $V(x,y) = w(x)\cdot z(y)$ is a product.

- Introduce the scale-splitting operator $\mathcal{Q}_\varepsilon$. [Cioranescu/Damlamian/Griso 2008]
- Use that $x \mapsto (\mathcal{Q}_\varepsilon w)(x)\cdot z(\frac{x}{\varepsilon})$ is an admissible test function in $\mathrm{H}^1(\Omega)$.

Note that for classical diffusion **and** exactly periodic coefficients we have

$$U(x,y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x)\cdot z_i(y).$$

Proposition [R., PhD thesis, 2015]

For every $V \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ it holds

$$\| \mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V \|_{\mathrm{L}^2(\Omega)} + \| \varepsilon \nabla(\mathcal{G}_\varepsilon V) - \mathcal{F}_\varepsilon(\nabla_y V) \|_{\mathrm{L}^2(\Omega)} \leq (\varepsilon + \varepsilon^{1/2}) C \|V\|.$$

Outline of proof: Step 1. Derive boundary estimates of order $O(\varepsilon^{1/2})$.

Step 2. Assume $V(x, y) = w(x) \cdot z(y)$ is a product.

Step 3. Prove the estimate for general $V(x, y)$.

- For an orthonormal basis $\{\Phi_i\}$ of $\mathrm{H}^1_{\mathrm{per}}(Y)$ use the decomposition

$$V(x, y) = \sum_{i=1}^{\infty} v_i(x) \cdot \Phi_i(y) \quad \text{with} \quad v_i(x) = \int_Y V(x, y) \cdot \Phi_i(y) \, \mathrm{d}y.$$

- Show that $\Phi_i \perp \Phi_j$ in $\mathrm{H}^1_{\mathrm{per}}(Y)$ implies $\mathcal{G}_\varepsilon \Phi_i \perp \mathcal{G}_\varepsilon \Phi_j$ in $\mathrm{H}^1(\Omega)$ for $i \neq j$.
- Exploit Step 2 with $w = v_i$ and $z = \Phi_i$. $\square$

Classical diffusion: admisible test function for $(\mathrm{P}_\varepsilon)_{\mathrm{class}}$ is $\varphi_\varepsilon = u_\varepsilon - [u + \varepsilon\,\mathcal{G}_\varepsilon U]$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon - u\|^2_{\mathrm{L}^2(\Omega)} = \int_\Omega (\dot{u}_\varepsilon - \dot{u})\cdot\varphi_\varepsilon + \underbrace{(\dot{u}_\varepsilon - \dot{u})\cdot\varepsilon\,\mathcal{G}_\varepsilon U}_{\sim O(\varepsilon)}\,\mathrm{d}x.$$

Classical diffusion: admisible test function for $(\mathrm{P}_\varepsilon)_{\text{class}}$ is $\varphi_\varepsilon = u_\varepsilon - [u + \varepsilon\,\mathcal{G}_\varepsilon U]$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon - u\|^2_{\mathrm{L}^2(\Omega)} = \int_\Omega (\dot u_\varepsilon - \dot u)\cdot\varphi_\varepsilon + \underbrace{(\dot u_\varepsilon - \dot u)\cdot\varepsilon\,\mathcal{G}_\varepsilon U}_{\sim O(\varepsilon)}\,\mathrm{d}x.$$

Slow diffusion: We require improved time-regularity, i.e. $\sup_{\varepsilon>0}\|\dot v_\varepsilon\|_{\mathrm{L}^2(\Omega)} < \infty$.

An admisible test function for $(\mathrm{P}_\varepsilon)_{\text{slow}}$ is $\varphi_\varepsilon = v_\varepsilon - \mathcal{G}_\varepsilon V$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\varepsilon - \mathcal{F}_\varepsilon V\|^2 = \int_\Omega (\dot v_\varepsilon - \mathcal{F}_\varepsilon \dot V)\cdot\varphi_\varepsilon + \underbrace{(\dot v_\varepsilon - \mathcal{F}_\varepsilon \dot V)\cdot(\mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V)}_{\sim O(\varepsilon+\varepsilon^{1/2})\text{ folding mismtach}}\,\mathrm{d}x.$$

Classical diffusion: admisible test function for $(\mathrm{P}_\varepsilon)_{\mathrm{class}}$ is $\varphi_\varepsilon = u_\varepsilon - [u + \varepsilon\, \mathcal{G}_\varepsilon U]$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon - u\|^2_{\mathrm{L}^2(\Omega)} = \int_\Omega (\dot u_\varepsilon - \dot u)\cdot\varphi_\varepsilon + \underbrace{(\dot u_\varepsilon - \dot u)\cdot\varepsilon\,\mathcal{G}_\varepsilon U}_{\sim O(\varepsilon)}\,\mathrm{d}x.$$

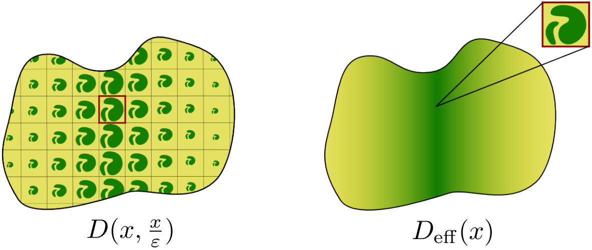
Slow diffusion: We require improved time-regularity, i.e. $\sup_{\varepsilon>0}\|\dot v_\varepsilon\|_{\mathrm{L}^2(\Omega)} < \infty$.

An admisible test function for $(\mathrm{P}_\varepsilon)_{\mathrm{slow}}$ is $\varphi_\varepsilon = v_\varepsilon - \mathcal{G}_\varepsilon V$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\varepsilon - \mathcal{F}_\varepsilon V\|^2 = \int_\Omega (\dot v_\varepsilon - \mathcal{F}_\varepsilon \dot V)\cdot\varphi_\varepsilon + \underbrace{(\dot v_\varepsilon - \mathcal{F}_\varepsilon \dot V)\cdot(\mathcal{G}_\varepsilon V - \mathcal{F}_\varepsilon V)}_{\sim O(\varepsilon+\varepsilon^{1/2})\ \text{folding mismtach}}\,\mathrm{d}x.$$

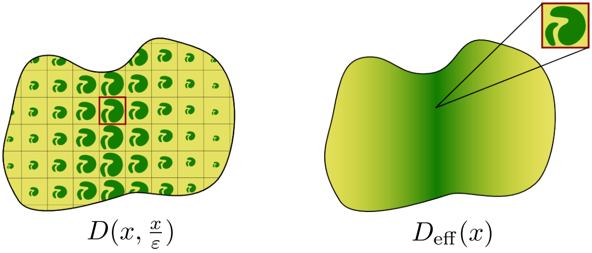
- To obtain the convergence rate $\varepsilon^{1/2}$, carry out estimations on Ω_ε^+ such that $\Delta^\varepsilon_{\mathrm{fold}} \sim O(\varepsilon)$.

- Two-scale convergence “makes” asymptotic expansion rigorous.
- Not exactly periodic microstructures are possible.



- Periodic unfolding method gives rise to error estimates for possibly discontinuous data (w.r.t. $y \in Y_{\text{per}}$).
- For homogenization without error estimates discontinuities of the data w.r.t. $x \in \Omega$ are allowed, too.

- Two-scale convergence "makes" asymptotic expansion rigorous.
- Not exactly periodic microstructures are possible.



- Periodic unfolding method gives rise to error estimates for possibly discontinuous data (w.r.t. $y \in Y_{\text{per}}$).
- For homogenization without error estimates discontinuities of the data w.r.t. $x \in \Omega$ are allowed, too.

Thank you for your attention.