Nonlinear Approximation
- An Idiot Abroad

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KAAS Colloquium
Overview: objectives

Objectives of this talk

1. informally discuss some basic ideas from nonlinear approximation and their applications in computation;
2. vaguely describe the results of [1], with minimal use of black magic from theory of function spaces

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Overview: approximation and computation

Approximation Theory - resolve a complicated target function by a sequence of functions of small complexity (approximants).

Computation - in a sense the same goal.

What assumptions?

Approximation: some direct knowledge of the target function, typically values of simple functionals acting on it (e.g. point evaluations).

Computation: here knowledge of the target function is usually indirect, e.g. it satisfies a PDE.

Still, the subjects are closely connected.
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Notation

$\Omega \subseteq \mathbb{R}^d$ domain, $p \in (0, \infty]$.

$L^p(\Omega)$ - space of functions such that

$$\|f\|_p := \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} < \infty$$

or

$$\|f\|_\infty = \text{ess sup}_{x \in \Omega} |f(x)| < \infty.$$ 

$H^1_0(\Omega)$ - space of functions $f$ such that $f = 0$ on $\partial \Omega$ and

$$\|f\|_{H^1_0(\Omega)} := \|f\|_2 + \|\nabla f\|_2 < \infty$$
Linear approximation

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Sequence \(\mathcal{F} = \{X_j\}\) of finite-dimensional subspaces

\[ X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X \]

(approximation scheme)
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Error of best approximation of \(f \in X\)

\[
E_n(f)_X = E_n(f, \mathcal{F})_X = \inf_{g \in X_n} \|f - g\|_X
\]

Note \(E_n(f)_X\) decreasing sequence of real numbers
Example: Weierstrass’ theorem

**Example:** $X = C(0, 1)$ with norm $\| \cdot \|_X = \| \cdot \|_{\infty}$

$$X_n = \mathcal{P}_n = \{ \text{polynomials of degree } \leq n \}$$

and

$$E_n(f)_{\infty} = \inf_{p \in \mathcal{P}_n} \| f - p \|_{\infty}$$
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\[ E_n(f)_\infty = \inf_{p \in \mathcal{P}_n} \| f - p \|_\infty \]

Weierstrass’ theorem:

\[ \lim_{n \to \infty} E_n(f)_\infty = 0 \]
One main question

\[ E_n(f)_X \] encodes quality of approximation - central object.
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**Given**
- approximation scheme (several examples below);
- norm $\| \cdot \|_X$ to measure error of best approximation (e.g. $X = L^p$).
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**Question**

relationship between

intrinsic properties of \( f \) \( \iff \) behaviour of \( E_n(f)_X \)?
One main question

Main focus of our research in this area

Interested in results of the following type

\[ \frac{E_n(f)}{X} = O\left(n^{-\gamma}\right) \] (direct result/Jackson estimate)

\[ \frac{E_n(f)}{X} = O\left(n^{-\gamma}\right) \] (inverse result/Bernstein estimate)

Ideally, direct and inverse results should match (i.e. \( \square = \square \)); not always the case.
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Nonlinear approximation: set-up

Sequence $\mathcal{F} = \{X_j\}$ of sets/manifolds (not vector spaces) such that:

$X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$

$aX_j = X_j (\forall a \in \mathbb{R})$ and

$X_{n+1} + X_n = \{x + y : x, y \in X_n\} \subset X_{cn}$ some fixed $c \in \mathbb{N}$ (bounded nonlinearity)

As before, $E_n(f)_{X} = E_n(f, \mathcal{F})_X = \inf_{g \in X_n} \|f - g\|_X$
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Example: $N$-term approximation

\[ \Psi = \{ \psi_k \} \text{ a sequence of functions (think of as basis)} \]
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**Compression:** approximate a signal having $\#(spectrum) = M$ by using $N \ll M$ frequencies.
Example: Free knot spline approximation

Set of points ("knots")

\[ T = \{ 0 = x_0 < x_1 < \ldots < x_n = 1 \} \]
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A \textit{k-th order spline on} \ \mathcal{T}

= function \( s \) such that \( s \) is a polynomial of degree \( \leq k \) on each \( (x_j, x_{j+1}) \).
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\( S_0(\mathcal{T}) = \) all step functions (uniform step \( 1/n \))
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Spline manifolds

\[ X_n = S(n, k) = \{ s : \exists T \text{ such that } s \in S_k(T) \text{ and } \#(T) \leq n + 1 \} \]
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What is the point?! Point is that partitions are allowed to adapt to target function ⇒ better approximating power
A comparison

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Philosophy: place the knots where they are useful!
(≈ equidistribute error/local variation)
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Error rates for approximation of \( f(x) = x^\alpha \) (0th order spline, \( L^\infty \)-norm):

\[
E_n^L(f)_{\infty} \asymp \frac{1}{n^\alpha} \quad \text{and} \quad E_n^{NL}(f)_{\infty} = \frac{1}{n}
\]

Nonlinear method has faster convergence!
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Error rates for approximation of $f(x) = x^\alpha$ (0th order spline, $L^\infty$-norm):

$$E^L_n(f)_\infty \simeq \frac{1}{n^\alpha} \quad \text{and} \quad E^{NL}_n(f)_\infty = \frac{1}{n}$$

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**Direct and inverse theorem** (0th order spline, $L^\infty$)

$$E^L_n(f)_\infty = O(1/n) \quad \iff \quad f' \text{ bounded}$$

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**General** Optimal rate attained for wider class of functions.
Nonlinear approximation and computations

Extended example (applications)

Let $\Omega \subset \mathbb{R}^2$ and consider Dirichlet problem for Poisson's equation:

\[
\begin{align*}
-\Delta u &= f & \text{in } \Omega \\
 u &= 0 & \text{on } \partial \Omega
\end{align*}
\]

Weak formulation of (⋆):

\[
\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx \quad \forall v \in H^1_0(\Omega)
\]

Galerkin method:

Solve (⋆⋆) in a finite-dimensional subspace $V \subset H^1_0(\Omega)$

How to choose $V$?
Nonlinear approximation and computations

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How to choose $V$?
Nonlinear approximation and computations

- $\mathcal{T} =$ triangulation of $\Omega$, i.e. $\Omega \approx \bigcup_{\Delta \in \mathcal{T}} \Delta$
- $\mathcal{V} =$ vertices of triangles of $\mathcal{T}$
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**Courant elements**: for each $P \in \mathcal{V}$, define a continuous function $\varphi_P$ by

1. $\varphi_P(P) = 1$;
2. $\varphi_P(Q) = 0$ for $Q \in \mathcal{V} \setminus \{P\}$;
3. the restriction of $\varphi_P$ to each $\Delta \in \mathcal{T}$ is affine
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$$V := S^0_1(\mathcal{T}) = \text{span}\{\varphi_P : P \in \mathcal{V}\},$$

1st degree splines (restrictions to $\Delta$’s have degree $\leq 1$) with smoothness 0 (i.e. continuous).
Nonlinear approximation and computations

Why is this choice of \( V \) good?
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$\mathcal{T} := \mathcal{T}_h$, Galerkin method gives approximate solution

\[ u_h \in V_h := S^0_1(\mathcal{T}_h) \]
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A priori estimate: $u$ exact solution to $(\star)$

$$\|u - u_h\|_{H^1_0(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

if $\partial\Omega$ smooth.
Nonlinear approximation and computations: AFEM

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Scheme Use a posteriori error estimator to refine triangulation at necessary places

\[ h = \max(\text{diam}\Delta) \] loses its value as measure of coarseness

Substitute: \( n = \text{number of triangles} \)
Nonlinear approximation and computations: AFEM

Adaptive schemes 'seems' reasonable, but rigorous derivation?

For any Galerkin solution $u$ on $n$ triangles:

$$E_n(u)_{H^1_0(\Omega)} \leq \|u - U\|_{H^1_0(\Omega)}$$

but $\|u - U\|_{H^1_0(\Omega)}$ may be much larger.

Theorem (Binev, Dahmen, DeVore '04)

Let $u$ be the solution to $(\star)$. If $u$ can be approximated (nonlinearly) by 1st order continuous splines with rate $E_n(u)_{H^1_0(\Omega)} = O(n^{-\gamma})$ then there is an explicit adaptive algorithm that in $O(n)$ steps constructs a triangulation $T_n$ with $\#T_n = O(n)$ and a Galerkin solution $u_n \in S_{0,1}(T_n)$ such that $\|u - u_n\|_{H^1_0(\Omega)} = O(n^{-\gamma})$.
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$$E_n(u)_{H_0^1(\Omega)} \leq \| u - U \|_{H_0^1(\Omega)}$$

but $\| u - U \|_{H_0^1(\Omega)}$ may be much larger

Theorem (Binev, Dahmen, DeVore ’04)

Let $u$ be the solution to $(\star)$. If $u$ can be approximated (nonlinearly) by 1st order continuous splines with rate

$$E_n(u)_{H_0^1(\Omega)} = \mathcal{O}(n^{-\gamma})$$
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*then there is an explicit adaptive algorithm that in $\mathcal{O}(n)$ steps constructs a triangulation $T_n$ with $\|T_n\| = \mathcal{O}(n)$ and a Galerkin solution $u_n \in S_1^0(T_n)$ s.t.*

$$\|u - u_n\|_{H^1_0(\Omega)} = \mathcal{O}(n^{-\gamma})$$
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Program: from approximation to computation
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Moral of the story: nontrivial computational information obtained from rate of approximation.
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Roughly: $f \in B^s_{\tau,\tau}$ means that $f$ has partial derivatives up to order $s$ in $L^\tau(\Omega)$.

Since $s$ may be fractional, definition is not so simple.
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$B^s_{\tau,\tau}$ closely related to nonlinear approximation in $L^p(\Omega)$ ($\Omega \subset \mathbb{R}^d$) when

$$\frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}$$

(Critical line; Sobolev embedding theorem $B^s_{\tau,\tau} \hookrightarrow L^p(\Omega)$.)
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**Notation** $S(n, 1, 0)$: set of continuous functions $S$ on $\Omega$ such that there exists a 'triangulation' $\mathcal{T} = \{\Delta\}$ with

$$S|_\Delta \text{ is affine and } \#\mathcal{T} \leq n$$
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For continuous piecewise linear spline approximation on 'triangles' and parameters satisfying

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**Inverse estimate** Assume that \( S_1 \in S(n, 1, 0) \) and \( S_2 \in S(Kn, 1, 0) \), then

\[ |S_2|_{B^s_{\tau,\tau}} \leq |S_1|_{B^s_{\tau,\tau}} + cn^{s/d} \|S_1 - S_2\|_p \]
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\( S_1 \in B^s_{\tau,\tau} \) 'simple' function, \( S_2 \) 'complex' function;
If error \( \|S_2 - S_1\|_p = O(n^{-s/d}) \), then
\[
|S_2|_{B^s_{\tau,\tau}} \leq |S_1|_{B^s_{\tau,\tau}} + cn^{s/d} \times n^{-s/d} < \infty.
\]